Topological excitations and long-range order

Neelima Gupte* and Subodh R. Shenoy
School of Physics, University of Hyderabad, Hyderabad 500134, India
(Received 24 October 1985)

Within a mean-field approximation, we show that the topological excitations of the three-dimensional XY model (vortex loops) break rotational invariance below the critical temperature. The onset of long-range order shows up as an orientational transition of the vortex loops perpendicular to the direction picked out by the mean field.

I. INTRODUCTION

The plane rotator model

$$\beta H = -K_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j), \quad -\pi \le \theta_i \le \pi \tag{1.1}$$

has been the focus of much interest both in field theory[1-3] and statistical mechanics[4-6] over the past decade. θ_i represents the angular deviation of the classical planar spin at the ith site of a hypercubic lattice in d dimensions, while $\langle ij \rangle$ denotes nearest neighbors, counted once. The model has a symmetry under the change $\theta_i \to \theta_i + 2n_i\pi$. It has an interesting topological structure related to this symmetry, the detailed nature of which depends on the dimensionality of the lattice.

In two dimensions (2D) the topological excitations are vortex points. These are circular deviations of θ, centered at some point and falling off slowly, across the lattice. They exist in addition to the spin-wave collective modes that are small oscillatory deviations of θ. The unbinding of topological excitations of opposite sign[7-9] drives the phase transition in two dimensions, where long-range spin ordering of the conventional type is forbidden.[10]

In three dimensions (3D) by contrast, long-range order does exist. Below a critical temperature, the spins lock on average in a common direction. This is the spontaneous breaking of a global (rotational) symmetry. The topological excitations in the 3D XY model are vortex loops that correspond to toroidal configurations of spins.[2,11]

In this paper, we consider the general question of the interplay between long-range order (LRO) and the topological excitations of the 3D XY model. How are the vortex loops affected by the transition and how do they participate in it? Are vortex loops alone sufficient to describe the transition?[12]

The most convenient framework for discussing these questions is in terms of a Hamiltonian directly describing vortex loops, rather than the spin Hamiltonian of (1.1). The most direct way of extracting the topological Hamiltonian is by the "dual transformation"[1-3,6] variables $J_\mu(r), \phi_\mu(r)$ defined on the bonds of a dual lattice. This is a lattice penetrating the original lattice and displaced by half a lattice spacing in each direction. The original site variables θ_i are integrated out. The two representations are exactly equivalent descriptions of the same model. The partition function has the form

$$Z = \int \prod_{r,\mu} d\phi_\mu(r) \sum_{\substack{J_\mu(r) \\ \text{loops only}}} \exp\left[-\frac{1}{2K_0} \sum_{r,\mu} [\epsilon_{\mu\nu\lambda} \Delta_\nu \phi_\lambda(r)]^2 + 2\pi i \sum_{r,\mu} J_\mu(r)\phi_\mu(r) \right] . \tag{1.2}$$

The partition function is analogous to that of a current $J_\mu(r)$ linearly coupled to a vector potential $\phi_\mu(r)$ but with a pure imaginary coupling constant. The current and the vector potential are *topological* in nature.

Continuing the analogy, the topological dipole moment $M_\mu(R)$ of the current loop can be defined, and the discrete curl of the vector potential is a magnetic field $B_\mu(r)$. It is more convenient to work with this set of dual variables $\{B_\mu(r), M_\mu(R)\}$.

Since LRO is most directly seen in the mean-field approximation, the questions raised previously take the following form: How does one do mean-field theory in terms of $M_\mu(R), B_\mu(r)$ variables? How are topological excitations affected by long-range order (LRO)?

These questions are also relevant in another context. A much studied[4,5,12] generalization of (1.1) describes a matter field θ_i minimally coupled to a gauge field A_{ij}. The Hamiltonian has the form

$$\beta H = -K_0 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j - eA_{ij}) .$$

For the charge $e=0$, this clearly reduces to the XY spin model of (1.1). Monte Carlo simulations have been used to study the full charge-temperature phase diagram.[12] Our investigation corresponds to the temperature axis $e^2=0$.

The $e^2=0$, T axis can be related to the $T=0$, e^2 axis by a variation of the dual transformation.[5] Hence, our results are of relevance to this 3D lattice gauge theory. We elsewhere investigate the behavior of the vortex loops for $e^2 \neq 0$ in the rest of the phase diagram.

In this work we perform the simplest symmetry-breaking truncation on the Hamiltonian in dual $M_\mu(R)$, $B_\mu(r)$ variables. This might be termed "topological mean-field theory." The vortex loop, or circulating topological current is characterized by a strength, size, and direction of circulation. This is related to the magnitude and orientation of $M_\mu(R)$.

We find a (mean-field) transition that involves the onset of orientation of the very smallest vortex loops. The topological dipole moment $M_\mu(R)$ perpendicular to the plane of the loop, tends to line up with others below a critical temperature T_c, so that $\langle M_\mu(R)\rangle=0$. The orientational transition involves the spontaneous breaking of a global rotational symmetry, just as in the spin description. It does not contradict Elitzur's theorem:[13] no local gauge symmetry is broken. See Fig. 1 for a pictorial illustration of this. The transition is like that of a dipolar ferromagnet but with itinerant spins, $M_\mu(R)=0,\pm 1$.

The nature of the self-consistent solution is affected by the pure imaginary coupling constant in the bilinear coupling between the real variable $M_\mu(R)$ and $B_\mu(r)$. For the $\langle B_\mu(r)\rangle=0$ solution, one obtains the usual[14] flip in the sign of the Biot-Savart law between the currents $J_\mu(r)$, because the square of the pure imaginary coupling constant is negative, $i^2=-1$. Then ferromagnetic tendencies are forbidden and $\langle M_\mu(R)\rangle=0$. However, for $T<T_c$, one finds the symmetry-breaking solution that lowers the free energy, namely, $\langle M_\mu(R)\rangle\neq 0$ in a particular direction, with the auxiliary field $\langle B_\mu(r)\rangle\neq 0$ in the same direction but pure imaginary.

The idea that topological excitations in 3D participate crucially in a phase transition, has, of course, been considered by others.[1-3] The 2D Kosterlitz-Thouless (KT) idea of an unbinding of vortex points has been carried over in 3D to a picture of a sudden expansion of vortex ring size at the transition. As mentioned, however, the 3D and 2D cases are qualitatively different: the 3D case has LRO while the 2D case does not. The transition that we find involving vortex loops is distinct from previous ideas[1,3] and involves an orientation directly associated with the onset of LRO both in the spin-spin correlation and the M-M correlation. One cannot, of course, rule out an independent KT-like size expansion transition at this or other transition temperatures. This is discussed further in the last section.

