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Quenched massive Schwinger model in the infrared approximation

F. Guerin
Laboratoire de Physique Théorique, Université de Nice, Parc Valrose, 06034 Nice Cedex, France

H. M. Fried
Physics Department, Brown University, Providence, Rhode Island 02912
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In the quenched, continuum version of the Schwinger model, a gauge-invariant summation over soft photons exchanged across a fermion loop is performed for the order parameter $\langle \bar{\psi}\psi \rangle$ and the correlation function $\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y) \rangle$ with a fermion mass $m \neq 0$ and photon momentum $k < m$. The limit $m \rightarrow 0$ leads to a finite, nonzero value for $\langle \bar{\psi}\psi \rangle$; in the correlation function the leading terms, proportional to $\langle \bar{\psi}\psi \rangle^2$, cancel between the two types of diagrams leaving free massless propagation.

I. INTRODUCTION

Numerical simulations with fermions[1,2] use the lattice version of two-dimensional models as a testing ground. Comparisons are made between the "quenched" approximation, where no virtual fermion loops are included, and the full version of the theory; also results for $m=0$ are obtained from an extrapolation from $m \neq 0$. The conventional wisdom concerning the Schwinger model[3,4] suggests that the photon field gets a mass from fermion-loop vacuum polarization, and one might imagine that much of the physics is lost in the quenched version of the theory. A first numerical simulation[1] found that the chiral-symmetry-breaking parameter has about the same value in both versions; however, a recent calculation[5] found a different value for smaller values of m.

Exact results for the full Schwinger model have been obtained by performing a chiral transformation on the fermion field[3] or by bosonization methods.[4] Three recent papers address the question of the physics of the quenched model; two use bosonization methods,[5,6] while the other estimates the density of zero-energy states for a massless fermion propagating in a random background field.[7] In the present paper, the problem is approached in a simple way, by summing up the contributions of the soft photons exchanged across a closed fermion loop. We use a continuum, infrared (IR) method that has elsewhere[8] been applied to certain strong-coupling (SC) problems in field theory; this IR approximation is only the first step of a systematic approximation scheme, but it should be relevant in continuum SC situations. As the several machine groups have used QED_2 as a theoretical laboratory to explore possible techniques, so we apply here the IR method to the same problems. We compute the order parameter and the correlation function for the case $m \neq 0$, without internal fermion loops, and study the limit as the fermion mass vanishes. Our aim is to answer the following questions in the context of the IR method: How is chiral symmetry broken as $m \rightarrow 0$? Does one encounter infrared divergences as $m \rightarrow 0$?

Also for the correlation function, with an eye on numerical simulations, one will attempt to answer these questions: At what distance scale should one sit to obtain the large-distance results of the $m=0$ case? Are there cancellations between the different types of diagrams?

The basic ideas underlying the IR method are easily explained. Quantities under consideration involve the exchange of an infinite number of virtual photons across a closed fermion loop. Let $G_c(A)$ describe the causal propagator of the fermion in the presence of a background field $A_\mu(x)$; as described by Schwinger[9,10] a long time ago, the closed fermion loop can be constructed in terms of $G_c(A)$, with the linkage of all A dependence (via the free photon propagator) generating all the desired photon insertions. If one wishes to restrict the calculation to "soft" virtual photons, a first guess would be to replace $G_c(A)$ by the Bloch-Nordsieck no-recoil propagator[11,8] $G_v(A)$, where v represents an "averaged" fermion four-velocity. Our method makes use of an exact representation due to Fradkin[12] for $G_c(A)$ in terms of Gaussian fluctuations of $G_v(A)$ over the four-vector v_μ which is now dependent upon the fermion's proper time s. Then a definition of a "soft photon" which preserves gauge invariance can be given, in terms of an upper cutoff μ_c to the virtual-photon spectrum. This, together with the further simplification of calculating only the most important IR part of the resulting expressions, defines the method of "IR extraction." The choice $\mu_c = c/\sqrt{\tau}$ with $s=-i\tau$ and c a real, positive constant of the order of unity, reproduces for the fermion loop the correct τ dependence of the exact form. An alternative choice is $\mu_c = cm$.

The freedom of choice of the constant $c \simeq 1$ represents the basic arbitrariness in the definition of soft photons; different choices correspond to a different separation of soft and "hard" effects. Any exact result must be independent of c; but this parameter will appear in any approximation. An appropriate numerical value to c can then be assigned by comparison with a known physical parameter of the theory, or, lacking that, by the minimization of a ground-state energy, or some reasonable physical requirement external to the computation. In our case, it can be trivially specified by a comparison with the known, exact value of the order parameter, so that $\langle \bar{\psi}\psi \rangle$ calculated in the quenched IR approximation becomes exact in the SC limit. One would then expect that other, more complicated correlation functions would be accurately reproduced in their SC regions by the same choice of the constant c.

The picture that emerges from this approximation may be described in the following way: the fermion moves in a constant background field whose scale is $F_{\mu\nu}\sim\mu_c$; one subsequently performs a Gaussian average over the field strength; this background field is imaginary. In other terms, one may undo the linkage introduced by the soft photons exchanged and replace it by an effective imaginary constant field, the linkage being effected by the subsequent Gaussian integration. One may ask what physical picture is associated with the propagation of a fermion in a constant background field, when it is analytically continued to the case of an imaginary field. One important point is that the gauge-covariant multiplicative phase factor of the propagator always disappears for a fermion loop; for an imaginary field this phase factor would become a real factor. The remaining part of the propagator is smoothly behaved, sharing some of the properties of a real field. If one accepts this result, all of our further results can be obtained from Schwinger's proper-time representation for a fermion propagating in a constant background field.[9]

It is no surprise that an IR method generates forms very close to those found in Schwinger's constant-background-field solution, but it should be noted that the appearance of our constant, imaginary background field is just one way of representing the output of the IR approximation, *after* the fluctuations of the virtual photons are summed over. We most certainly do not replace the photon fields $F_{\mu\nu}(z)$ appearing in the relevant Green's functions by constants; rather, the IR method replaces them by functions whose Fourier components are suitably limited to frequencies less than m. In the limit as $m\to 0$ the graphs containing virtual-photon exchange, each of which individually vanish, all sum together to give a nonzero result. This is the heart of the IR approximation in this problem, but how it relates to other approaches, for example, studies of localization, we do not know. In our method we sum over a continuous spread of low frequencies, and estimate the quantum fluctuations due to such components; but the final results are going to resemble Schwinger's, and in fact turn out to be expressible as Gaussian quadratures over his forms.

The arrangement of these remarks is as follows. In the next section we state and discuss some of the relevant Schwinger-model results, noting how one could have predicted Van den Doel's IR divergence of the quenched, massless order parameter. In Sec. III we first (and very briefly) reproduce the Schwinger-Fradkin representation for $G_c(A)$, and apply it (very briefly) to the special case of a constant electromagnetic field. In Sec. IV we apply the representation to the computation of $\langle\bar\psi\psi\rangle$ in the quenched IR approximation; as $m\to 0$ no IR divergence is found. The correlation function $\langle\bar\psi(x)\psi(x)\bar\psi(y)\psi(y)\rangle$ is discussed in Secs. V and VI, in terms of quadratures over fermions propagators defined in a constant background field; the derivations here are confined to Appendix A. Again, as $m\to 0$, no IR divergences are found. Section VII contains our conclusions, as well as a simple illustration of why Refs. 5 and 6 were able to find IR singularities, and we do not. In Appendix B, written by one of us (F.G.) the nonrelativistic form of the fermion propagator is discussed, along with its relation to the bound states of the e^+-e^- pair in the exact Coulomb and IR approximation calculations.

II. SOME RESULTS FOR THE SCHWINGER MODEL

We recall some results for the original, $m=0$ Schwinger model, to be compared to our results for the quenched version of the model as $(m/g)\to 0$.

A. Lagrangian

The Lagrangian of the model

$$\mathscr{L}=-\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}-\bar\psi(i\partial\!\!\!/-gA\!\!\!/)\psi \tag{2.1}$$

possesses the global symmetry $\psi(x)\to e^{i\theta\gamma_5}\psi(x)$. The set of possible vacuum states obeys

$$\langle 0\,|\,[\bar\psi\psi(x)]^2+[\bar\psi i\gamma_5\psi(x)]^2\,|\,0\rangle=C^2\ ,$$

where the constant C is obtained via bosonization methods.[4] One vacuum state is characterized by its θ value:

$$\langle 0\,|\,\bar\psi\psi(x)\,|\,0\rangle_\theta=C\cos\theta\ .$$

For such a vacuum, the cluster decomposition is valid,

$$\langle 0\,|\,\bar\psi\psi(x)\bar\psi\psi(y)\,|\,0\rangle_\theta\to\langle 0\,|\,\bar\psi\psi(x)\,|\,0\rangle_\theta^{\,2}\quad\text{as } (x-y)^2\to\infty\ .$$

The $m=0$ result amounts to an average over θ values whereas the $m\to 0$ limit selects the $\theta=0$ vacuum so that[1]

$$\langle 0\,|\,\bar\psi\psi(x)\bar\psi\psi(y)\,|\,0\rangle_{m=0}\to\tfrac{1}{2}[\langle\bar\psi\psi(x)\rangle_{m\to 0}]^2\ . \tag{2.2}$$

B. Derivation of correlation function

A rapid derivation of the correlation function for the case $m=0$ is as follows.

If one performs a chiral transform of the fermion field in (2.1),

$$\tilde\psi(x)=e^{i\theta(x)\gamma_5}\psi(x)$$

with (2.3)

$$\partial_\nu\theta(x)=g\epsilon_{\mu\nu}A_\mu(x)\ ,$$

one obtains[13,14] an equivalent action of a free massless fermion $\tilde\psi(x)$ and a free massive vector field $\partial_\nu\theta(x)$ whose (mass)^2 is the anomaly g^2/π. Indeed an infinitesimal chiral transform should generate in the action the correct conservation law for the chiral current; the successive infinitesimal transforms build up a mass for the vector field.

