# Renormalization of QCD with nonvanishing vacuum angle, the $\mathrm{U}_A(1)$ Ward-Takahashi identity, and decoupling

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Renormalization of QCD in the presence of a nonvanishing vacuum angle and with a general quark mass matrix is considered to all orders in a loop expansion. The generalized Landau gauge is assumed. By incorporating the anomalous $\mathrm{U}_A(1)$ Ward-Takahashi identity into our renormalization procedure, we establish (i) the nonrenormalizability of $\theta$ and (ii) the validity of the well-known formula giving the vacuum angle change induced through quark-mass-matrix diagonalization to all orders. We then show how the decoupling theorem applies to QCD with nonvanishing $\theta$. Renormalization of the QCD $\theta$ term in the context of larger theories is also briefly discussed.

## I. INTRODUCTION

Most physicists believe that quantum chromodynamics (QCD) is the correct underlying field theory of strong interactions. QCD is a renormalizable non-Abelian gauge theory[1] based on the SU(3) color group and exhibits asymptotic freedom[2] at the short-distance scale. The latter property makes the (suitably arranged) perturbation theory a useful tool for studying short-distance aspects of the theory[3] and the theoretical predictions indeed compare quite well with high-energy experimental data. At a long-distance scale, on the other hand, QCD is supposed to be highly nonperturbative with the effective coupling constant of order 1. Elementary fields of QCD, colored quarks, and gluons, are thus not directly relevant for the physical spectrum; they are permanently bound to form mesons and baryons, and possibly glueballs, as the only physically accessible asymptotic states. In the present work, various issues related to renormalization of the QCD vacuum angle[4] will be our primary concern.

The (unrenormalized) QCD action is given by

$$S^0_{\mathrm{GI}}=\int d^4x\left\{-\tfrac{1}{4}F^{\mu\nu a}(A)F^a_{\mu\nu}(A)+\bar{\psi}_f\left[i\gamma^\mu\left(\partial_\mu-igA^a_\mu\frac{\lambda^a}{2}\right)\delta_{ff'}-M_{ff'}\right]\psi_{f'}+\frac{\theta g^2}{32\pi^2}\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)\right\}, \tag{1.1}$$

where

$$F^a_{\mu\nu}(A)\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf_{abc}A^b_\mu A^c_\nu(a=1,2,\ldots,8)\ ,$$

$\lambda^a$ and $f_{abc}$ are the $3\times3$ SU(3)-color generator matrices and structure constants (normalized by $\mathrm{tr}(\lambda^a\lambda^b)=2\delta_{ab},[\lambda^a/2,\lambda^b/2]=if_{abc}\lambda^c/2)$, respectively, and ${}^*F^{\mu\nu a}(A)\equiv\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}F^a_{\lambda\delta}(A)$ (with $\epsilon^{0123}=1$). We follow the metric and $\gamma$-matrix convention of Ref. 5. The indices $f,f'$ $(=1,2,\ldots,N_f)$ denote quark flavors, and for the flavor-dependent quark mass matrix $M=(M_{ff'})$ we will assume the general nondiagonal form (which may involve $\gamma_5$)

$$M=\mathcal{M}\frac{1+\gamma_5}{2}+\mathcal{M}^\dagger\frac{1-\gamma_5}{2}\equiv M^{(S)}+i\gamma_5M^{(P)}\ . \tag{1.2}$$

The origin of this (current) quark-mass term lies in the theory of weak interactions. Note that the explicit form of $M$ depends on the quark field basis chosen in the flavor space.

It should be noted that all free parameters of the theory—gauge coupling $g$, quark mass matrix $M$, and the vacuum angle $\theta$—are intrinsically connected with symmetries or symmetry breakings characteristic of QCD. The coupling $g$ is of course related to the SU(3)-color gauge symmetry; this exact non-Abelian gauge symmetry plays a fundamental dynamical role in QCD by endowing the theory with both asymptotic freedom and (probably) quark confinement. The quark mass matrix $M$ introduces the explicit, but soft, symmetry-breaking terms for the global $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R\times\mathrm{U}_A(1)$ flavor symmetries (which are believed to be spontaneously broken via quark condensates in the QCD vacuum). The vacuum angle $\theta$, which determines relative phases for topologically distinct gauge-field contributions to the QCD partition function,[4,6] is related to the global $\mathrm{U}_A(1)$ via the Adler-Bell-Jackiw anomaly.[7] Namely, to describe the same theory, the $\mathrm{U}_A(1)$ phase change for quark-field variables must be accompanied by suitable translation in $\theta$. This implies that, even with $M\equiv0$, the global $\mathrm{U}_A(1)$ becomes anomalous since we are supposed to assign a certain *specific value* for $\theta$ to define our theory; this apparently solves the $\eta$ problem.[6]