The plan of the paper is as follows: In Sec. II we state the standard mean-field results in the spin representation θ_i. In Sec. III this is rewritten in terms of discrete variables $M_\mu(R)$ and a continuous auxiliary field $B_\mu(r)$ by a dual transformation. A mean-field-like truncation is made, and coupled self-consistency equations for $\langle B_\mu(r)\rangle$ and $\langle M_\mu(R)\rangle$ are obtained. These are solved close to a transition temperature T_c to obtain $\langle M_\mu(r)\rangle \propto (1-T/T_c)^{1/2}$. This T_c is estimated. The lowering of the free energy is calculated and turns out to be the same as the standard mean-field result in Sec. II, indicating that

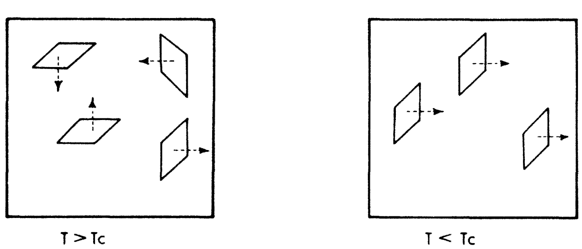

FIG. 1. Schematic physical picture at transition: (a) For temperatures above transition, loops tumble freely. Dashed arrows denote topological moments of loops. (b) For temperatures below transition, the (smallest) loops orient, on average.

the original and dual lattice descriptions are equivalent. This point is confirmed in Sec. IV by evaluating the long-range spin-spin correlation $\langle\cos(\theta_i-\theta_k)\rangle$ in the dual representation. It turns out to be proportional to $\langle M_\mu(R)\rangle^2\propto(1-T/T_c)$ implying that spins are ordered only if vortex rings do not tumble randomly about, as is physically reasonable. The results are discussed in Sec. V.

II. SPIN VARIABLES ARE MEAN-FIELD THEORY

In this section we outline the standard mean-field results for comparison later.

The XY Hamiltonian is

$$\beta H(\theta)=-K_0\sum_{\langle ij\rangle}\mathbf{s}_i\cdot\mathbf{s}_j \quad (|\mathbf{s}_i|=1)$$
$$=-K_0\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\,, \tag{2.1}$$

where $K_0=J/k_BT$ and J is the ferromagnetic coupling constant. Clearly (2.1) is invariant under the spin rotations $\theta_i\to\theta_i+\alpha$.

We get the following symmetry-breaking average:

$$\mathbf{n}\cdot\mathbf{s}_i=\langle\cos\theta_i\rangle$$
$$=\frac{1}{Z}\int_{-\pi}^{\pi}\prod_i\frac{d\theta_i}{2\pi}e^{-\beta H(\theta)}\cos\theta_i\,. \tag{2.2}$$

The partition function is

$$Z=\int_{-\pi}^{+\pi}\prod_{i=1}^{n}\frac{d\theta_i}{2\pi}e^{-\beta H(\theta)}\,, \tag{2.3}$$

where $\mathbf{n}$ is a unit vector in the direction of spontaneous magnetization.

The mean-field truncation of the weighting factor corresponds to a partition function given by

$$Z_{\rm MF}(\{\theta\})=e^{K_0zN\langle\cos\theta\rangle^2}\prod_{i=1}^{N}\exp\left[K_0z\langle\cos\theta\rangle\sum_i\cos\theta_i -K_0zN\langle\cos\theta\rangle^2\right]. \tag{2.4}$$

Here the first exponent is the averaged Hamiltonian and the second exponent in large parentheses is linear in the fluctuations with a zero average. The number of nearest neighbors is $z=6$ for a cubic 3D lattice.

Using the weighting factor of (2.4), the self-consistently nonzero solution of (2.2) is found to be, for $T<T_c$,

$$\langle\cos\theta_i\rangle=\sqrt{2}\left[1-\frac{T}{T_c}\right]^{1/2}, \tag{2.5}$$

where the mean-field transition temperature is

$$T_c=\frac{3K_B}{J},\quad \frac{J}{K_BT_c}=0.333\ldots\,. \tag{2.6}$$

The free energy lowering from (2.4) is

$$\frac{\beta\Delta F}{N}=-\left[1-\frac{T}{T_c}\right]^2. \tag{2.7}$$

The spin-spin correlation function exhibits long-range order for large separations:

$$\langle \mathbf{S}_{\mathbf{r}_i}\cdot\mathbf{S}_{\mathbf{r}_k}\rangle=\langle\cos(\theta_i-\theta_k)\rangle \underset{|r_i-r_k|\to\infty}{\longrightarrow} \langle\cos\theta_i\rangle^2 ,$$
$$\langle\cos\theta_i\rangle^2=2\left[1-\frac{T}{T_c}\right] . \tag{2.8}$$

III. DUAL VARIABLES AND MEAN-FIELD THEORY

An alternative description of the XY system in three dimensions is in terms of dual variables $J_\mu(r),\phi_\mu(r)$ defined on the bonds of the dual lattice.

This is obtained by making a dual transformation[1–3] on the partition function defined on the original lattice. The weighting factor $e^{-\beta H(\theta)}$ is Fourier analyzed and the θ integrations are performed. These give a set of Kronecker δ constraints on the integer-valued Fourier labels $k_\mu(i)$. The partition function of (2.3) is

$$Z=\sum_{\{k_\mu(i)\}}\exp\left[\sum_{i,\mu}V(k_\mu(i))\right]\prod\delta_{\Delta_\mu k_\mu(i),0} , \tag{3.1}$$

where $e^{V(x)}$ is the Fourier coefficient of $e^{-\beta H(\theta)}$. The constraint can be satisfied as an identity by the introduction of integer-valued variables on a dual lattice. Using the Poisson summation formula, the partition function takes the form

$$Z=\sum_{\{J\}}{}'\int_{-\infty}^{+\infty}\prod_{r,\mu}d\phi_\mu(r)\exp\left[\sum_{r,\mu}V(\epsilon_{\mu\nu\lambda}\Delta_\nu\phi_\lambda(r)) +2\pi iJ_\mu(r)\phi_\mu(r)\right] . \tag{3.2}$$

The prime on the sum in (3.2) denotes that the $\{J\}$ variables form closed loops,[4] obeying the constraint $\Delta\mu J_\mu(r)=0$. The integer-valued current variables $J_\mu(r)$ are physical variables describing topological excitations of the original θ field while $\phi_\mu(r)$ is an auxiliary field variable. [Note that (3.2) clearly differs from the contact interaction model (3.1), that has also been called a loop model.[12]]

The result (3.2) is formally exact. The weighting factor can, however, be approximated by a Gaussian to give

$$Z\simeq\sum_{\{J\}}{}'\int_{-\infty}^{+\infty}\prod_{r,\mu}d\phi_\mu(r)\exp\left[-\frac{1}{2K_0}\sum_{r,\mu}[\epsilon_{\mu\nu\lambda}\Delta_\nu\phi_\lambda(r)]^2 +2\pi i\sum_{r,\mu}J_\mu(r)\phi_\mu(r)\right] . \tag{3.3}$$

Clearly the exponent in (3.3) contains an analogy to electromagnetism, with an auxiliary field

$$B_\mu(r)=\epsilon_{\mu\nu\lambda}\Delta_\nu\phi_\lambda(r) \tag{3.4}$$

playing the role of a magnetic field. Then in (3.3) one has a Hamiltonian

$$(2K_0)^{-1}\sum_{\mu,r}B_\mu{}^2(r)-2\pi i\sum_{r,\mu}J_\mu(r)\phi_\mu(r)$$

with a field energy and a $\mathbf{J}\cdot\mathbf{A}$-type current vector-potential interaction.