For a later purpose, the anomaly will be written g'^2/π. With the help of the chiral transform (2.3) the correlation function becomes

$$\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle = -\mathrm{tr}G_0(x-y)G_0(y-x) \times \langle \exp\{2i\gamma_5[\theta(x)-\theta(y)]\}\rangle = -\mathrm{tr}G_0(x-y)G_0(y-x) \times \exp\{-\langle 0|[\theta(x)-\theta(y)]^2|0\rangle\} ,$$

where $G_0(x-y)$ is the propagator of a massless fermion

$$G_0(x-y)=\frac{-i}{2\pi}[\gamma\cdot(x-y)]^{-1} , \tag{2.4}$$

so that

$$\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle = \frac{1}{2\pi^2(x-y)^2}\exp\left[g^2\int\frac{d^2k}{k^2}\frac{|e^{ik\cdot(x-y)}-1|^2}{k^2+g'^2/\pi}\right] = \frac{1}{2\pi^2}\frac{1}{(x-y)^2}\exp\left\{2\frac{g^2}{g'^2}\left[\gamma+\ln\left[\frac{g'|x-y|}{2\sqrt{\pi}}\right]+K_0(g'|x-y|/\sqrt{\pi})\right]\right\} , \tag{2.5}$$

where γ is Euler's constant.

Other relations are

$$\langle\bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(y)\rangle = -\langle\bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle , \quad \sum_\mu\langle\bar{\psi}\gamma_\mu\psi(x)\bar{\psi}\gamma_\mu\psi(y)\rangle=0 . \tag{2.6}$$

For the realistic case $g'=g$ and $g'|x-y|\gg 1$ one obtains[14]

$$\langle\bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle=\frac{g^2}{8\pi^3}e^{2\gamma}[1+2K_0(g|x-y|/\sqrt{\pi}) + \cdots] . \tag{2.7}$$

The first term of the right-hand side of Eq. (2.7) exhibits the property (2.2), the second term is the free propagator of the scalar state of the model with mass $g/\sqrt{\pi}$.

As soon as the strength of the anomaly is decreased, the correlation function increases at large distances; for $g'<g$ and $g'|x-y|\gg 1$

$$\langle\bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle\simeq\frac{1}{2\pi^2}(x-y)^{-2(1-g^2/g'^2)} \times\exp\left\{2\frac{g^2}{g'^2}\left[\gamma+\ln\left[\frac{g'}{2\sqrt{\pi}}\right]\right]\right\} ,$$

which suggests that $\langle\bar{\psi}\psi(x)\rangle_{m\to 0}$ could be infinite. That this might be the case has been argued recently by Van den Doel;[6] but one should note that the limit $m=0$ has been taken here at the very beginning, before any computation is performed

III. SOME FORMALISM AND THE IR APPROXIMATION

A. The Fermion propagator in an arbitrary background field

The essential techniques have been discussed in the second paper of Ref. 8; one writes $G_c(x,y|gA) = \langle x|G_c(gA)|y\rangle$ and uses

$$G_c(gA)=i[m+\gamma\cdot\Pi(x)][(\gamma\cdot\Pi)^2-m^2]^{-1} \tag{3.1}$$

with $\Pi_\mu=i\partial_\mu-gA_\mu$ and $(\gamma\cdot\Pi)^2=\Pi^2-(g/2)\sigma_{\mu\nu}F_{\mu\nu}$. One introduces the proper-time representation suggested by Schwinger[3]

$$G_c(gA)=(m+\gamma\cdot\Pi)\int_0^\infty ds\, e^{-ism^2}\cdot e^{is(\gamma\cdot\Pi)^2} . \tag{3.2}$$

The representation of Fradkin[12] replaces the $\exp[+is(\gamma\cdot\Pi)^2]$ factor by the quantity

$$U(s)=\left\{\exp\left[i\int_0^s ds'\left[\Pi^2-v_\mu(s')\Pi_\mu-\frac{g}{2}\sigma\cdot F\right]\right]\right\}_+ , \tag{3.3}$$

where $v_\mu(s)$ denotes an arbitrary vector function of the proper time. The symbol $\{\ \}_+$ represents an ordering with respect to the variable s'. Then

$$G_c(ga)=\int_0^\infty ds\, e^{-ism^2}\left[m+i\gamma\cdot\frac{\delta}{\delta v(s)}\right]U(s)\bigg|_{v_\mu\to 0} . \tag{3.4}$$

The advantage of doing this is that all the complexity of the problem, the lack of commutativity of Π_μ and Π_ν, can be represented in terms of Gaussian fluctuations of the $v_\mu(s)$:

$$U(s)=\exp\left[-i\int_0^{s-\epsilon}ds'\frac{\delta^2}{\delta v^2(s')}\right] \times\left\{\exp\left[-i\int_0^s ds'\left[v_\mu\Pi_\mu+\frac{g}{2}\sigma\cdot F\right]\right]\right\}_+ , \tag{3.5}$$

where ε is a small, positive parameter, subsequently set equal to zero. To demonstrate that (3.5) provides a representation of (3.3) one need only examine the equations obtained from (3.3) for $\partial U/\partial s$ and $\delta U/\delta v_\mu(s)$. The essential and useful point is that the ordered bracket of (3.5) has an explicit representation:

$$\langle x|\left\{\exp\left[-i\int_0^s ds'\left[v_\mu(s')\Pi_\mu+\frac{g}{2}\sigma\cdot F\right]\right]\right\}_+|y\rangle = \delta^{(2)}\left[x-y+\int_0^s ds' v(s')\right] \times\exp\left[ig\int_0^s ds'[v_\mu(s')+\sigma_{\mu\nu}\partial_\nu] \times A_\mu\left[y-\int_0^{s'}ds''v(s'')\right]\right] . \tag{3.6}$$

In obtaining (3.6) the quantity

$$\exp\left[-i\int_0^s ds' v_\mu(s') i\partial_\mu^x\right]$$

has been recognized as a translation operator. The constant v_μ of the Block-Nordsieck approximation has been replaced by the variable $v_\mu(s)$, the four-velocity of the fermion as it propagates from y to x, gaining a phase by interaction with the background field A_μ at each point linking y and x as specified by the coordinate difference $\int_0^s ds' v(s')$; and one subsequently sums in (3.6) over all possible points or paths, between y and x. The effect of Fradkin's representation is to write exact quantities in terms of Gaussian variations of the parameter $v_\mu(s)$, and this can be useful in a wide variety of physical problems.[15]

It will be convenient to adopt an integral version of (3.5),

$$\exp\left[-i\int_0^s ds' \frac{\delta^2}{\delta v^2(s')}\right]\mathscr{F}(v)\Big|_{v\to 0}$$

$$=N(s)\int d[\phi]\exp\left[-\frac{i}{4}\int_0^s ds' \sum_\mu \phi_\mu{}^2(s')\right]\mathscr{F}(\phi) ,$$

$N(s)$ in the normalization factor so that the quantity is unity for $\mathscr{F}=1$. Substitution of these equations into (3.4) then yields

$$\langle x\,|\,G_c(gA)\,|\,y\rangle=\int_0^\infty ds\, e^{-ism^2}N(s)\int d[\phi]\exp\left[-\frac{i}{4}\int_0^s ds'\phi^2(s')\right]$$
$$\times[m-\tfrac{1}{2}\gamma\cdot\phi(s)]\delta^{(2)}\left[x-y+\int_0^s ds'\phi(s')\right]\exp\left[i\int f_\mu(z)A_\mu(z)d^2z\right] , \quad (3.7)$$

where

$$f_\mu(z)=g\int_0^s ds'[\phi_\mu(s')+\sigma_{\mu\nu}\partial_\nu^z]\delta\left[z-y+\int_0^{s'} ds''\phi(s'')\right] . \quad (3.8)$$

An integration by parts has replaced $[m+i\gamma\cdot\delta/\delta\phi(s)]$ by $[m-\frac{1}{2}\gamma\cdot\phi(s)]$ in (3.7). Equation (3.7) is an exact expression for the fermion propagator in an arbitrary background field; it will be our starting point in Sec. IV and Appendix A. Note that this equation is gauge covariant. Under a gauge transformation $A_\mu(z)\to A_\mu(z)+\partial_\mu\Lambda(z)$,

$$G_c(x,y\,|\,gA)\to e^{-ig\Lambda(x)}G_c(x,y\,|\,gA)e^{+ig\Lambda(y)} ,$$

because from (3.8)

$$ds'\phi_\mu(s')\frac{\partial}{\partial z_\mu}\Lambda(z)=-dz_\mu\frac{\partial}{\partial z_\mu}\Lambda(z)=-d\Lambda ,$$

so that

$$\int d^2z\, f_\mu(z)\partial_\mu\Lambda(z)=g\left[\Lambda(y)-\Lambda\left[y-\int_0^s ds'\phi(s')\right]\right]$$
$$=g[\Lambda(y)-\Lambda(x)] .$$

B. The case of a constant background field

We use Eq. (3.7) to derive the well-known result for the fermion propagator in a constant background $F_{\mu\nu}$ field:[9,16]

$$A_\mu(z)=-\tfrac{1}{2}F_{\mu\nu}z_\nu \quad \text{in the gauge } \partial_\mu A_\mu=0 .$$

From (3.7) one writes

$$y-\int_0^{s'} ds''\phi(s'')=x+\int_{s'}^s ds''\phi(s'')$$
$$=\tfrac{1}{2}(x+y)$$
$$-\int_0^s ds''\tfrac{1}{2}[\theta(s'-s'')$$
$$-\theta(s''-s')]\phi(s'') ,$$

so that for a constant $F_{\mu\nu}$ field the phase is

$$\int f_\mu(z)A_\mu(z)d^2z=\frac{g}{4}(x-y)_\mu F_{\mu\nu}(x+y)_\nu$$
$$+\frac{g}{2}F_{\mu\nu}[O_{\mu\nu}(s,\phi)-s\sigma_{\mu\nu}] , \quad (3.9)$$

with

$$O_{\mu\nu}(s,\phi)=\int_0^s\int_0^s ds'ds''\phi_\mu(s')\phi_\nu(s'')$$
$$\times\tfrac{1}{2}[\theta(s'-s'')-\theta(s''-s')] . \quad (3.10)$$

This form for the phase will also be obtained as a result of the IR approximation.