The connection between the global $\mathrm{U}_A(1)$ and $\theta$ may also be characterized using the quark mass matrix. Given a quark mass matrix of the general form (1.2), one may wish to redefine quark fields suitably such that the resulting mass matrix, say

 

$$M'=\mathcal{M}'\frac{1+\gamma_5}{2}+\mathcal{M}'^{\dagger}\frac{1-\gamma_5}{2}\ ,$$

may become diagonal and $\gamma_5$-free (i.e., real and diagonal $\mathcal{M}'$). Because of the $U_A(1)$ anomaly, such quark-field redefinition in the flavor space should be accompanied by a suitable $\theta$ translation. There is a well-quoted formula for that:[8]

$$\theta'=\theta+\arg(\det\mathcal{M})\ . \tag{1.3}$$

In the quark-field basis where $M$ is diagonal and $\gamma_5$-free, a nonvanishing value for $\theta$ implies $P$ and $CP$ violations and the observed $CP$ invariance to high degrees of accuracy in strong-interaction physics puts a very stringent limit to its magnitude. Explaining "naturally" the vanishingly small QCD vacuum angle still remains as one of the most important theoretical problems (the strong $CP$ problem).[8]

In this paper we will consider anew renormalization of QCD and some related issues, in the presence of the vacuum angle $\theta$ (as an *a priori* arbitrary dimensionless free parameter of the theory) and for a general quark mass matrix of the form (1.2). Does the vacuum angle $\theta$ get renormalized just like any other free parameters in the theory? Is the mass matrix $\mathcal{M}$ in Eq. (1.3) a bare one or a renormalized one? Can we expect a simple formula such as Eq. (1.3) to be valid to all orders in a fully renormalized theory? These are some of the questions to which we shall provide definite answers. We hope that our analysis enhances our understanding on the physical role of the vacuum angle $\theta$ when higher-order quantum effects are taken into account, especially in connection with the strong $CP$ problem. In any case, it should be important to have renormalization structure of any given field theory—let alone QCD—fully understood; only then one can formulate renormalization-group equations correctly.

We shall assume the loop expansion. Note that, unlike strict weak-coupling perturbation theory, loop expansion can accommodate in principle contributions due to topologically nontrivial gauge-field configurations such as instantons. Another point to mention is the gauge choice. Recently some authors[9] have discussed the $\theta$-term (non-)renormalization on the basis of the background-gauge background-field method.[10] But, their discussions depend rather crucially on the special characters of the background-gauge background-field method. We shall here work in more conventional gauges—the generalized Landau (or covariant) gauges.

Throughout our discussions, strong emphasis will be laid on the symmetry aspects of the theory, viz., how softly broken or/and anomalous global symmetries of QCD mentioned above, together with SU(3)-color gauge symmetry, can be implemented manifestly through renormalization. This naturally leads to the simplest possible multiplicative renormalization scheme of QCD, and hence also to simplest renormalization-group equations. In our text it will be called the $(M,\theta)$-independent renormalization scheme, essentially an adaptation of Weinberg's zero-mass renormalization scheme[11] to our purpose. It turns out that the global $SU(N_f)_L\times SU(N_f)_R$-symmetry considerations, in the form of appropriate Ward-Takahashi (WT) identities, restrict counterterms to the quark mass matrix to a very simple multiplicative structure in our renormalization scheme. Also, a careful use of the $U_A(1)$ WT identity allows us to fix renormalization of the $\theta$-dependent term in the action; the vacuum angle $\theta$ is unrenormalized if one uses the properly *renormalized* Pontryagin density to define the $\theta$ term in the action. Furthermore, on the basis of the renormalized $U_A(1)$ WT identity, we can establish that Eq. (1.3), with the mass matrix $\mathcal{M}$ representing the renormalized values, is in fact valid to all orders in the $(M,\theta)$-independent renormalization scheme. These will be discussed through Secs. II–IV.