The Hamiltonian exhibits a continuous local U(1) gauge symmetry, under $\phi_\mu(r)\to\phi_\mu(r)+\Delta_\mu X(r)$. The ring constraint $\Delta_\mu J_\mu(r)=0$ is thus seen as a conservation law associated with this gauge symmetry. The Hamiltonian also exhibits a global rotational symmetry $\phi_\mu(r)\to R_{\mu\nu}\phi_\nu(r)$, $J_\mu(r)\to R_{\mu\nu}J_\nu(r)$, where R is an orthogonal matrix, $R^2=1$. We make a mean-field truncation in terms of variables $M_\mu(\mathbf{R})$ related to $J_\mu(r)$ by Eq. (3.5) below. Our mean-field truncation breaks the second, global rotational symmetry. This is as in the spin Hamiltonian case. The first, local gauge symmetry is left untouched—rings remain rings. There is no violation of Elitzur's theorem[13] at any stage.

Carrying the electromagnetic analogy further, the circulating topological current J may be regarded as producing a topological dipole moment $M_\mu(\mathbf{R})$ defined by

$$M_\mu(\mathbf{R})=\frac{1}{2}\sum_{\substack{\mathbf{r}_l\text{ on}\\ \text{loop }\mathbf{R}}}\epsilon_{\mu\nu\lambda}(\mathbf{r}_l)_\nu J_\lambda(\mathbf{r}_l) . \tag{3.5}$$

Here $\mathbf{R}$ is a vector denoting an origin attached to, and labeling, the vortex ring. $\mathbf{r}_l=\mathbf{r}_l(\mathbf{R})$ is a vector from this origin to the site from which the current $\mathbf{J}(r_l)$ emerges. Notice that the previous site labeling $J_\mu(l)$ has been replaced by vector separation labeling $J_\mu(r_l)$ for currents leaving site r_l. For an elementary loop, Fig. 2 shows the relative components of $J_\mu(\mathbf{r})$, $M_\mu(\mathbf{R})$, with $\mathbf{R}$ at the plaquette center. We later find $\langle B_\mu(\mathbf{r})\rangle$ is pure imaginary. However the direction of this imaginary part is parallel to M_μ and originates at the loop corner. $J_\mu(\mathbf{r})$ takes values $0,\pm1$. Using (3.5), this gives $M_\mu(\mathbf{R})=0,\pm1$.

Since $M_\mu(R)$ is a vortex loop variable that carries the relevant information regarding the position and overall orientation of the loop, we cast the thermodynamics in terms of it. A formal inversion of (3.5) is required

Equation (3.5) can be written in an obvious Dirac braket notation as

$$M_\mu(\mathbf{R})=\langle\mathbf{R}\,|\,M_\mu\rangle$$
$$=\sum_{\mathbf{r}}\left[\epsilon_{\mu\nu\lambda}\tfrac{1}{2}(\mathbf{r})_\nu\sum_{\substack{\mathbf{r}_0\text{ on}\\ \text{loop }\mathbf{R}}}\delta(\mathbf{r}-\mathbf{r}_0)\right]J_\lambda(\mathbf{r}) \tag{3.6a}$$
$$=\sum_{\mathbf{r}}\langle\mathbf{R}\,|\,\hat{L}_{\mu\lambda}\,|\,\mathbf{r}\rangle\langle\mathbf{r}\,|\,J_\lambda\rangle , \tag{3.6b}$$

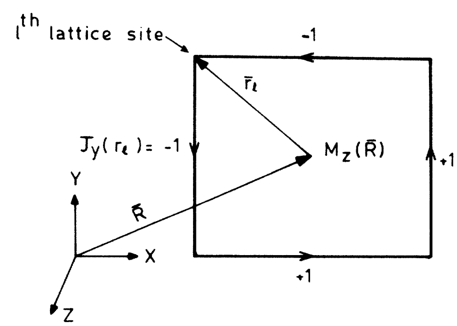

FIG. 2. Topological current loops centered at R with respect to the origin 0 and parallel to the plane of paper. The topological dipole moment $\mathbf{M}(\mathbf{R})$ will point out of the plane of paper and is located at $\mathbf{R}$. The topological current $J_\mu(\mathbf{r})$ has a sign convention as shown.

where $\hat{L}$ defines an operator with matrix elements as in the large parentheses above. $|\mathbf{R}\rangle$ and $|\mathbf{r}\rangle$ are position kets such that they are eigenstates of corresponding position operators:

$$\mathbf{R}_{\rm OP}|\mathbf{R}\rangle=\mathbf{R}|\mathbf{R}\rangle,\quad \mathbf{r}_{\rm OP}|\mathbf{r}\rangle=\mathbf{r}|\mathbf{r}\rangle\ .$$

The operator L_μ is formally

$$\hat{L}_{\mu\lambda}=\tfrac{1}{2}\epsilon_{\mu\nu\lambda}(\mathbf{r}_{\rm OP})_\lambda\sum_{\mathbf{r}_0}\delta(\mathbf{r}_0-\mathbf{r}_{\rm OP}) \tag{3.7}$$

and has matrix elements as in the large parentheses of (3.6a). Then formally $|J_\mu\rangle=L_{\mu\nu}{}^{-1}|M_\mu\rangle$ and we find, from (3.3), that

$$Z=\sum_{M_\mu}\int\prod_{r,\mu}dB_\mu(r)\exp\left[-\sum_{r,\mu}\frac{B_\mu{}^2(\mathbf{r})}{2K_0}+2\pi i\sum_{r,\mu}M_\mu^\dagger(\mathbf{R})\times U_{\mu\nu}(\mathbf{R},\mathbf{r})B_\nu(\mathbf{r})\right], \tag{3.8}$$

where the interaction $U_{\mu\nu}(\mathbf{R},\mathbf{r})$ is the matrix element

$$\langle\mathbf{R}|U_{\mu\nu}|\mathbf{r}\rangle=U_{\mu\nu}(\mathbf{R},\mathbf{r})=\langle\mathbf{R}|(L_{\mu\lambda}{}^{-1})^\dagger(\epsilon_{\lambda\mu'\nu}\Delta_{\mu'})^{-1}|\mathbf{r}\rangle\ . \tag{3.9}$$

The inverse curl operator $(\epsilon_{\lambda\mu'\nu}\Delta_{\mu'})^{-1}$ can be understood from the Maxwell's equation $\nabla\times\mathbf{B}=4\pi\mathbf{J}/c$ that has the solution

$$\mathbf{B}(\mathbf{r})=\frac{1}{c}\int d^3r'\frac{\mathbf{J}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\ .$$

Note that the interaction terms in (3.8) can, in an obvious symbolic notation, be written in two equivalent ways:

$$\sum_{r,\mu}J_\mu(r)\phi_\mu(r)=M_\mu^\dagger U_{\mu\nu}B_\nu=B_\mu^\dagger U_{\mu\nu}^\dagger M_\nu=\sum_{r,\mu}\phi_\mu(r)J_\mu(r)\ .$$

(Here the dagger, or its absence refers to the use of bras or kets as appropriate.) This will become important when considering spontaneous symmetry breaking.