The functional integration over the ϕ_μ field may be performed in (3.7). One writes

$$\frac{-i}{4}\int_0^s ds'\phi^2+i\frac{g}{2}F_{\mu\nu}O_{\mu\nu}(s,\phi)$$
$$=-\frac{i}{2}\int_0^s\int_0^s ds_1ds_2\phi_\mu(s_1)K_{\mu\nu}(s_1,s_2)\phi_\nu(s_2) ,$$

where

$$K_{\mu\nu}(s_1,s_2)=K_{\nu\mu}(s_2,s_1)$$
$$=\tfrac{1}{2}g_{\mu\nu}\delta(s_1-s_2)-\frac{g}{2}F_{\mu\nu}[\theta(s_1-s_2)$$
$$-\theta(s_2-s_1)] .$$

Then

$$N(s)\int d[\phi]\{1;\phi_\rho(s)\}\exp\left[-\frac{i}{2}\int\int\phi\cdot K\cdot\phi\right]\int\frac{d^2p}{(2\pi)^2}\exp\left[ip\cdot(x-y)+ip_\mu\int_0^s ds_1\phi_\mu(s_1)\right]$$

$$=\int\frac{d^2p}{(2\pi)^2}\{1;S_{\rho\sigma}p_\sigma\}\exp\{ip\cdot(x-y)+ip_\mu R_{\mu\nu}p_\nu-\tfrac{1}{2}\mathrm{Tr}\ln[KK(0)^{-1}]\}$$

$$=-\frac{1}{4\pi}\{1;-iS_{\rho\sigma}\partial_\sigma^x\}\exp\left[-\frac{i}{4}(x-y)_\mu R^{-1}{}_{\mu\nu}(x-y)_\nu\right]\exp\{-\tfrac{1}{2}\mathrm{tr}\ln R-\tfrac{1}{2}\mathrm{Tr}\ln[KK^{-1}(0)]\}\ ,$$

with

$$R_{\mu\nu}(s)=\int_0^s\int_0^s ds_1ds_2\tfrac{1}{2}K^{-1}{}_{\mu\nu}(s_1,s_2)$$

and

$$S_{\rho\sigma}(s)=\int_0^s ds_2K^{-1}{}_{\rho\sigma}(s,s_2)\ .$$

Introducing $\mathcal{X}_{\mu\nu}=-gF_{\mu\nu}=-\varepsilon_{\mu\nu}H$, the explicit expressions are

$$\tfrac{1}{2}K^{-1}{}_{\mu\nu}(s_1,s_2)=g_{\mu\nu}\delta(s_1-s_2)-\left[\mathcal{X}e^{-2(s_1-s_2)\mathcal{X}}\left[\theta(s_1-s_2)-\theta(s_2-s_1)+\frac{1-e^{-2s\mathcal{X}}}{1+e^{-2s\mathcal{X}}}\right]\right]_{\mu\nu}\ ,$$

$$R_{\mu\nu}=\left[\frac{1}{\mathcal{X}}\frac{1-e^{-2s\mathcal{X}}}{1+e^{-2s\mathcal{X}}}\right]_{\mu\nu},\quad S_{\mu\nu}=\left[\frac{4e^{-2s\mathcal{X}}}{1+e^{-2s\mathcal{X}}}\right]_{\mu\nu},$$

and

$$\mathrm{Tr}\ln[KK(0)^{-1}]=\mathrm{Tr}\ln(1+\tilde{\theta}K)=\int_0^1\frac{d\lambda}{\lambda}\mathrm{Tr}\left[\lambda\tilde{\theta}\mathcal{X}\frac{1}{1+\lambda\tilde{\theta}\mathcal{X}}\right]$$

$$=\int_0^1\frac{d\lambda}{\lambda}\mathrm{Tr}[1-\tfrac{1}{2}K^{-1}(\mathcal{X}\to\lambda\mathcal{X})]$$

$$=\int_0^1\frac{d\lambda}{\lambda}\mathrm{Tr}\left[s\lambda\mathcal{X}\frac{1-e^{-2\lambda s\mathcal{X}}}{1+e^{-2\lambda s\mathcal{X}}}\right]=2\ln[\cosh(sH)]\ ,$$

so that

$$\tfrac{1}{2}\mathrm{Tr}\ln R+\tfrac{1}{2}\mathrm{Tr}[KK(0)^{-1}]=\ln\left[\frac{\sinh(sH)}{H}\right]\ .$$

For a constant $gF_{\mu\nu}=g\varepsilon_{\mu\nu}H$ field (3.7) becomes

$$G_c(x,y\mid H)=-\frac{1}{4\pi}\exp\left[\frac{i}{4}(x-y)_\mu\epsilon_{\mu\nu}(x-y)_\nu H\right]$$

$$\times\int_0^\infty ds\frac{H}{\sinh(sH)}e^{-ism^2}\{m+\gamma_\mu[g_{\mu\nu}+\epsilon_{\mu\nu}\tanh(sH)]i\partial_\nu^x\}\exp\left[-\frac{i}{4}(x-y)^2\frac{H}{\tanh(sH)}\right]e^{-i/2\sigma\cdot\epsilon sH}\ ,\qquad(3.11)$$

which is the two-dimensional version of Eq. (2.146) of Ref. 16 or Eqs. (3.20) and (3.13) of Ref. 9. As $H\to0$, the free propagator is obtained

$$G_c(x,y\mid 0)=-\frac{1}{4\pi}\int_0^\infty\frac{ds}{s}e^{-ism^2}(m+i\gamma\cdot\partial^x)e^{-i(x-y)^2/4s}$$

$$=-\frac{1}{2\pi}(m+i\gamma\cdot\partial)K_0(m\mid x-y\mid)\ .$$

The analytic continuation of (3.11) to the case of an imaginary field is defined as

$$s=-i\tau,\quad H=iH'\ ,$$

and the result is well behaved for Euclidean distance $(x-y)^2=-(x-y)_E{}^2$:

$$G_c(x,y\mid H')=-\frac{1}{4\pi}\exp[-(x-y)_\mu\epsilon_{\mu\nu}(x+y)_\nu H'/4]\int_0^\infty d\tau\,e^{-\tau m^2}\frac{H'}{\sinh(\tau H')}\{m+\gamma_\nu[g_{\mu\nu}+\epsilon_{\mu\nu}\tanh(\tau H')]i\partial_\nu^x\}\ ,$$

$$\times\exp\left[\frac{(x-y)^2}{4}H'\coth(\tau H')\right]\exp\left[-i\frac{\sigma\cdot\epsilon}{2}\tau H'\right]\ .\qquad(3.12)$$

The first multiplicative factor is the path-dependent phase factor which exhibits explicitly the gauge covariance of the propagator. It has been evaluated in the gauge $\partial_\mu A_\mu = 0$ and analytically continued to the case of an imaginary field; this factor will be equal to unity for the fermion loops.

C. Large timelike distance with a constant field

For large timelike distance $D^2 = (x-y)^2 > 0$ and constant real H, a stationary point s_0 exists for the phase of the integrand in (3.11):

$$F(s) = -\left[sm^2 + \frac{D^2}{4}\frac{H}{\tanh(sH)} \right],$$

$$F'(s_0) = 0 \quad \text{or} \quad \frac{1}{\tanh^2(s_0 H)} = 1 + \frac{4m^2}{H^2 D^2}.$$

$s_0 H$ increases logarithmically for large D and the phase behaves as

$$F(s_0) = \frac{HD^2}{4} + \frac{m^2}{H}\ln\left[\frac{HD}{m}\right]. \tag{3.13}$$

Note that no stationary point exists for the analytic continuation of the propagator to Euclidean $D^2 < 0$.

If one now considers the analytic continuation to the case of an imaginary field $s = -i\tau$ and $H = iH'$ in (3.11),

$$e^{iF(s)} \to e^{-F(\tau)}.$$

A saddle point now exists for Euclidean distances $D^2 = -D_E{}^2$ and

$$F(\tau_0) = \frac{H' D_E{}^2}{4} + \frac{m^2}{H}\ln\left[\frac{HD_E}{m}\right]. \tag{3.14}$$

In the alternative case for the background field $H' = f\tau^{-1/2}$, a saddle point also exists:

$$\tau_0 = (D_E{}^2 f / 8m^2)^{2/3}$$

and

$$F(\tau_0) = -\tfrac{3}{4}(fmD_E{}^2)^{4/3}.$$

IV. CALCULATION OF $\langle\bar{\psi}\psi\rangle$

The quenched approximation for $\langle\bar{\psi}\psi(x)\rangle$ is

$$\langle\bar{\psi}\psi(x)\rangle = -e^{\mathscr{D}}\,\mathrm{tr}[G_c(x,x\,|\,gA) - G_c(x,x\,|\,0)]\,|_{A\to 0}, \tag{4.1}$$

where $G_c(x,y\,|\,gA)$ is the fermion propagator in the background field $A_\mu(x)$, tr denotes the fermion-loop summation over Dirac indices, and

$$\mathscr{D} = \tfrac{1}{2}\int\int du\,dv \frac{\delta}{\delta A_\mu(u)} D_c^{\mu\nu}(u-v)\frac{\delta}{\delta A_\nu(v)} \tag{4.2}$$

is the functional differential linkage operator of all virtual photons within the loop, where $D_c^{\mu\nu}$ is the contraction of two A_μ fields, i.e., the bare photon propagator.