In Sec. V, we consider how the decoupling theorem[12] works in the context of QCD with nonvanishing $\theta$. The decoupling theorem in renormalizable field theories states that leading effects of virtual heavy particles (or, a bit loosely, interactions confined to very small distance scales) on low-energy physics of light particles are indistinguishable from renormalization counterterms (involving light particle fields). Thus, virtual heavy-particle effects are suppressed (at low energy) by power, other than possible readjustments of free parameters of the theories. In most past works on decoupling, however, people have not paid much attention to the $\theta$ term in non-Abelian gauge theories. It is clearly important to know (in connection with the strong $CP$ problem especially) what the low-energy effective vacuum angle would be in comparison to the high-energy value of $\theta$ defined in the full theory. We consider in some detail the case when the full theory includes certain quarks with very large Lagrangian mass terms [but of the general form (1.2)]. The low-energy effective vacuum angle is determined by tracking down symmetry constraints carefully.

Section VI is devoted to the summary and discussions of our work. In the future, we hope to study various issues discussed in this paper in the context of "larger" theories [such as the standard $SU(2)\times U(1)\times SU(3)$ model] where the quark mass matrix is generated as a result of spontaneous symmetry breaking. Of course, as long as QCD constitutes a part of low-energy effective field theory from such larger theory, information obtained in this paper would not become entirely irrelevant. We shall briefly discuss some new problems arising in these cases in Sec. VI.

## II. THE $\theta$ TERM IN LOOP EXPANSION AND SYMMETRIES OF THE QCD ACTION

The Feynman path-integral language will be used to study quantized gauge theories. Here the ($c$-number) action plays the fundamental role. The gauge-invariant action $S^0_{\rm GI}$ given in Eq. (1.1) is singular and, as is well known, the corresponding nonsingular action can be found by adding a suitable gauge-fixing term and appropriate Faddeev-Popov ghost terms to it.[13] We shall assume the generalized Landau gauge characterized by a gauge-fixing parameter $\alpha$. Then the full unrenormalized action for the theory, from which (unrenormalized) Feynman rules may be read off, can be written as

$$S^0=\int d^4x\left\{-\tfrac{1}{4}F^{\mu\nu a}(A)F^a_{\mu\nu}(A)+\bar{\psi}_f\left[i\gamma^\mu\left[\partial_\mu-igA^a_\mu\frac{\lambda^a}{2}\right]\delta_{ff'}-M_{ff'}\right]\psi_{f'}-\frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2\right.$$
$$\left.-\partial^\mu\bar{\chi}^a(\partial_\mu\delta_{ac}+gf_{abc}A^b_\mu)\chi^c+h\frac{\theta g^2}{32\pi^2}\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)\right\}, \tag{2.1}$$

where $\chi^a,\bar{\chi}^a$ are Faddeev-Popov ghost fields. Here, $(\psi,\bar{\psi},\chi^a,\bar{\chi}^a)$ belong to elements of the Grassmann algebra and $A^a_\mu$ are ordinary functions. At the front of the $\theta$ term we have inserted the loop-expansion parameter $h$, which may be eventually set to 1 (after making the expansion). The reason for providing the $\theta$ term with an extra loop-expansion parameter is to make the $U_A(1)$ WT identity look natural in order-by-order loop expansion.[14] (Remember that sources of anomalies are one-loop spinor diagrams.)

Before one starts to consider higher-order loop effects with the action (2.1), one must face the following dilemma: ${}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$ corresponds to a total derivative and thus, in ordinary perturbation theory, it is difficult to distinguish the quantity

$$\int d^4x\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$$

from zero. This of course reflects the fact that

$$\int d^4x\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$$

may be nonzero only for Yang-Mills potentials of nontrivial topological character. Does this imply that one can study renormalization of the $\theta$ term (or $\theta$-dependent loop corrections) only by explicitly considering quantization of the theory around a certain, topologically nontrivial, background classical solution?[15] That we are very much afraid of—we want to apply the power-counting method which has long been a standard tool for discussing renormalization of quantum field theories developed around the trival classical solution. It is now generally held that the renormalization structure of a field theory does not depend on the classical solution (topologically trivial or not) which has been chosen to perturb around. The most well-known example of this is the renormalization of spontaneously broken gauge theories.[1] So this should not be a real problem. Our problem is only that, if the theory is analyzed with the perturbation theory developed around the trivial vacuum solution [i.e., $A^a_\mu(x)=0$] in mind, no distinction between $\int d^4x\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$ and zero can be made and thus it will be virtually impossible to determine the $\theta$-renormalization structure. Below we shall describe a simple way to overcome this fix.