If $\langle B_\mu\rangle=0$, completing the square in (3.8) gives

$$Z\propto\sum_{\{M_\mu\}}\exp\left[-(2\pi^2K_0)\sum_{\mathbf{R},\mathbf{R}'}M_\mu(\mathbf{R})V_{\mu\nu}(\mathbf{R},\mathbf{R}')\mathbf{M}_\nu(\mathbf{R}')\right], \tag{3.10}$$

where $V_{\mu\nu}$, the interaction between the moments, is

$$V_{\mu\nu}(\mathbf{R},\mathbf{R}')=\langle\mathbf{R}|L_{\mu\lambda}{}^{-1}(\epsilon\Delta\epsilon\Delta)^{-1}{}_{\lambda\lambda'}L^{-1}{}_{\lambda'\nu}|\mathbf{R}'\rangle\ . \tag{3.11}$$

In terms of the current variables $\{J\}$, (3.5) in (3.10) gives

$$Z\propto\sum_{\{J\}'}{}'\exp\left[-(2\pi^2K_0)\sum_{r,r'}J_\mu(\mathbf{r})G_{\mu\nu}(\mathbf{r},\mathbf{r}')J_\nu(\mathbf{r}')\right]. \tag{3.12}$$

The inversion can be made in a gauge-invariant fashion,[15] where $G^{-1}{}_{\mu\nu}=\epsilon_{\mu\alpha\beta}\Delta_\alpha\epsilon_{\nu\alpha'\beta'}\Delta_{\alpha'}$. In coordinate space, the Green's function is asympotically $G_{\mu\nu}(\mathbf{r},\mathbf{r}')=(1/4\pi)(\delta_{\mu\nu}/|\mathbf{r}-\mathbf{r}'|)$. Then (3.12) is seen to represent a Biot-Savart-type current-current interaction, but with the sign flipped[14] due to the imaginary coupling, as $i^2=-1$. This agrees with the direct integration of (3.3) with $\langle\phi_\mu\rangle=0$. Equations (3.10) and (3.12) have overall factors from B integrals

Because of the sign flip, (3.10) and (3.12) tend to align two vortex loops with J circulating in opposite directions, i.e., with $M-s$ antiparallel. Thus $\langle M\rangle=0$. Going back to (3.8), for $\langle M\rangle$ to be nonzero, there must be an associated nonzero average of the auxiliary field, $\langle B\rangle=0$. The imaginary coupling constant plays a central role in the nature of this symmetry breaking as explained below.

We formally regard the integrand of the partition function (3.8) as a "Boltzmann factor" describing two fields $\mathbf{B},\mathbf{M}$ interacting bilinearly through a pure imaginary coupling constant. If we consider a complex B plane and a complex M plane, fluctuations in the high-temperature phase occur along the real axes around the origin: $\langle B\rangle=0$, $\langle M\rangle=0$ as in Fig. 3(a).

Now suppose the physical variable M acquires a nonzero real average value $\langle M\rangle$ that lowers the free ener-

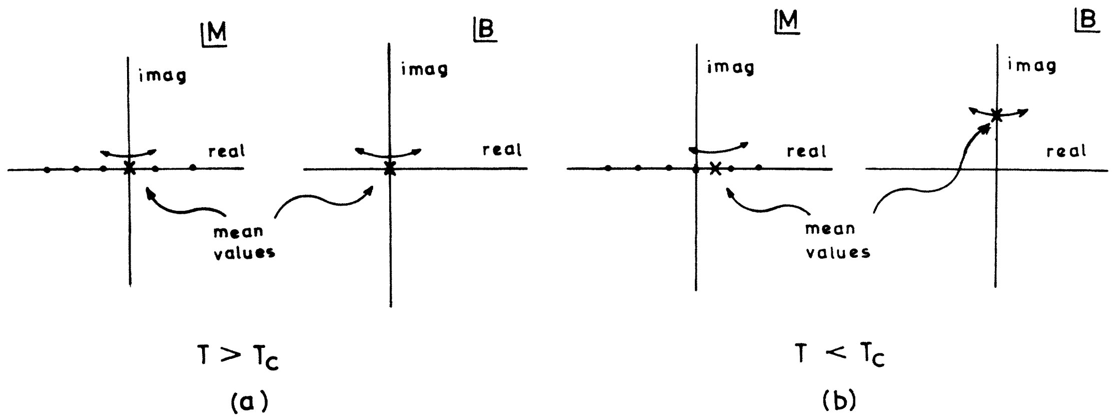

FIG. 3. (a) $T>T_c$ fluctuations, in complex B and M planes about mean values $\langle B\rangle=\langle M\rangle=0$. (b) $T<T_c$ fluctuations about nonzero mean values, $\langle M\rangle$ real, $\langle B\rangle$ pure imaginary.

gy. Then fluctuations occur along the real axis, around this $\langle M\rangle \neq 0$ value [Fig. 3(b)]. The interaction term then shows that B is directed by a pure imaginary mean field through $\{2\pi i\langle M\rangle U\}B$. This will tend to force B in the direction $-i\langle M\rangle U$ in the complex B plane since the Boltzmann weight in Z will be increased. Thus, if M acquires a real mean value $\langle M\rangle$, then iB acquires a real mean value $i\langle B\rangle$. Then below T_c, the situation of Fig. 3(b) becomes more probable.

Thus, because of the pure imaginary coupling, the solution of the coupled self-consistency equations that lowers the free energy below T_c has $\langle M\rangle$, $i\langle B\rangle$ both nonzero and real. The partition function, and the term $\exp(2\pi i\langle B\rangle U\langle M\rangle)$, are of course, real.

The symmetry breaking proposed is admittedly unusual. But a detailed evaluation of the B integral overall factor in (3.12) also shows that B must be extended to complex values, as done henceforth.

We now justify the symmetry breaking discussed by a mean-field evaluation of the anomalous averages. Dropping arguments and subscripts for simplicity,

$$\langle M\rangle=\frac{1}{Z}\sum_{M}\int dB\exp\left[-\frac{1}{2K_0}\sum|B|^2\right]M \times\exp\left[-2\pi i\sum B^{\dagger}U^{\dagger}M\right], \quad (3.13a)$$

$$\langle B^*\rangle=\frac{1}{Z}\sum_{M}\int dB\exp\left[-\frac{1}{2K_0}\sum|B|^2+2\pi i\sum M^{\dagger}UB\right] \quad (3.13b)$$

with the partition function Z as in (3.8). A mean-field approximation then amounts to approximating variables by their averages, namely, $B^{\dagger}\to\langle B\rangle^*$, $M\to\langle M\rangle$ in (3.13a) and (3.13b), respectively. The partition function is approximated by

$$Z\simeq Z_{\rm MF}=\exp\left[-2\pi i\sum\langle M\rangle U\langle B\rangle\right]\sum_{M}\int\prod dB\exp\left[-\frac{1}{2K_0}\sum|B|^2+2\pi i\sum(MU\langle B\rangle+\langle M\rangle UB)\right]. \quad (3.14)$$