Substitution of Eq. (3.7) for the fermion propagator yields

$$\langle\bar{\psi}\psi(x)\rangle = \int_0^\infty ds\, e^{-ism^2} N(s)\int d[\phi]\exp\left[-\frac{i}{4}\int_0^s ds'\phi^2\right] \times(-m)\delta\left[\int_0^s ds'\phi\right], \times \mathrm{tr}T(\phi,s). \tag{4.3}$$

All the photon linkages are expressed by the formula

$$T(\phi,s) = e^{\mathscr{D}}\exp\left[i\int fA\right]\Big|_{A\to 0} = \exp\left[-\tfrac{1}{2}\int f_\mu D_c^{\mu\nu} f_\nu\right], \tag{4.4}$$

where $D_c^{\mu\nu}$ represents the bare photon propagator in an arbitrary gauge and f_μ is given by (3.8). Because of the δ function in (4.3) one has

$$\frac{\partial}{\partial z_\mu} f_\mu(z) = 0$$

ensuring the independence of the result on the gauge-dependent part of the propagator; one may take

$$\tilde{D}_c^{\mu\nu}(k) = -i\left[\frac{g_{\mu\nu}}{k^2}\right]. \tag{4.5}$$

Equation (3.8) becomes in momentum space

$$\tilde{f}_\mu(n) = g\int_0^s ds'[\phi_\mu(s') - i\sigma_{\mu\nu}k_\nu] \times\exp\left[-ik\cdot y + i\int_0^{s'} ds'' k\cdot\phi(s'')\right]$$

so that

$$\ln T(\phi,s) = \frac{ig^2}{2}\int_0^s ds_1\int_0^s ds_2\int\frac{d^2k}{(2\pi)^2}\frac{1}{k^2}[\phi_\mu(s_1) + i\sigma_{\mu\nu}k_\nu] \times[\phi_\mu(s_2) - i\sigma_{\mu\rho}k_\rho] \times\exp\left[i\int_{s_1}^{s_2} ds' k\cdot\phi(s')\right]. \tag{4.6}$$

Equation (4.3) with (4.6) is an exact expression for $\langle\bar{\psi}\psi\rangle$ in the quenched approximation.

Several remarks are in order.

(i) The condition $\int_0^s ds'\phi_\mu(s') = 0$ eliminates from (4.6) the usual IR divergence of the free photon propagator in two dimensions. This is a consequence of the gauge invariance of the closed fermion loop.

(ii) By a scaling argument one can determine the form of the exact dependence on the proper time s, or its continuation τ, where $s = -i\tau$. From the exponential factor

$$\exp\left[-\tfrac{1}{4}\int_0^s ds'\phi^2(s')\right]$$

or (4.3) it follows that ϕ scales as $\tau^{-1/2}$, and then (4.6) scales as τ.

(iii) Equation (4.6) together with the physical interpretation of s_1 and s_2 as labeling the point where the interaction takes place [see (3.6)], shows how the linkage of photon lines produces an effective four-fermion-like interaction. This is most apparent in the $\sigma_{\mu\nu}\sigma_{\mu\rho}$ term which can be thought of as a contact, four-fermion interaction with dimensionless coupling constant $g^2\tau$. The analogy with the Gross-Neveu model[17] and with the strong-coupling limit of lattice gauge theories[18] suggests that this effective interaction may cause chiral-symmetry breaking.

The momentum integration of (4.6) may be performed explicitly, and leads to a functional dependence on ϕ so complicated that one must resort to an approximation. The IR model used here has as its motivation the idea that the soft part of the virtual-photon exchange should play the dominant role in the strong-coupling limit $g/m \gg 1$ of this two-dimensional model, as it has in the other contexts of Ref. 8. Our approximation will consist in retaining only the "most IR" part of the soft-photon contribution, achieved by the following steps.

(1) Isolate the soft photons in (4.5) by replacing

$$\frac{1}{k^2} \text{ by } \left[\frac{1}{k^2}\right]\exp\left[\frac{k^2}{\mu_c^2}\right],$$

where this form of cutoff already assumes a continuation to a Euclidean metric $k^2=-k_E^2$. As explained in the second paper of Ref. 8, μ_c may not be allowed to depend upon the fermion-loop momentum, if rigorous gauge invariance is to be preserved. One possible choice is

$$\mu_c=c/\sqrt{\tau}\ ,$$

where c is a real, positive constant of order unity, and for two reasons, (i) (4.6) will then scale as τ, the form of the exact proper-time dependence mentioned above and (ii) the magnitude of the phase dependence of (4.6) is then typically less than unity, $O(\tau k\phi)\lesssim O(c)\sim 1$.

Another possible choice for the photon cutoff is

$$\mu_c=cm\ .$$

It should give equivalent results because of the presence in the fermion propagator (3.7) of the factor $\exp(-ism^2)=\exp(-\tau m^2)$.

(2) With either choice of μ_c, the magnitude of this phase term is $\lesssim 1$, and it is sensible to define a "multipole expansion" of the exponential $\exp(i\int ds' k\cdot\phi)$. We here retain only the first nonzero terms of this expansion. Every k integration then reduces to

$$\int \frac{d^2k}{(2\pi)^2}\frac{k_\sigma k_\rho}{k^2}e^{k^2/\mu_c^2}=ig_{\rho\sigma}\mu_c^2/8\pi\ ,$$

and (4.6) contains terms at most quartic in ϕ.

These steps lead to the following result for all the soft-photon linkages:

$$\ln T(\phi,s)=-\frac{J^2}{2}(O_{\mu\nu}-s\sigma_{\mu\nu})(O_{\mu\nu}-s\sigma_{\mu\nu})\ , \tag{4.7}$$

where $O_{\mu\nu}(s,\phi)$ is defined by (3.10) and

$$J^2=g^2\mu_c^2/8\pi\ . \tag{4.8}$$

Indeed, if use is made of the δ function of (4.3), $O_{\mu\nu}$ is antisymmetric in its indices, so that $O_{\mu\nu}O_{\mu\nu}=-(\varepsilon_{\mu\nu}O_{\mu\nu})^2$.

One then introduces the convenient representation

$$T(\phi,s)=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}\exp\left[-\frac{\alpha^2}{4}-\alpha\frac{J}{2}(O_{\mu\nu}-s\sigma_{\mu\nu})\epsilon_{\mu\nu}\right]. \tag{4.9}$$

Comparing (4.8) to the phase gained by the fermion in a constant background field (3.9), one finds that the result of our approximation to the photon exchange can be described as the propagation of the fermion in an effective imaginary background field

$$gF_{\mu\nu}=i\varepsilon_{\mu\nu}\alpha J,\quad J=g\mu_c/\sqrt{8\pi}\ . \tag{4.10}$$

A Gaussian average is subsequently made over the parameter α. Note that the sign in (4.7) is crucial for the background field to be imaginary. One then substitutes (4.9) into (4.3); the functional integration over the ϕ field has been performed in Sec. III with the result

$$\langle\bar\psi\psi\rangle=-\frac{m}{4\pi^{3/2}}\int_{-\infty}^{+\infty}d\alpha\, e^{-\alpha^2/4}\times\int_0^\infty ds\, e^{-ism^2}\left[\frac{H}{\tan(sH)}-\frac{1}{s}\right], \tag{4.11}$$

where

$$H=\alpha g\mu_c/\sqrt{8\pi}\ .$$

One then analytically continues $s=-i\tau$, and obtains

$$\langle\bar\psi\psi\rangle=-\frac{m}{2\pi^{3/2}}\int_0^\infty d\alpha\, e^{-\alpha^2/4}\int_0^\infty d\tau\, e^{-\tau m^2}\left[H\coth(\tau H)-\frac{1}{\tau}\right]. \tag{4.12}$$

where $H=\alpha gmc/\sqrt{8\pi}$. For $m/g\ll 1$ this is

$$-\langle\bar\psi\psi\rangle=\frac{gc}{\sqrt{8}\pi^2}+\frac{m}{2\pi}\ln\left[\frac{m}{gc}\right]+O(m)\ .$$

As discussed above, a choice exists for the photon cutoff.

(i) $\mu=cm$, $c\sim 1$. Then the τ integration can be performed:

$$\langle\bar\psi\psi\rangle=-\frac{m}{2\pi^{3/2}}\int_0^\infty d\alpha\, e^{-\alpha^{2/4}}\left[\frac{H}{m^2}+\ln\left[\frac{m^2}{2H}\right]-\psi\left[1+\frac{m^2}{2H}\right]\right], \tag{4.13}$$

(ii) $\mu=c\tau^{-1/2}$, $c\sim 1$. Then the α integration can be performed after the change of variable $\beta=\tau H$, leading to

$$\langle\bar\psi\psi\rangle=-\frac{m}{2\pi}\int_0^\infty\frac{d\beta}{\beta}(\beta\coth\beta-1)e^{-\beta/p}\ , \tag{4.14}$$

where $\rho = cg/m\sqrt{8\pi}$. This yields

$$\langle \bar\psi\psi \rangle = -\frac{m}{2\pi}\left[\rho + \ln\left[\frac{1}{\rho}\right] - \psi\left[1+\frac{1}{2\rho}\right]\right],$$

and, in the limit $(m/g) \ll 1$,

$$\langle \bar\psi\psi \rangle \simeq -\frac{gc}{2(2\pi)^{3/2}}\left[1 + O\left[\frac{m}{gc}\right]\ln\left[\frac{m}{gc}\right]\right].$$

The result is that the IR approximation to the quenched version generates a finite, nonzero order parameter as $(g/m) \to \infty$, a nonperturbative result. The contribution comes from the large-size loops of a fermion propagating in a constant imaginary field whose strength is of order gm.

In the absence of the $\exp(-i\sigma\cdot\varepsilon\tau H/2)$ factor of (3.12) for the fermion propagator, the $\coth(\tau H)$ factor of (4.12) would have been replaced by $[\sinh(\tau H)]^{-1}$ [which is the result of scalar QED (Ref. 9)] and the corresponding $\langle \bar\psi\psi \rangle$ would vanish in the limit $m \to 0$. A similar factor plays a crucial role in a description[7] of the zero-energy states of the quenched model.

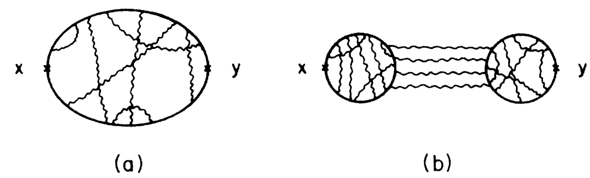

FIG. 1. The two types of diagram contributions to the correlation function in the quenched approximation. (a) The sum of all virtual photons exchanged across a single closed fermion loop. (b) The sum of all virtual photons exchanged across and between a pair of closed fermion loops.