The idea is very simple. We may study the theory with the $\theta$ term in the action replaced by

$$\int d^4x\,h\frac{g^2}{32\pi^2}\Theta(x)\,{}^*F^{\mu\nu a}(A(x))F^a_{\mu\nu}(A(x)), \tag{2.2}$$

where $\Theta(x)$ is an arbitrary (supposedly vanishing at space-time infinity) externally given function. The theory may be renormalized in the presence of this arbitrary externally given (one may say background) function $\Theta(x)$, and only at final stage we may let ($\epsilon$ is an infinitesimal positive number)

$$\Theta(x)\to\theta e^{-\epsilon(|x|+|y|+|z|+|t|)} \tag{2.3}$$

to make the connection with the physical vacuum angle. Note that this procedure is gauge invariant and, in the Euclidean path-integral description, will retain the topological character of the term

$$\int d^4x\,h\frac{\theta^2}{32\pi^2}\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$$

intact when $A^a_\mu(x)$ describe topologically nontrivial field configurations. With the expression (2.2) in the action, its presence cannot be missed even if we use the loop expansion developed around a trivial vacuum solution. This procedure should be effective in finding renormalization counterterms which depend on $\theta$. But how about possible subtractive renormalization counterterms proportional to

$$\int d^4x\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A),$$

which might be generated as purely higher-order loop effects even if one starts with $\theta=0$? [For the moment, disregard renormalization of Yang-Mills field strengths. See Eq. (2.7).] This can be quite a delicate problem, and we here elect to resolve the case by invoking the symmetry principle: viz., with $\Theta(x)=0$ and $M=0$, $P$ (the parity) should correspond to an exact symmetry of quantum chromodynamics. This will be sufficient to guarantee no subtractive renormalization counterterm proportional to $\int d^4x\,{}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$ from higher loops.

The fact that ${}^*F^{\mu\nu a}(A)F^a_{\mu\nu}(A)$ corresponds to a dimension-four term will not put us in the position of altogether reconsidering the renormalization procedure for $\theta\neq0$. Upon integrating by parts, the expression (2.2) is identical to

$$-\int d^4x\,h\frac{g^2}{32\pi^2}[\partial_\mu\Theta(x)]K^\mu(A(x)), \tag{2.4}$$

where

$$K^\mu(A(x))=2\epsilon^{\mu\nu\lambda\delta}(A^a_\nu\partial_\lambda A^a_\delta+\tfrac{1}{3}gf_{abc}A^a_\nu A^b_\lambda A^c_\delta). \tag{2.5}$$

In other words, the term (2.2) can be rewritten in the form of a superrenormalizable interaction involving the dimension-three function, the Chern-Simons form $K^\mu(A)$. Gauge invariance is manifest in the form (2.2) while the form (2.4) stresses that it can be treated as a soft term in (ultraviolet) power counting.[16] Renormalization counterterms may not involve more than four powers of $R_\mu(x)\equiv\partial_\mu\Theta(x)$. This simple observation tells us that the $\theta$ term will have no effect on the coupling-constant renormalization or other logarithmically divergent renormalization counterterms of the theory. Hence the QCD $\beta$ function can be taken to be independent of $\theta$; assuming $N_f \leq 16$, QCD is asymptotically free for all $\theta$. (But we do not exclude possible $\theta$ dependences in the running coupling constant from finite-renormalization effects which may include nonperturbative contributions.)

Possible counterterms to the bare action are constrained by symmetries of the theory. We may here first state the correct form of the renormalized QCD action [in the presence of an externally given function $\Theta(x)$]. It reads

$$S=\int d^4x\left\{-\tfrac{1}{4}Z\mathcal{F}^{\mu\nu a}(A)\mathcal{F}^a_{\mu\nu}(A)-\frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2-Y\partial_\mu\bar{\chi}^a\mathcal{D}^{ac}_\mu\chi^c+Z_F\bar{\psi}_f\left[i\gamma^\mu\left(\partial_\mu-ig_BZ^{1/2}A^a_\mu\frac{\lambda^a}{2}\right)\delta_{ff'}-Z_MM_{ff'}\right]\psi_{f'}\right.$$
$$\left.+\Theta(x)h\left[\frac{g_B{}^2Z}{32\pi^2}{}^*\mathcal{F}^{\mu\nu a}\mathcal{F}^a_{\mu\nu}(x)-Z_F(\delta X)\partial^\mu(\bar{\psi}_f\gamma_\mu\gamma_5\psi_f)(x)\right]\right\}, \tag{2.6}$$