Restricting ourselves to integer values $J_\mu=0,\pm1$ or $M_\mu=0,\pm1$, the anomalous averages in particular symmetry-breaking directions, $\langle M_\mu\rangle,\langle B_\nu\rangle$, are

$$\langle B_\nu(r)\rangle^*=\frac{\int e^{-(1/2K_0)|B_\nu|^2(\mathbf{r})}\exp\left[2\pi i\sum_{R,\mu}\langle M_\mu(\mathbf{R})\rangle U_{\mu\nu}(\mathbf{R},\mathbf{r})B_\nu(\mathbf{r})\right]B_\nu^*(r)}{\int dB_\nu(r)\exp\left[-\frac{1}{2K_0}|B_\nu|^2(\mathbf{r})+2\pi i\sum_{R,\mu}\langle M_\mu(\mathbf{R})\rangle U_{\mu\nu}(\mathbf{R},\mathbf{r})B_\nu(\mathbf{r})\right]}, \quad (3.15a)$$

$$\langle M_\mu(R)\rangle=\frac{\sum_{M_\mu=0,\pm1}\exp\left[+2\pi i\sum_{r,\nu}\langle B_\nu(\mathbf{r})\rangle U^{\dagger}_{\nu\mu}(\mathbf{r},\mathbf{R})M_\mu(\mathbf{R})\right]M_\mu(\mathbf{R})}{\sum_{M_\mu(R)=0,\pm1}\exp\left[+2\pi i\sum_{r,\nu}\langle B_\nu(r)\rangle U^{\dagger}_{\nu\mu}(\mathbf{r},\mathbf{R})M_\mu(\mathbf{R})\right]}. \quad (3.15b)$$

There are three solutions: $\langle M\rangle=\langle B\rangle=0$, $\langle M\rangle\sim -i\langle B\rangle$, and $\langle M\rangle\sim +i\langle B\rangle$. The first is the disordered state, the second raises the free energy relative to this, and the third lowers the free energy and is hence preferred.

Notice that two conjugate forms of the interaction term have been used, (3.13a) and (3.13b), before doing the approximation. The $\langle M\rangle,\langle B\rangle$ averages are, in fact, on the same free energy lowering branch: if M is oriented by $+i\langle B\rangle$ then B must be oriented by $-i\langle M\rangle$.

For $\langle M\rangle$ small, (3.15a) yields, with dominant contributions from $\langle M\rangle$ on the same loop R (Fig. 2),

$$\langle B_\nu(r)\rangle=-2\pi iK_0\sum_{R',\nu}\langle M_\mu(\mathbf{R}')\rangle U_{\mu\nu}(\mathbf{R}',\mathbf{r})$$

$$\simeq -2\pi iK_0\sum_{\nu}\langle M_\mu(R)\rangle U_{\mu\nu}(R,r)\ . \quad (3.16)$$

This is the pure imaginary solution conjectured above. The real $\langle M\rangle$ root is given by

$$|\mathbf{M}|=\frac{2\sinh\left[4\pi^2K_0\sum_{r,\nu}U_{\mu\nu}(\mathbf{R},\mathbf{r})U^{\dagger}_{\nu\mu}(\mathbf{r},\mathbf{R})\,|\mathbf{M}|\right]}{1+2\cosh\left[4\pi^2K_0\sum_{r,\nu}U_{\mu\nu}(\mathbf{R},\mathbf{r})U^{\dagger}_{\nu\mu}(\mathbf{r},\mathbf{R})\,|\mathbf{M}|\right]}, \quad (3.17)$$

where $\langle M_\mu\rangle=|\mathbf{M}|\hat{\mu}$ and $\hat{\mu}$ is a unit vector. Since $K_0(T)\propto 1/T$ it is convenient to define a temperature T_c by

$$\frac{8\pi^2}{3}K_0(T)\sum_{r,\nu}U^{\dagger}_{\mu\nu}(\mathbf{R},\mathbf{r})U_{\nu\mu}(\mathbf{r},\mathbf{R})=\frac{T_c}{T} \quad (3.18)$$

so that the self-consistent solution obtained from (3.18) is

$$|\mathbf{M}|=\left[\frac{8}{3}\right]^{1/2}\frac{T}{T_c}\left[1-\frac{T}{T_c}\right]^{1/2},\quad T<T_c$$

$$=0,\quad T>T_c\ . \quad (3.19)$$

The free energy lowering, from the partition function of (3.14), is

$$\frac{\beta\Delta F}{N}=\frac{3}{4}\frac{T_c}{T}\mathbf{M}^2-\ln\left[1+\frac{2}{3}\left[\cosh\frac{3}{2}\frac{T_c}{T}\overline{M}-1\right]\right]$$
$$=-\left[1-\frac{T}{T_c}\right]^2 . \tag{3.20}$$

Thus the self-consistent $\langle B\rangle,\langle M\rangle\neq 0$ solution is indeed preferred. Note that the expression (3.20) is the same as the mean-field result of (2.7) in the θ representation.

The relation between the mean fields (3.16) can be rewritten in a more familiar form:

$$\epsilon_{\mu\nu\lambda}\Delta_\nu\langle B_\lambda\rangle=2\pi i K_0\langle J_\mu\rangle . \tag{3.21}$$

This is of course the analogue of a Maxwell equation, and can be obtained by treating the exponent in the weighting factor of (3.3) as a Lagrangian, and varying with respect to ϕ_μ.

The vortex ring orients in a particular direction, and moves freely with its plane perpendicular to it. If $\langle M_\mu\rangle$ is nonzero in the z direction, then $\langle J_\mu\rangle$ will have components only in the x and y directions. $J_\mu(r)$ tend to cancel for all r inside the system and will be nonzero at the surface, taken to be solenoidal for convenience.

In the thermodynamic limit, from (3.21), $-i\langle B\rangle$ is in the direction of a field inside such a solenoid, namely, along the z axis. [The most transparent way of seeing this is to write (3.21) in terms of a vector potential $\mathbf{A}(r)=\sum_{r'}\mathbf{J}(r)/|\mathbf{r}-\mathbf{r}'|$ with $\mathbf{B}=\nabla\times\mathbf{A}$.]

We now estimate the transition temperature T_c in (3.18),

$$T_c=\frac{8\pi^2}{3}\frac{J}{k_B}\sum_{r,\nu}U^\dagger_{\mu\nu}(\mathbf{R},\mathbf{r})U_{\nu\mu}(\mathbf{r},\mathbf{R}) . \tag{3.22}$$

Here, as is implicit in (3.9) and (3.6), the sum over r is over the sites that participate in the loop labeled by R. For the square elementary loop, from Fig. 2, we use the convention[15] $J_\mu(r)=\pm 1$ for $\pm x$ and $\pm y$ directions. The $\mathbf{r}$ label is from the start of the J_μ arrows, and $\mathbf{R}$ at the center of the plaquette is taken as the origin. Since $r_x=r_y=\frac{1}{2}$,

$$M_z(\mathbf{R})=\tfrac{1}{2}[\tfrac{1}{2}(+1)+\tfrac{1}{2}(+.1)+(-\tfrac{1}{2})(-1)+(-\tfrac{1}{2})(-1)]$$
$$=1 . \tag{3.23}$$

Thus the effect of $\hat{L}_\mu$ acting on a unit magnitude J in the XY plane is to produce a unit M in the perpendicular z direction, exactly as in a magnetic dipole moment of a square current loop. (Of course, these are topological current and dipole variables here.)