V. THE CORRELATION FUNCTION, DIAGRAMS OF TYPE A

In the quenched approximation, the diagrams contributing to the correlation function fall into two classes. In the first type of diagram, type A, the fermion propagates from y to x and back, as pictured in Fig. 1. It is shown in Appendix A that this contribution to the IR approximation for $\langle \bar\psi\psi(x)\bar\psi\psi(y) \rangle$ yields

$$I_a = -\int_{-\infty}^{+\infty} \frac{d\alpha}{2\sqrt{\pi}} e^{-\alpha^2/4} \mathrm{tr}[G_c(x,y\,|\,H)G_c(y,x\,|\,H)], \quad (5.1)$$

with $H = \alpha g \mu_c$.

Substituting Eq. (3.12) for the fermion propagator, I_a becomes, after analytic continuation to Euclidean space,

$$I_a = \int_0^\infty \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/4} \int_0^\infty d\tau_1 H\, e^{-\tau_1 m^2} \int_0^\infty d\tau_2 H\, e^{-\tau_2 m^2} \exp\left[-(x-y)^2 \frac{H}{4}[\coth(\tau_1 H) + \coth(\tau_2 H)]\right]$$
$$\times\left[-\frac{2}{(4\pi)^2}\right]\left[m^2\left[1+\frac{1}{\tanh(\tau_1 H)\tanh(\tau_2 H)}\right] - \frac{(x-y)^2}{4}\frac{H^2}{\sinh^2(\tau_1 H)\sinh^2(\tau_2 H)}\right]. \quad (5.2)$$

In (5.2) and subsequently, $(x-y)^2$ denotes the positive Euclidean distance.

If the cutoff μ_c is chosen proportional to m, I_a may be expressed in terms of the function

$$I\left[\lambda = \frac{m^2}{2H}, z = (x-y)^2 \frac{H}{2}\right] = \int_0^\infty d\tau_1 H\, e^{-\tau_1 m^2} \exp\left[-(x-y)^2 \frac{H}{4}\coth(\tau_1 H)\right], \quad (5.3)$$

where $I(\lambda, z)$ obeys the recursion formula

$$\frac{\partial^2 I}{\partial z^2} = \left[\frac{1}{4} + \frac{\lambda}{z}\right] I .$$

Then I_a may be written as

$$I_a = \left[\frac{-2}{(4\pi)^2}\right]\int_0^\infty \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/4}\left\{m^2\left[I^2 + 4\left[\frac{\partial I}{\partial z}\right]^2\right] - \frac{4}{(x-y)^2}(2\lambda I)^2\right\}. \quad (5.4)$$

We shall examine several properties of this function.

A. Proper limit for the case of free propagation

As $H \to 0$, $\partial I/\partial z \to -K_0(m\,|x-y\,|)$ so that

$$I_a\,|_{H=0} = -\frac{8}{(4\pi)^2}\left[m^2 K_0^2(m\,|x-y\,|) - \left[\frac{\partial K_0}{\partial\,|x-y\,|}\right]^2\right] \quad (5.5)$$

and the Euclidean version for free propagation is recovered since

$$I_{a,\mathrm{free}}=-\frac{1}{(2\pi)^2}\mathrm{tr}[(m+i\gamma\partial_x)K_0(m\,|x-y|)\times(m-i\gamma\partial_{x'})K_0(m\,|x'-y|)]_{x'=x}\ .$$

We emphasize that since for $mz>1$,

$$K_0(mz)\simeq\frac{\sqrt{\pi/2}}{\sqrt{mz}}e^{-mz}$$

and for $mz \ll 1$, $K_0(mz)\simeq -\ln(mz)$, the free propagation for the case $m=0$ is obtained in the region $m\,|x-y|\ll 1$, where

$$I_{a,\mathrm{free}}=-\frac{1}{2\pi^2}\left[m^2\ln^2(m\,|x-y|)-\frac{1}{(x-y)^2}\right]+O(m^2(x-y)^2)\ . \qquad (5.6)$$

So the proper distance to look for the $m\to 0$ limit is the region

$$m\,|x-y|\ll 1\ .$$

B. The large-distance scale

The scale showing up in the correlation function for large separations is set by the scale appearing in the fermion propagator in the background field. As shown in Sec. III, for large Euclidean distance a saddle-point argument shows that the scale is set by the background field H. Equation (3.12) for the fermion propagator behaves as $\exp[-(x-y)^2H/4]$ as the effective imaginary proper time τ increases as $\ln(x-y)^2$. With a background field $H=\alpha g\mu_e$, with μ_c proportional to m, the distance scale is $(mg)^{-1/2}$.

C. The results

The function $I(\lambda,z)$ defined in (5.3) is known in terms of Whittaker's function $W_{-\lambda,1/2}(z)$ if use is made of formula (3.547) of Ref. 19:

$$I(\lambda,z)=\tfrac{1}{2}\Gamma(\lambda)W_{-\lambda,1/2}(z),\quad z>0\ . \qquad (5.7)$$

Defining

$$I(\lambda,z)=\frac{e^{-z/2}}{2\lambda}Y(\lambda,z)$$

and $u=z/\alpha$, (5.3) yields

$$I_a=-\frac{2}{(4\pi)^2}\int_0^\infty\frac{d\alpha}{\sqrt{\pi}}e^{-\alpha^2/4-\alpha u}\left\{\left[\frac{4}{m}\right]^2\left[Y^2+\left[Y-2\frac{\partial Y}{\partial z}\right]^2\right]-\frac{4}{(x-y)^2}Y^2\right\}, \qquad (5.8)$$

where $u=g(\mu/2)(x-y)^2$, $H=\alpha g\mu$, and $\mu=cm/\sqrt{8\pi}$.

For $u\ll 1$, the following equation for Y,

$$Y(\lambda,z)=1-\lambda z[-\ln z+\psi(1)+\psi(2)-\psi(1+\lambda)]+\lambda z^2\ln z+O(z^2)\ ,$$

leads to

$$I_a=-\left[\frac{gc}{\sqrt{8\pi}}\right]^2\frac{1}{2\pi^2}+\frac{1}{2\pi^2}\frac{1}{(x-y)^2}+O(8m(x-y)^2)\ . \qquad (5.9)$$

Comparison of (5.9) with (5.6) and (5.1) shows that in the region $(x-y)^2mg\ll 1$, I_a becomes the sum of the contribution of free propagation of massless fermions plus a constant term which is $-\langle\bar\psi\psi\rangle^2(\pi/2)$.

For $u\gg 1$,

$$Y=\Gamma(\lambda+1)z^{-\lambda}[1-\lambda(\lambda+1)/z+O(z^{-2})]\ ,$$

and the integration over $z=\alpha u$ gives

$$I_a=\frac{1}{(x-y)^2}\frac{1}{u}\left[-\frac{C'}{m^2(x-y)^2}+C''\right], \qquad (5.10)$$

where C' and C'' are constants.

How do these results compare with expectations? One would expect the scale of the large Euclidean distance to be determined by the lowest e^+-e^- bound state if such state is generated by the diagrams of Fig. 1, when $m\to 0$. Such a state would have a mass of order g, corresponding to the exchange of photons with momentum $k\sim g$. The exchange of these photons has not been taken into account by the IR approximation which only sums up photons whose momentum is $k<m$.

In Appendix B one looks at the opposite limit, i.e., nonrelativistic fermions with $g\ll m$, where exact analytical results exist for both the full problem and the IR approximation. Diagrams of type A describe mostly the Coulomb interaction $g^2|x_1-x_2|$ which generate bound states with a binding energy $E\sim g(g/m)^{1/3}$ corresponding to the exchange of photons with momentum $(mg^2)^{1/3}$. On the other hand, the IR approximation, i.e., propagation in an effective imaginary background field whose strength is of order gm, generates $(e^+\text{-}e^-)$ states which are not localized, resulting in a power-law dropoff rather than an exponential one for the nonrelativistic Green's function. So the approximation of keeping only the most IR part of the photons with momentum $k<m$ fails to generate bound states.

From this discussion it emerges that the interesting results of the IR approximation lie in the region $(mg)(x-y)^2\ll 1$, where the correlation function $\langle\bar\psi\psi(x)\bar\psi\psi(y)\rangle$ behaves as a constant term related to $\langle\bar\psi\psi\rangle^2$, plus free propagation of massless fermions.

With the other choice for the cutoff, $\mu_c=c\tau^{-1/2}$, the constant field H now depends on the total proper time of the loop, $H=\alpha g(\tau_1+\tau_2)^{-1/2}c/\sqrt{8\pi}$. The integration over the α variable may be performed in (5.2), after the change of variables $\tau_1 H=\beta_1$, and $\tau_2 H=\beta_2$, with the resulting exponential cutoff for the integrand:

$$\exp\left[-\left[m^2(x-y)^2(\beta_1+\beta_2)(\coth\beta_1+\coth\beta_2)+\frac{m^2}{g^2}(\beta_1+\beta_2)^2\frac{8\pi}{c^2}\right]^{1/2}\right].$$

The parameter $mg(x-y)^2$ controls the shrinking of the effective domain (β_1,β_2) of integration in the region $m^2(x-y)^2 \ll 1$. Again the scale $mg(x-y)^2 \sim 1$ appears in the correlation function. No simple analytical answer has been found in the region $mg(x-y)^2 \ll 1$.

VI. DIAGRAMS OF TYPE B AND THE TOTAL CORRELATION FUNCTION

A. Diagrams of type B

In diagrams of type B, one-fermion loops around each point x and y; in the IR approximation, both fermions propagate in an effective background field H:

$$I_b=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\mathrm{tr}[G_c(x,x\mid H)-G_c(x,x\mid 0)]\times\mathrm{tr}[G_c(y,y\mid H)-G_c(y,y\mid 0)]\;. \quad (6.1)$$

The subtracted terms correspond to the singular contribution of diagrams where one fermion, at least, propagates freely around point x or y. In (6.1), no scale appears for the distance $(x-y)$, i.e., the contribution is a constant as a function of distance. In Appendix A, the IR approximation is shown to approximate $\exp[-m^2(x-y)^2]\sim 1$, valid in the region $m^2(x-y)^2 \ll 1$, of interest to limit $m\to 0$. The strength of the background field may either be the same for both fermions loops if the photon cutoff is chosen $\mu_c=cm$, or may be different if it is related to the proper time of each loop. In this latter case an expression more general than (6.1) is obtained in Appendix A:

$$I_b=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_{-\infty}^{+\infty}\frac{d\beta}{2\sqrt{\pi}}e^{-\beta^2/4}\int_0^\infty d\tau_1\int_0^\infty d\tau_2 e^{-m^2(\tau_1+\tau_2)}\times\frac{m^2}{(2\pi)^2}\left[H_1\coth(\tau_1 H_1)-\frac{1}{\tau_1}\right]\left[H_2\coth(\tau_2 H_2)-\frac{1}{\tau_2}\right], \quad (6.2)$$

where

$$H_1=\frac{g\mu_1}{\sqrt{8\pi}}\frac{1}{\sqrt{2}}(\alpha\sqrt{1+\omega}+\beta\sqrt{1-\omega})\;,$$

$$H_2=\frac{g\mu_2}{\sqrt{8\pi}}\frac{1}{\sqrt{2}}(\alpha\sqrt{1+\omega}-\beta\sqrt{1-\omega})\;,$$

$$\omega=2\mu_1\mu_2/({\mu_1}^2+{\mu_2}^2)\;,$$

where $\mu_1\mu_2$ are the photon cutoffs associated with each loop. It is shown in Appendix A that setting $\omega=0$ in (6.2) gives the contribution of diagrams where no photon is exchanged between the two loops; in this case, with the rotation $\alpha'=(\alpha+\beta)/2$ and $\beta'=(\alpha-\beta)/2$ one obtains correctly

$$I_b(\omega=0)=\langle\bar\psi\psi\rangle^2 \quad (6.3)$$

where $\langle\bar\psi\psi\rangle$ is defined by (4.11).

With the choice for the cutoff $\mu_1=\mu_2=cm$, one has $\omega=1$, $H_1=H_2=H$. Comparison of Eq. (6.2) with Eqs. (3.11) and (3.12) leads to

$$I_b(\omega=1)=\left[\frac{m}{2\pi}\right]^2\int_0^\infty d\alpha\, e^{-\alpha^2/4}\left[\frac{H}{m^2}+\ln\left[\frac{m^2}{2H}\right]-\psi\left[1+\frac{m^2}{2H}\right]\right]^2. \quad (6.4)$$

For $m/g \ll 1$ the following result is obtained:

$$I_b(\omega=1)=\frac{\pi}{2}I_b(\omega=0)=\frac{\pi}{2}\langle\bar\psi\psi\rangle^2\;, \quad (6.5)$$

with $\langle\bar\psi\psi\rangle$ given by (4.13). The result is valid for $(x-y)^2m^2 \ll 1$. This constant term exactly cancels the constant term of I_a, (5.9), in the region $(x-y)^2mg \ll 1$.

With the other choice of the cutoff, no simple analytical answer exists for the constant term I_b.

B. Results for the total correlation function

With the choice of the photon cutoff $\mu_c=cm$, the sum of the contributions I_a and I_b, from diagrams of type A and B, leads to the following properties.

The scale of the correlation function is $(mg)^{-1/2}$ for diagrams of type A, m^{-1} for diagrams of type B.

In the region $(x-y)^2 \ll (mg)^{-1}$ the correlation function $\langle\bar\psi\psi(x)\bar\psi\psi(y)\rangle$ reduces to the free propagation of massless fermions, because of an exact cancellation between the leading terms in the two types of diagrams:

$$\langle\bar\psi\psi(x)\bar\psi\psi(y)\rangle\to\frac{1}{2\pi^2}\frac{1}{(x-y)^2}\;.$$

In the region $(mg)^{-1} \ll (x-y)^2 \ll m^{-2}$ diagrams of type A drop off and diagrams of type B contribute a constant term:

$$\langle\bar\psi\psi(x)\bar\psi\psi(y)\rangle=\frac{\pi}{2}\langle\bar\psi\psi(x)\rangle^2\;.$$

The other channels give similar results.

For the case $\langle\bar\psi\gamma_\mu\psi(x)\bar\psi\gamma_\mu\psi(y)\rangle$, $I_b=0$ and

$$I_a=-\frac{1}{4\pi^2}\int_0^\infty\frac{d\alpha}{\sqrt{\pi}}e^{-\alpha^2/4}m^2\left[4\left[\frac{\partial I}{\partial z}\right]^2-I^2\right],$$

with I defined by (5.3). In the region $(x-y)^2mg \ll 1$, one finds $I_a \sim 0$ using the approximate form for I valid in this region. For the case $\langle \bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(y)\rangle$,

$$I_a = -\frac{1}{8\pi^2}\int_0^\infty \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/4}\left\{ m^2\left[I^2 + 4\left[\frac{\partial I}{\partial z}\right]^2\right] + \frac{4}{(x-y)^2}(\lambda I)^2 \right\},$$

and for I_b, $H\coth(\tau_j H)$ is replaced by H in (6.1) and no subtraction term appears:

$$I_b = \frac{1}{(2\pi)^2}\int_0^\infty \frac{d\alpha}{\sqrt{\pi}} e^{-\alpha^2/4} \times \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 e^{-(\tau_1+\tau_2)m^2}(mH)^2 .$$

Comparison with Eqs. (5.4) and (6.4) gives, for $(x-y)^2 \ll (mg)^{-1}$,

$$\langle \bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(y)\rangle = -\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle ,$$

for $(mg)^{-1} \ll (x-y)^2 \ll m^{-2}$, where only diagrams of type B survive,

$$\langle \bar{\psi}\gamma_5\psi(x)\bar{\psi}\gamma_5\psi(y)\rangle = +\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y)\rangle .$$

This is further evidence that the interesting region is $(x-y)^2 \ll (mg)^{-1}$ in this IR model.

With the other choice for the cutoff, when each fermion loop provides a cutoff proportional to its proper time, the addition of the contributions of the two types of diagrams can only be made numerically. However, the same qualitative features hold, the distance scale is $(mg)^{-1/2}$ for diagrams of type A and m^{-1} for diagrams of type B.

VII. CONCLUSIONS

The approximation of keeping only the most IR part of the photon exchanged across a fermion loop plus the choice of a photon cutoff on the order of the fermion mass m leads to the following results for the quenched version of the Schwinger model.

For the correlation function, we have stressed that the results for the $m=0$ case should be looked for in the region $(x-y)^2 m^2 \ll 1$. In that region, the large-distance scale is set by the background field for the diagrams where the fermion propagates the full distance $(x-y)$ and back; the scale is set by m for the diagrams where there is one-fermion loop around each point x or y. In the region $(x-y)^2 mg \ll 1$, we find a cancellation of the leading terms, which are proportional to $\langle \bar{\psi}\psi\rangle^2$, between the two types of diagrams, leaving only free massless fermion propagation, i.e., the effect of the exchange of the most IR photons just cancels in the correlation function. This result must be taken as a warning of some numerical simulations of correlation functions in the quenched approximation, when only diagrams of type A are computed, although both types A and B exist.

A finite, nonzero value $\langle \bar{\psi}\psi\rangle$ for the order parameter is obtained as $m \to 0$. The effect arises from the large loops of a fermion propagating in an effective imaginary constant field whose scale is m. The exact, gauge-invariant expressions for the order parameter and the correlation function show no IR divergences as long as $m \neq 0$. In our approximation, we do not encounter IR divergences as $m \to 0$ or any other growth that appears in the $m=0$ case when the strength of the anomaly is described.

One can understand the reason why a calculation of $\langle \bar{\psi}\psi\rangle$ in quenched approximation might yield a divergent result by an argument based upon a very simple and unambiguous computation. Consider that particular contribution to the quenched $\langle \bar{\psi}\psi\rangle$ which results from the quantum fluctuations of only the zero-frequency component of

$$F(x) = (1/\sqrt{LT})\sum_k F_k \exp(ik\cdot x) ;$$

that is, $F(x) \to F_0/\sqrt{LT}$. By discussing the fluctuations of individual frequency components one is implicitly introducing a third mass scale into the problem, $1/\sqrt{LT}$, and new possibilities arise for subsequent limits. The relevant expression for the trace of a (normal-ordered) Green's function in a constant background field $F^0/\sqrt{LT}$ is certainly well known; and to compute the corresponding zero-frequency mode contribution to the quenched $\langle \bar{\psi}\psi\rangle$ one must simply perform an ordinary Gaussian, Euclidean integration over the F_0 dependence in $G_c(F_0)$. Omitting all relevant constants, including the normalization, one has

$$\langle \bar{\psi}\psi\rangle_0^Q \sim m \int_{-\infty}^{+\infty} dF_0 e^{-F_0{}^2/2}\int_0^\infty \frac{d\tau}{\tau} e^{-\tau m^2}(\beta\coth\beta - 1) , \tag{7.1}$$

where $\beta = g\tau F_0/\sqrt{LT}$, and τ (again) denotes the proper time associated with the electron loop. That (7.1) is correct cannot be disputed. In order to estimate these integrals it is simplest to replace the parentheses of (7.1) by

$$\int_0^\infty d\beta(\beta\coth\beta - 1)\delta(\beta - g\tau F_0/\sqrt{LT}) ,$$

and then to write a representation of the δ function, so that the F_0 integration can be performed. Again dropping all multiplicative constants, one finds

$$\langle \bar{\psi}\psi\rangle_0^Q \sim m \int_0^\infty d\beta(\beta\coth\beta - 1)\int_0^\infty \frac{d\tau}{\tau} e^{-\tau m^2}\int_{-\infty}^{+\infty} d\omega \exp(i\omega\beta - \omega^2 g^2\tau^2/2LT)$$

$$\sim \frac{m^3\sqrt{LT}}{g}\int_0^\infty d\beta(\beta\coth\beta - 1)\int_0^\infty \frac{du}{u^2}\exp(-u - \beta^2 m^4 LT/2g^2u^2) \tag{7.2}$$

after rescaling the proper-time integral. Another change of variable, $u = 1/r$, brings the latter into the form

$$\int_0^\infty dr \exp(-1/r - \beta^2 r^2 m^4 LT/2g^2) ,$$

which can qualitatively be approximated as

$$\int_1^\infty dr \exp(-\beta^2 r^2 m^4 LT/2g^2) ,$$

thereby generating

$$\langle \bar{\psi}\psi \rangle_0^Q \sim m \int_0^\infty \frac{dA}{A} [(RA)\coth(RA) - 1][1 - \Phi(A)] , \tag{7.3}$$

where $R = \sqrt{2}g/m^2\sqrt{LT}$ and $\Phi(z)$ denotes the probability integral $(2/\sqrt{\pi}) \int_0^z dt\, e^{-t^2}$.