Similarly, if we approximate $(\epsilon_{\mu\nu\lambda}\Delta_\nu)^{-1}$ by its discrete analogue, then for a point r at the lower right-hand corner of the square loop, for B of unit magnitude

$$(\epsilon\Delta)_{xz}{}^{-1}\hat{\mathbf{B}}_z=\frac{-1}{4\pi}\sum_{r'}\frac{\hat{\mathbf{B}}_z\epsilon_{xyz}(\mathbf{r}-\mathbf{r}')_y}{|\mathbf{r}-\mathbf{r}'|^3}=\frac{1}{2\pi} . \tag{3.24}$$

For the elementary plaquette, $U_{\mu\nu}=\delta_{\mu\nu}U_{\mu\mu}$ and (3.16) becomes

$$\langle B_\mu\rangle=-2\pi i K_0 U_{\mu\mu}\langle M_\mu\rangle,\quad U_{\mu\mu}=\frac{1}{2\pi} . \tag{3.25}$$

Thus, approximately, the critical temperature T_c for the orientation of elementary loops is

$$T_c=\frac{8}{3}\frac{J}{k_B},\quad \frac{J}{k_BT_c}=0.377 . \tag{3.26}$$

This estimate satisfies the Myerson bound[16] for the XY model of $J/k_BT_c\geq 0.32$. It differs slightly from the original lattice estimate of (2.4), as the duality procedure contains an extra Gaussian truncation and a continuum expression has been used in our estimate.

As far as larger loops with N links are concerned, it is easy to see that $M\sim N^2$, $L^{-1}\sim N^{-2}$, and so $T_c\sim N^{-4}$. The transition temperature is lower for larger loops: it is the smallest loops that spontaneously orient themselves first, as depicted in Fig. 1. The transition is somewhat analogous to the isotropic-to-nematic transition in liquid crystals.

IV. CORRELATION FUNCTION AND LONG-RANGE ORDER

The spin-spin correlation function $\langle S_iS_k\rangle=\langle\cos(\theta_i-\theta_k)\rangle$ exhibits long-range order below T_c as noted in (2.8). We have seen above that in the language of topological excitations, the spin-ordering transition is an orientation of the smallest vortex rings. Continuing this analysis, we do a dual transformation on the correlation function that can be written as a (nonunique) projection

$$\langle\cos\theta_{ik}\rangle=\frac{\partial^2}{\partial h_i\partial h_k}\ln\left[\int\prod_i d\theta_i\exp\left[K_0\sum_{\langle ij\rangle}\cos\theta_{ij}\right]\times e^{h_ih_k\cos\theta_{ik}}\right]_{h=0}$$
$$=\frac{\partial^2}{\partial h_i\partial h_k}\ln Z(h),\quad \theta_{ij}=(\theta_i-\theta_j) . \tag{4.1}$$

(Although all possible projections must be equivalent, with the exact form, they need not agree after some general approximations.) Now $\cos\theta_{ik}$ can be written in terms of nearest-neighbor phase differences θ_{rs} by adding and subtracting neighboring phase angles along some path from site i to site k. The weighting factor in $Z(h)$ is thus a periodic function of nearest-neighbor phase angles only, and can be Fourier analyzed and dual transformed as before:

$$Z(h)=\sum_{M_\mu}\int\prod_{r,\mu}dB_\mu(r)\exp\left[2\pi i\sum M_\mu(\mathbf{R})U_{\mu\nu}(\mathbf{R},\mathbf{r})B_\nu\right]e^{V(B_\mu,h)} , \tag{4.2}$$

where the weighting factor is defined by the inverse transform,

$$e^{v(n_{rs},h)}=\int\prod_i d\theta_i \exp\left[-i\sum_i n_{rs}(i)\theta_{rs}(i)+K_0\sum_{\langle rs\rangle}\cos\theta_{rs}+h_ih_k\cos\theta_{ik}\right]$$

$$=I_{\{n_{rs}\}}(h)\simeq I_{\{0\}}(0)e^{\Delta I/I_{\{0\}}(0)}\ , \tag{4.3}$$

$$\frac{\Delta I}{I_{\{0\}}(0)}=\frac{\int\prod_i d\theta_i\left[\exp\left[-i\sum n_{rs}(i)\theta_{rs}(i)\right]e^{h_ih_k\cos\theta_{ik}}-1\right]\exp\left[K_0\sum_{\langle ij\rangle}\cos\theta_{ij}\right]}{\int\prod_i d\theta_i\exp\left[K_0\sum\cos\theta_{ik}\right]}$$

$$\simeq-\sum_{\langle rs\rangle}n_{rs}{}^2\langle\theta_{rs}{}^2\rangle-\tfrac{1}{2}h_ih_k\sum n_{rs}n_{r's'}\langle\theta_{rs}\theta_{r's'}\cos\theta_{ik}\rangle\ . \tag{4.4}$$

For widely separated sites i and k, the second average is nonzero if rs and $r's'$ are in the immediate neighborhood of i and k. Then

$$\sum_{\substack{\langle rs\rangle\\ \langle r's'\rangle}}n_{rs}n_{r's'}\langle\theta_{rs}\theta_{r's'}\cos\theta_{ik}\rangle$$

$$=\sum_\mu n_\mu(i)n_\mu(k)\left\langle\theta_\mu(r)\theta_\mu(r')\cos\sum_{l=i}^{k}\theta_\mu(l)\right\rangle$$

$$=\sum_\mu n_\mu(i)n_\mu(k)\langle\theta_i\sin\theta_i\rangle\langle\theta_k\sin\theta_k\rangle\ , \tag{4.5}$$

where it is convenient to switch to the vector compact notation. Writing $\theta_i=\frac{1}{2}(\theta_i-\theta_{i+\hat{\mu}})+\frac{1}{2}(\theta_i+\theta_{i+\hat{\mu}})$, where $\hat{\mu}$ is the direction of some nearest neighbor, one obtains

$$\frac{\Delta I}{I_{\{0\}}(0)}=-\frac{1}{2K_0}\sum_{r,\mu}n_\mu{}^2(r)$$

$$-\frac{h_ih_k}{2}\langle\cos(\theta_i-\theta_{i+\hat{\mu}})\rangle^2\sum_\mu n_\mu(i)n_\mu(k)\ . \tag{4.6}$$

The correlation function of (4.2) then yields, with (4.1),

$$\langle\cos\theta_{ik}\rangle=-\tfrac{1}{2}\langle B_\mu(i)B_\mu(k)\rangle\ , \tag{4.7}$$

where the average on the right-hand side is with respect to the dual variables B,M.

Thus long-range order in the spin variables is associated with long-range order in the auxiliary field B and hence in the order parameter M. Using Eq. (3.25) we find

$$\langle\cos\theta_{ik}\rangle=-2\mathbf{B}^2=\frac{3}{2}K_0\frac{T_c}{T}\mathbf{M}^2$$

$$=4K_0\frac{T}{T_c}\left[1-\frac{T}{T_c}\right]\ . \tag{4.8}$$

The equivalence of the two orientational orderings is further confirmed by introducing an external dual lattice field that breaks the symmetry and then reversing the dual transformation[11] to see what it looks like on the original lattice.