The computation is now essentially finished. Consider the strong-coupling (SC) limit $(g/m) \gg 1$, with $m\sqrt{LT}$ fixed, for which (7.3) becomes

$$\langle \bar{\psi}\psi \rangle_0^Q \sim mR \int_0^\infty dA [1 - \Phi(A)] . \tag{7.4}$$

The integral of (7.4) is perfectly convergent, so that one finds for this SC case

$$\frac{\langle \bar{\psi}\psi \rangle_0^Q}{g} \sim 1/m\sqrt{LT} . \tag{7.5}$$

If one holds fixed the total volume of quantization, LT, and takes the limit $m \to 0$, the result is clearly divergent; but if one first takes LT infinite, the result is zero (as one might instinctively expect as the contribution of a single mode). Which limit should one take first? In spite of computer limitations, the only physically sensible answer is $LT \to \infty$ first, to be followed by any other desired sequence of parameter limits.

In the bosonization calculation of Ref. 6, IR regularization is introduced "by hand," with factors of $\mu^2 \sim 1/LT$ inserted in an essentially *ad hoc* way; and one calculates limits by first setting $m = 0$ and then $\mu \to 0$. (These quenched calculations are effectively performed in the continuum, with momentum-space integrals sensitive to low frequencies; were these integrals calculated in configuration space, they would be sensitive to large distances, to finite-volume effects, which is what is meant here by the relation $\mu^2 \sim 1/LT$.)

A more meaningful comparison would be to the work of Carson and Kenway,[5] but this is beyond our ability because of the two steps taken by them, bosonization and the replica trick, which are completely different from our somewhat more direct approach. No one can object to bosonization; but it could conceivably be that in the process of taking the number of fermion species N to zero, that one has introduced a divergence. We do not know if this reason is correct, or if another reason associated with the process of quenching after bosonization is at fault; but we do know that our estimates give a divergence-free result, of the correct order of magnitude and sign as that of the exact calculation. One does not want to be in the position of apologizing for having achieved a finite result; but we are sure of the validity of our result, as one that extracts the proper dependence of a continuous spread of very low frequencies.

Our calculation introduces no *ad hoc* IR regularization and requires no UV lattice spacing parameter; it begins with a properly normal-ordered $\bar{\psi}\psi$ and it straightforwardly gives a finite, nonzero estimate of the quenched $\langle \bar{\psi}\psi \rangle$. The reason this happens can easily be understood on the basis of the elementary single-mode computation: a typical summation over many virtual-photon modes $(1/LT)$. $\sum_k$ (appearing, e.g., in an exponentiated propagator), which yielded just the factor $1/LT$ in the zero-frequency mode calculation, is here replaced by the continuum integral

$$\int_0^{\mu_c} d^2k/(2\pi)^2$$

with μ_c a special cutoff of (eventual) order m. In this way the $1/LT$ factor of the single-mode estimate is, after a sequence of finite integrations, effectively replaced by $\xi^2 m^2$, where ξ is some fixed, numerical constant. Hence the $(m/\sqrt{LT})^{-1}$ factor of (7.5) is replaced by a constant $(\xi)^{-1}$ generating a finite, quenched $\langle \bar{\psi}\psi \rangle$ in the SC limit, of the same form and order of magnitude as the exact, unquenched answer. Further, when the quenched approximation is partially removed by including all closed fermion loops in a chain-graph approximation,[20] one finds a reasonable diminution (25%) of the quenched result obtained in this paper; and this diminution of magnitude is in agreement with estimates of older machine calculations.[1]

Questions of comparison with other people's divergences aside, one of the purposes of the present IR estimates is to illustrate the possibility of obtaining SC effects by the simple and straightforward method of IR extraction, applicable directly in the continuum and in any number of dimensions. Every such successful SC estimate made in this way enhances the probability that the method will be useful where it really counts, in QCD and beyond. Most recently, the IR method has been applied to viscous Navier-Stokes fluids,[21] and should find a wide range of applicability in nonlinear physical problems of continuous media.

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APPENDIX A: COMPUTATION OF $\langle \bar{\psi}\psi(x)\bar{\psi}\psi(y) \rangle$

In the quenched approximation two types of diagrams contribute to the correlation function,

$$I_a=-e^{\mathscr{D}}\mathrm{tr}[G_c(x,y\,|\,gA)G_c(y,x\,|\,gA)]\,|_{A\to 0}\;, \tag{A1}$$

$$I_b=e^{\mathscr{D}}\mathrm{tr}[G_c(x,x\,|\,gA)]\mathrm{tr}[G_c(y,y\,|\,gA)]\,|_{A\to 0}\;, \tag{A2}$$

where $\mathscr{D}$ is the linkage operator of all virtual photons within each fermion loop and linking each loop, as defined in (4.2), and $G_c(x,y\,|\,gA)$ the fermion propagator in a background field $A_\mu(z)$. Substitution of the gauge-covariant form (3.7) for $G_c(x,y\,|\,gA)$ leads to the gauge-invariant expressions

$$I_{\begin{vmatrix} a \\ b \end{vmatrix}}=\int_0^\infty ds_1 e^{-is_1m^2}N(s_1)\int d[\phi]\exp\left[-\frac{i}{4}\int_0^{s_1}ds'\phi^2\right]\int_0^\infty ds_2 e^{-is_2m^2}N(s_2)\int d[\psi]\exp\left[-\frac{i}{4}\int_0^{s_2}ds'\psi^2\right]\begin{vmatrix} A \\ B\end{vmatrix}\;, \tag{A3}$$

where

$$A=-\,\mathrm{tr}\{[m-\tfrac{1}{2}\gamma\phi(s_1)][m-\tfrac{1}{2}\gamma\psi(s_2)]T\}$$
$$\times\delta\left[x-y+\int_0^{s_1}\phi ds'\right]\delta\left[y-x+\int_0^{s_2}\psi ds'\right]\;,$$
$$B=-\,\mathrm{tr}^{(1)}\mathrm{tr}^{(2)}\{[m-\tfrac{1}{2}\gamma\phi(s_1)][m-\tfrac{1}{2}\gamma\psi(s_2)]T\}$$
$$\times\delta\left[\int_0^{s_1}\phi ds'\right]\delta\left[\int_0^{s_2}\psi ds'\right]\;,$$

and $T(\phi,s_1;\psi,s_2)$ is the result of all the photon linkages; i.e., it contains the self-interaction of each fermion propagator and the photon exchanges between the two propagators.

With the formula (4.4) one obtains

$$\ln T=-\tfrac{1}{2}\int d^2u\int d^2v[f_\mu^{(1)}(u)+f_\mu^{(2)}(u)]D_c^{\mu\nu}(u-v)$$
$$\times[f_\nu^{(1)}(v)+f_\nu^{(2)}(v)]\;, \tag{A4}$$

where

$$f_\mu^{(1)}(u)=g\int_0^{s_1}ds'[\phi_\mu(s')\pm\sigma_{\mu\nu}\partial_\nu^u]$$
$$\times\delta\left[u-y+\int_0^{s'}ds''\phi(s'')\right]\;,$$
$$f_\mu^{(2)}(u)=g\int_0^{s_2}ds'[\psi_\mu(s')\pm\sigma_{\mu\nu}\partial_\nu^u]$$
$$\times\delta\left[u-x+\int_0^{s'}ds''\psi(s'')\right]\;.$$

The $\pm$ sign accounts for the anticommuting property of the γ_ρ and $\sigma_{\nu\mu}$ matrices. Equation (A4) is exact. As a consequence of the δ functions appearing in (A3) one has the property

$$\frac{\partial}{\partial u_\mu}[f_\mu^{(1)}+f_\mu^{(2)}(u)]=0\;,$$

ensuring the independence of the result on the gauge-dependent part of the photon propagator.

As explained in Sec. III, the IR extraction method retains only the first nonzero term in the expansion of the argument of the photon propagator $D_c^{\mu\nu}$:

$$e^{ik(u-v)}=1+ik\cdot(u-v)+\frac{i^2}{2!}[k\cdot(u-v)]^2+\cdots\;.$$

The terms 1 of the expansion cancel each other in (A4). It is apparent for case B and can be verified for case A; because it is crucial for this cancellation to occur, one expands the full argument $(u-v)$ in the four terms appearing in (A4). The terms bilinear in k are the first nonzero terms; one obtains, for case A,

$$\ln T=-\frac{J^2}{2}[O_{\mu\nu}(s,\phi)+O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}(s_2\pm s_1)]$$
$$\times[O_{\mu\nu}(s_1,\phi)+O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}(s_2\pm s_1)]\;, \tag{A5}$$

where $O_{\mu\nu}(s_1,\phi)$, $O_{\mu\nu}(s_2,\psi)$ are defined as in (3.10) and J^2 by (4.8); (s_1+s_2) is associated with the m^2 term, (s_2-s_1) with the $\gamma\cdot\gamma$ term. As in Sec. IV, one then writes

$$T=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}\exp\left[-\frac{\alpha^2}{4}-\alpha\frac{J}{2}[O_{\mu\nu}(s_1,\phi)+O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}(s_2\pm s_1)]\epsilon_{\mu\nu}\right]\;. \tag{A6}$$

Insertion of (A6) into (A3) and comparison with (3.9) and (3.7) shows that (A3) becomes the product of two propagators in the constant background field $F_{\mu\nu}$ defined in (4.10); a Gaussian average is made over the α parameter. The ϕ and ψ integrations are performed as in Sec. III with the result

$$I_a=\frac{1}{(4\pi)^2}\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_0^\infty ds_1 e^{-is_1m^2}\int_0^\infty ds_2 e^{-is_2m^2}\frac{H^2\tilde{A}}{\sin(s_1H)\sin(s_2H)}\;, \tag{A7}$$

with

$$\tilde{A}=-\,\mathrm{tr}[G_c(x-y;s_1H)G_c(y-x;s_2H)]$$

and

$$G_c(x-y;s_jH)=\{m+\gamma_\mu[g_{\mu\nu}+i\epsilon_{\mu\nu}\tan(s_jH)]i\partial_\nu^x\}$$
$$\times\exp\left[-\frac{i}{4}(x-y)^2H\cot(s_jH)+\tfrac{1}{2}\sigma\epsilon S_jH\right]$$

with H defined in (4.12).

It is an easy matter to compute the trace, with the result

$$\tilde{A}=-2\left\{m^2\cos[(s_1+s_2)H]+\frac{(x-y)^2}{4}\frac{H^2}{\sin(s_1H)\sin(s_2H)}\right\}\times\exp\left\{-\frac{i}{4}(x-y)^2+[\cot(s_1H)+\cot(s_2H)]\right\}. \tag{A8}$$

The analytic continuation $s_1=-i\tau_1$, $s_2=-i\tau_2$ leads to (5.2) in its Euclidean version: $(x-y)^2=-(x-y)_E{}^2$.

For case B, the photon cutoff may be different for each fermion loop if it is related to the proper time of the loop. The result of the IR approximation is, more generally,

$$\ln T=\tfrac{1}{2}[O_{\mu\nu}(s_1,\phi)-\sigma_{\mu\nu}s_1]J_1{}^2+\tfrac{1}{2}[O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}s_2]^2J_2{}^2+[O_{\mu\nu}(s_1,\phi)-\sigma_{\mu\nu}s_1][O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}s_2]J_3{}^2\,, \tag{A9}$$

with $J_i{}^2=g^2\mu_i{}^2/8\pi$, where μ_1 (μ_2) is the cutoff for the photon exchanged within loop 1 (2), and μ_3 the cutoff for the photon exchanged between the loops: $\mu_3{}^{-2}=\frac{1}{2}(\mu_1{}^{-2}+\mu_2{}^{-2})$. Inserting (A9) into (A3) shows that I_0 is a constant in the IR approximation. The $|x-y|$ dependence is in the neglected terms; indeed, had one kept the full $e^{ik(x-y)}$ factor of the photon propagator, one would have obtained

$$g_{\rho\sigma}J_3{}^2=ig^2\int\frac{d^2k}{(2\pi)^2}e^{k^2/\mu_3{}^2}e^{ik(x-y)}k_\rho k_\sigma=g_{\rho\sigma}g^2\frac{\mu_3{}^2}{8\pi}e^{-\mu_3{}^2|x-y|^2/4}\,,$$

so the approximation describes incorrectly the region $|x-y|^2\mu_3{}^2\gtrsim 1$; this is of little consequence since μ_3 is of the order of m and one is interested in the $m\to 0$ limit.

Equation (A9) may be written as

$$T=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_{-\infty}^{+\infty}\frac{d\beta}{2\sqrt{\pi}}e^{-\beta^2/4}\exp[-AJ_1(\alpha\sqrt{1+\omega}+\beta\sqrt{1-\omega})/2]\exp[-BJ_2(\alpha\sqrt{1+\omega}-\beta\sqrt{1-\omega})]\,, \tag{A10}$$

where

$$A=[O_{\mu\nu}(s_1,\phi)-\sigma_{\mu\nu}s_1]\epsilon_{\mu\nu},\quad B=[O_{\mu\nu}(s_2,\psi)-\sigma_{\mu\nu}s_2]\epsilon_{\mu\nu}\,, \tag{A11}$$

and

$$\omega=\frac{J_3{}^2}{J_1J_2}=2\frac{\mu_1\mu_2}{\mu_1+\mu_2}\,. \tag{A12}$$

Insertion of (A10) into (A3) shows that there appear two propagators of a fermion in the background fields H_1 and H_2 with

$$H_1=\frac{g\mu_1}{4\sqrt{\pi}}(\alpha\sqrt{1+\omega}+\beta\sqrt{1-\omega}),\quad H_2=\frac{g\mu_2}{4\sqrt{\pi}}(\alpha\sqrt{1+\omega}-\beta\sqrt{1-\omega})\,. \tag{A13}$$

The ϕ and ψ integrations have been performed in Sec. III; one obtains

$$I_b=\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}\int_{-\infty}^{+\infty}\frac{d\beta}{2\sqrt{\pi}}e^{-\beta^2/4}\int_0^\infty ds_1e^{-is_1m^2}\int_0^\infty ds_2e^{-is_2m^2}\times\left\{H_1\cot(s_1H_1)-\frac{1}{s_1}\right\}\left\{H_2\cot(s_2H_2)-\frac{1}{s_2}\right\}, \tag{A14}$$

where the cases where one fermion, at least, propagates without interaction have been subtracted. One analytically continues $s_1=-i\tau_1$, $s_2=-i\tau_2$.

Note from (A11) that $\omega=0$ gives the contribution of diagrams where no photon is exchanged between the two-fermion loops.

APPENDIX B: NONRELATIVISTIC STUDIES

1. Nonrelativistic limit of the propagator

From the fermion propagator in a constant real field H, Eq. (3.11), we extract the boson propagator and study its nonrelativistic limit:

$$G(x,y)=\exp\left[i\frac{H}{4}\epsilon_{\mu\nu}(x-y)_\mu(x+y)_0\right]G'(x-y)\,,$$

$$G'(x-y)=-\frac{1}{4\pi}\int_0^\infty ds\frac{H}{\sinh(sH)}\times\exp[-ism^2-i(x-y)^2H/\tanh(sH)]\,. \tag{B1}$$

One may switch from the gauge $A_\mu = -\frac{1}{2}F_{\mu\nu}x_\nu$ to the gauge $gA_0 = Hx$, $A_1 = 0$ and modify accordingly the gauge-covariant phase factor. Let $T = x_0 - y_0$ and $X = x_1 - y_1$. From (B1) one obtains

$$G'(E,X) = \int_{-\infty}^{+\infty} dT\, e^{+iET} G'(T,X) .$$

One then writes $E = m + K$ and neglects the K^2 term in $G'(E)$:

$$G'(T,X) = e^{-imT}\int_{-\infty}^{+\infty} dK\, e^{-iKT} G'(E = m + K, X) .$$

The integration over K produces a factor $\delta(T/2m - \tanh(sH)/H)$. With constraint $TH/2m < 1$, the nonrelativistic result for the propagation of a particle in a constant field H is obtained[22] if one makes the further approximation $s_0 H \ll 1$:

$$e^{imT}G(T,x_1,y_1) = \left[\frac{m}{2\pi T}\right]^{1/2} \exp\left[-i\left[\frac{H}{2}(x_1+y_1)T - (x_1-y_1)^2\frac{m}{2T} + \frac{H^2}{m}\frac{T^3}{24}\right]\right] . \quad \text{(B2)}$$

An alternative expression is

$$e^{imT}G(T,x_1,y_1) = \int_{-\infty}^{+\infty} dK\, e^{-iKT}\psi_K(x_1)\psi_K^*(y_1) , \quad \text{(B3)}$$

where

$$\psi_K(x) = \frac{(2mH)^{1/3}}{H^{1/2}}\mathrm{Ai}((2mH)^{1/3}(x - K/H)) \quad \text{(B4)}$$

is the Schrödinger wave function of a particle in the potential $V(x) = Hx$. An analytical continuation of the form (B2) to $T = -i\tau$ explodes for large τ due to the spectrum of the Hamiltonian.

2. The nonrelativistic e^+-e^- system

Here we study the propagation of a nonrelativistic e^+-e^- pair, in order to compare the result for the exact Coulomb interaction to its IR approximation by the motion of both particles in an effective imaginary background field, subsequently averaged over. We first derive an exact result for the case of a real background field and perform the analytical continuation to the imaginary case at the last stage.

Let particle e^- propagate from x_a to x_b and e^+ propagate from x_a' to x_b' during the same time interval T. Making use of the nonrelativistic form (B2) of the propagator in a constant field, the product of the two propagators separates into the free motion of the center of mass and the propagation for the relative coordinate from $X = x_a - x_a'$ to $Y = x_b - x_b'$ during time T with mass parameter $\mu = m/2$.

The average over the strength of the field can be performed. Substituting αH for H in Eq. (B2) one obtains

$$\int_{-\infty}^{+\infty}\frac{d\alpha}{2\sqrt{\pi}}e^{-\alpha^2/4}G(T,X,Y;\alpha H) = \left[\frac{m}{2\pi T}\right]^{1/2}\left[1 + \frac{iH^2T^3}{6m}\right]^{-1/2} \times \exp\left[i(X-Y)^2\frac{m}{2T} - H^2(X+Y)^2T^2/4\left[1 + i\frac{H^2T^3}{6m}\right]\right] . \quad \text{(B5)}$$

Equation (B5) can be analytically continued to $T = -i\tau$. The large-τ limit, which selects the lowest-energy states of the system, is

$$\left[\frac{3}{\pi i}\right]^{1/2}\frac{m}{H\tau^2} . \quad \text{(B6)}$$

The constraints on the relative coordinates X and Y have disappeared; i.e., the $(e^+$-$e^-)$ system is not localized.

Equation (B6) indicates a density of states near the energy $K = 0$ proportional to K.

Equations (B3) and (B4) may be analytically continued to the case of an imaginary field $H = iH'$. The large-τ limit behaves identically to the case of a real field. This result has to be compared with the exact result for the propagation of an $(e^+$-$e^-)$ pair interacting via the Coulomb interaction $e\,|x_a - x_a'|$. For large τ

$$G_{\mathrm{Coulomb}}(T = -i\tau, X, Y) = e^{-K_0\tau}\psi_{K_0}(X)\psi_{K_0}(Y)$$

with $\psi_K(x)$ given by (B4) for $x > 0$, $\psi_K(-x) = \psi_K(x)$ and K_0 given by

$$(2mH)^{1/3}K_0/H = z_0 ,$$

where $-z_0$ is the first zero of the derivative of the Airy function $\mathrm{Ai}(z)$.

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