The physical picture of the ordered phase is as follows: The elementary vortex loops with topological dipole moments M float about freely. However, their M vectors point on average in a given direction, z, say. Thus the J currents circulate in the same sense on each loop. Since there is no preferred position, this corresponds to an external J_{ext} circulating around every plaquette in all planes perpendicular to z. Because of the cancellation between neighboring plaquettes, one has a J_{ext} circulating around the outer bonds of the system. If the system is taken as a giant cylinder for simplicity, one has a giant solenoidal current J_{ext} that produces an external dipole moment M_{ext} in the z direction.

Starting from the partition function

$$Z=\int\prod_{r,\mu}d\phi_\mu(r)$$

$$\times\sum_{J_\mu(r)}{}'\exp\left[-\frac{1}{2K_0}\sum_{r,\mu}[\epsilon_{\mu\nu\lambda}\Delta_\nu\phi_\lambda(r)]^2+2\pi i\sum_{r,\mu}[J_\mu(r)+J_\mu^{\text{ext}}(r)]\phi_\mu(r)\right] \tag{4.9}$$

and dualing backwards,[11] one gets

$$Z=\int\prod_i d\theta_i\sum_{m_\mu}\exp\left[-K_0\sum_{i,\mu}[\Delta_\mu\theta(i)-2\pi m_\mu(i)-2\pi b_\mu^{\text{ext}}(i)]^2\right]\ . \tag{4.10}$$

This is a Villain model in the original representation, b^{ext} is an external field orienting the phase angles. Using the asymptotic results for the Green's function

$$G=(\epsilon\Delta\epsilon\Delta)^{-1}\sim|\mathbf{r}-\mathbf{r}'|^{-1}\ ,$$

one has

$$b_\mu^{\text{ext}}=-\frac{1}{\epsilon_{\mu\nu\lambda}\Delta_\nu}J_\lambda^{\text{ext}}(\mathbf{r})$$

$$=\epsilon_{\mu\alpha\beta}\Delta_\alpha\frac{1}{\epsilon_{\mu\alpha\beta}\Delta_\alpha\epsilon_{\mu\nu\lambda}\Delta_\nu}$$

$$=\epsilon_{\mu\alpha\beta}\Delta_\alpha\sum_{r'}\frac{J_\beta^{\text{ext}}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\ . \tag{4.11}$$

Here b_μ^{ext} is related to the solenoidal J_μ^{ext} in the same way as a magnetic field is related to the current producing it.

Let the solenoid have length L and radius $a = L/2$ and $N = 1/a_0$ turns per unit length where a_0 is the lattice spacing: The field b^{ext} in the solenoid can be calculated by standard methods.[17] It serves to lock the spins in a common direction.

With an origin at the axis midpoint, the field in cylindrical coordinates, for large L, is

$$b_z^{\text{ext}}(\rho,z) \approx b_z^{\text{ext}}(0,z) \propto J^{\text{ext}} N \left[1 + \left[\frac{z}{L} \right]^2 + O\left[\left[\frac{\rho z}{L^2} \right]^2 \right] \right]$$

and $b_\rho^{\text{ext}}(\rho,z) \propto J^{\text{ext}} N \rho z / L^2$. The first could be gauged away by $\theta_i \to \theta_i + \int^{z_i} b_z^{\text{ext}}(0,z')dz'$, just like a Galilean transform $\mathbf{V}_s \to \mathbf{V}_s + \mathbf{u}$ that does not affect the essentials of superfluidity. This establishes Z as a reference direction. The second term locks the spins, with a ZX plane of rotation, say, to directions as in Fig. 4. The relative angles of tilt are $\Delta\theta \sim 1/L^2$ over a region $Z,\rho \le L$. As $L \to \infty$, the spins lock parallel over most of the volume.[18] (The volume $\sim L^3$.)

One can also add an orienting field term $-h \sum_l \cos\theta_l$ to (2.1) and repeat the standard dual transform argument.[18] The $h \cos\theta_l$ has the same periodicity in $2n_l\pi$ as (2.1) and hence carries similar vortex behavior in it. Thus one finds that h couples not just to small θ deviations, but also orients[18] vortex ring variables M.

From the above arguments, we conclude that the spin-ordering phase transition on the original lattice is equivalent to the vortex-loop-orientation transition on the dual lattice. This is in the spirit of a conjecture by Halperin.[12]

V. DISCUSSION

We have shown that the spin-locking ferromagnetic phase transition of the 3D XY model is associated with the orienting of elementary vortex rings, with topological currents circulating in the same direction. We identify the two transitions because (a) the (rotational) symmetry breaking in the same, (b) the free energy lowering is the same, (c) the spin-spin correlation function has long-range order produced spontaneously if and only if the topological dipole moments have long-range order. An external symmetry-breaking field in the dual representation also produces long-range order in the spin representation. (The estimated transition temperatures differ as the approximations made are slightly different.)

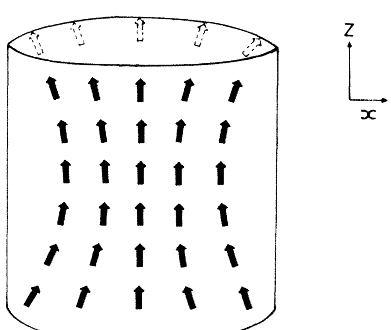

FIG. 4. Schematic picture of arrangement of spins equivalent to the oriented loop state on the dual lattice. See text.

The physical picture is as follows: above transition spins are disordered, and vortex loops, that are toroidal configurations of spins, tumble about freely. Below a critical temperature, the smallest vortex loops orient with the same sense of topological current circulation and with the dipole moments lined up. The small loops are otherwise free to float about in space perpendicular to the picked out direction (Fig. 1). Thus $\langle \cos\theta_i \rangle = 0$ means an orientation of both the rotation-free part (or the spin-wave deviations) and the spin toroids (which contain the periodicity). The toroids are loose extended object-spin configurations that may be hard to detect directly.

A Monte Carlo simulation to check this physical picture would be of considerable interest. One could start from (3.8), allow for the imaginary coupling constant, and look for $i\langle B \rangle \neq 0$, $\langle M \rangle \neq 0$.

A real-space renormalization-group calculation in terms of the topological excitations starting from (3.8) would also be of interest. One would expect a fixed point rather than a fixed line.[7] Monte Carlo investigations of the 3D XY model have of course been carried out.[12,19] However, these have been of the θ representations of (2.3),[19] and also the representations of (3.1) and a different dual representation.[12]

In 2D, when investigating the Kosterlitz-Thouless transition, a Coulomb gas picture corresponding to vortex points is found to be useful in Monte Carlo simulations since a single vortex point corresponds to a large configuration of spins.[20] The vortex unbinding transition is seen explicitly by depicting the configurations of $\pm$ charges. It would be interesting if vortex loop orientations could also be seen pictorially in simulations.

Monte Carlo studies of compact QED in four dimensions[21-26] have indicated that topological excitations (monopoles in the case of four dimensions) are essential for the phase transition[22] and that the monopole density is a suitable order parameter for studying the phase transition.[21] Of course, the nature of the phase transition in four dimensions may be quite different from three dimensions as the model is suspected to be at its marginal dimensions when $d = 4$.

Previous analyses of the role of topological excitations in the 3D XY model have been inspired by the 2D KT transition[7,8] whose novel feature is that it is not associated with long-range order. In the 2D KT transition, $\pm$ vortex pairs of disorder variables bind below a transition temperature T_{KT}. The size transition idea in 3D is to identify the vortex ring diameter with the analogue of the $\pm$ vortex separation. A sudden size shrinking of the randomly tumbling loops is postulated at a transition temperature, i.e., $T_c \leftrightarrow$ 'T_{KT}.' This approach has not yet been developed further by a renormalization-group calculation or even a mean-field truncation. As mentioned in the Introduction, the loop orientation transition discussed here is associated with the onset of long-range order, and is distinct from the size-change transition.

However there is some sense in which the spirit of the KT analogy is fulfilled. First, since the vortex loops no longer tumble about randomly stirring up the spins, some freezing out of disorder variables does indeed take place. Second, since the smallest loops orient first, at a given temperature below T_c, the lowering in free energy would be largest for these loops. Thus an associated and subsidiary size shrinkage along with the orientation transition cannot be ruled out. A Monte Carlo check of these questions would be useful.

Finally, we note that although we consider an XY-type model without a gauge field (charge $e^2=0$) here, the full lattice gauge model[4,5,12] may also be studied. Since the e^2 and T axes are dual to each other,[5] one expects an orientation of topological excitations at a critical coupling constant $e_c{}^2$. The explanation of the full e^2-T phase diagram, within the mean-field approach, is a broader problem and will be considered elsewhere. The topological viewpoint might also be applicable to the liquid crystal nematic to smectic A transition.[27]

An analytic approach of this kind where topological excitations are central, will also be useful in the context of periodic QED in four dimensions where Monte Carlo studies emphasizing topological excitations already exist.[21,22,25,26] It has also been observed[26] that topological excitations effectively divide the configuration space of the compact U(1) lattice gauge theory into separate sectors and that local Monte Carlo algorithms may have problems in moving from one sector to another and may produce misleading results. The utility of a theory where topological excitations are central is obvious in the above context.

Attempts to relate the bulk superfluid helium transition at T_λ to changes in vortex ring behavior were made much earlier.[28] But since the core size $\xi(T)$ diverges at T_λ, the physical definition and identity of the rings at transition is not entirely clear. A better physical realization of a 3D XY system, could be 3D $N\times N\times N$ arrays of Josephson junctions, where the lattice constant provides a core length scale. Such arrays, in 2D, are currently the subject of much interest.

ACKNOWLEDGMENTS

We thank P. Bhattacharyya for a stimulating question regarding Elitzur's theorem. S.R.S. thanks the Department of Atomic Energy, India for partial support by the Board of Research in Nuclear Studies. N.G. thanks the Council of Scientific and Industrial Research, India for support in the period when this work was started.

---

*Present address: Department of Physics, University of Poona, Pune 411 007, India

[1]R. Savit, Phys. Rev. B **17**, 1340 (1977).

[2]R. Savit, Rev. Mod Phys. **52**, 453 (1980).

[3]T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977); J. Cardy, *ibid.* **B170** [FS1], 369 (1980).

[4]M. Einhorn and R. Savit, Phys. Rev. D **17**, 2583 (1978).

[5]M. Peskin, Ann. Phys. (N.Y.) **113**, 122 (1978).

[6]J. José, L. Kadanoff, S. Kirkpatrick, and D. Nelson, Phys. Rev. B **16**, 1217 (1977).

[7]J. M. Kosterlitz and D. J. Thouless, J. Phys. C **6**, 1181 (1976); J. M. Kosterlitz, J. Phys. **7**, 1046 (1974).

[8]V. L. Berezinskii, Zh. Eksp. Teor. Fiz. **59**, 907 (1970) [Sov. Phys. JETP **32**, 493 (1971)]; **61**, 1144 (1971) [**34**, 610 (1972)].

[9]For a review see P. B. Wiegmann, in *Physics Review Vol. 2*, edited by I. M. Khalatnikov (Soviet Science Reviews) (Harwood Academic, New York, 1980), p. 41.

[10]P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); T. M. Rice, Phys. Rev. **140**, A1889 (1968); N. D. Mermin J. Math. Phys. **8**, 1061 (1967).

[11]N. Gupte and S. R. Shenoy, Phys. Rev. B **31**, 3150 (1985).

[12]C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. **47**, 1556 (1981); B. I. Halperin, in *Physics of Defects,* Les Houches Session XXXV, 1980, edited by R. Balian, M. Kléman, and J.-P. Poirier (North-Holland, Amsterdam, 1981).

[13]S. Elitzur, Phys. Rev. D **12**, 3978 (1978).

[14]H. Kleinert, Phys. Lett. **93A**, 86 (1982).

[15]N. Gupte, Phys. Rev. D **28**, 1515 (1983).

[16]R. Myerson, Phys. Rev. B **14**, 3203 (1977).

[17]J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1978), p. 205.

[18]On Fourier transforming $\exp(h\sum_l \cos\theta_l)$, extra Fourier labels $\{q_l\}$ appear, defined on the original lattice, as source terms in the constraint of (3.1). $k_\mu(i)$ can be split into a transverse and a longitudinal part, with the dual procedure going through for the former. The latter can be written as an electric field $E_\mu(r)$ related to $\{q_l\}$ by a solution of Poisson's equation. In addition to the B, M average, there is a new average over $\{q_l\}$. The weight factor for q_l is evaluated to first order in h. A mean-field analysis shows that $\langle M\rangle$ in (3.16) increases by $i\langle E\rangle\propto h\langle M\rangle$. Thus h orients $\langle M\rangle$.

[19]See, e.g., C. Ebner and D. Stroud, Phys. Rev. B **25**, 5711 (1983).

[20]J. Tobochnik and G. V. Chester, Phys. Rev. B **20**, 3781 (1979); Y. Saito and H. Muller-Krumbhaar, *ibid.* **23**, 308 (1981).

[21]J. S. Barber, R. E. Shrock, and R. Schrader, Phys. Lett. **152B**, 221 (1985).

[22]J. S. Barber, Phys. Lett. **147B**, 330 (1984).

[23]T. A. DeGrand and D. Toussaint, Phys. Rev. D **22**, 2478 (1980).

[24]J. Ranft, J. Kripfganz, and G. Ranft, Phys. Rev. D **28**, 360 (1983).

[25]R. Gupta, M. A. Novotny, and R. Cordery, Northeastern University Report No. NUB 2654 rev, 1985 (unpublished).

[26]V. Grosch, K. Jansen, J. Jersak, C. B. Lang, T. Neuhaus, and C. Rebbi, Phys. Lett. **162B**, 171 (1985).

[27]T. C. Lubensky, J. Chem. Phys. **80**, 31 (1983).

[28]F. Wiegel, Physica (Utrecht) **65**, 321 (1973).