where $A^a_\mu$, $\psi$, $\bar{\psi}$, $\chi$, and $\bar{\chi}$ now denote renormalized fields (obtained through suitable rescaling from bare fields) and we have defined

$$\begin{aligned}\mathcal{F}^a_{\mu\nu}(A)&\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g_BZ^{1/2}f_{abc}A^b_\mu A^c_\nu ,\\ \mathcal{D}^{ac}_\mu&\equiv\partial_\mu\delta_{ac}+g_BZ^{1/2}f_{abc}A^b_\mu .\end{aligned} \tag{2.7}$$

In Eq. (2.6), $M=(M_{ff'})$ represents the renormalized (and hence finite, by definition) mass matrix and $(Z,Y,g_B,Z_F,Z_M,\delta X)$ are appropriate renormalization constants which depend on specific regularization and normalization conditions. [A set of convenient normalization conditions are described in Eqs. (4.23a)–(4.23e)]. As will become clear as we proceed, the counterterm $\delta X$ is related to normalization of the $\mathrm{U}_A(1)$ current [via Eqs. (3.3) and (4.23e)] and the piece proportional to $\delta X$ in Eq. (2.6) is necessary only because we are here considering the theory in the presence of an arbitrary externally given function $\Theta(x)$. Aside from this term proportional to $\delta X$, the renormalized action (2.6) is precisely that obtained from the unrenormalized action (2.1) through usual rescaling of fields $A^a_\mu\to Z^{1/2}A^a_\mu$, $\psi\to Z_F{}^{1/2}\psi$, $\bar{\psi}\to Z_F{}^{1/2}\bar{\psi}$, $\chi\to Y^{1/2}\chi$, $\bar{\chi}\to Y^{1/2}\bar{\chi}$, combined with the replacements $g\to g_B$ and $M\to Z_MM$. Note that the full $\Theta(x)$-dependent terms in Eq. (2.6) can be also written in a superrenormalizable form, i.e., such as

$$-\int d^4x\,R_\mu(x)h\left[\frac{g_B{}^2Z}{32\pi^2}\mathcal{K}^\mu(A(x))-Z_F(\delta X)\bar{\psi}_f\gamma^\mu\gamma_5\psi_f(x)\right] \qquad [R_\mu(x)\equiv\partial_\mu\Theta(x)], \tag{2.8}$$

where

$$\mathcal{K}^\mu(A)\equiv 2\epsilon^{\mu\nu\lambda\delta}(A^a_\nu\partial_\lambda A^a_\delta+\tfrac{1}{3}g_BZ^{1/2}f_{abc}A^a_\nu A^b_\lambda A^c_\delta) . \tag{2.9}$$

There are some special features with our renormalized action (2.6). Note that the bare-quark mass matrix is simply written as $Z_MM$, with a *single multipicative constant* $Z_M$ renormalizing the full quark mass *matrix* $M$ of the general form (1.2). At the same time all renormalization constants $(Z,Y,g_B,Z_F,Z_M,\delta X)$ will be assumed to be independent of $M$ and $\theta$. The idea behind it is the same with the zero-mass renormalization scheme of Weinberg;[11] i.e., counterterms suitably defined with $M=\Theta(x)=0$ may be used to renormalize the theory with general nonzero values for $M$ and $\Theta(x)$. The $(M,\theta)$-independent renormalization scheme is especially convenient since we can here maximally utilize the softness of given global-symmetry-breaking terms in renormalizing QCD. If one wishes, connection to other renormalization schemes can be made by performing suitable finite renormalization from ours. We shall discuss the meaning of the $\theta$-term renormalization implied by the renormalized action (2.6) a short while later.

The form of the renormalized QCD action shown in Eq. (2.6) will be justified in Sec. IV, to all orders in loop expansion. Here, just accepting the form (2.6), we will identify exact or approximate symmetries of this ($c$-number) QCD action. First, we note that, after adding the gauge-fixing term and the Faddeev-Popov ghost term, the SU(3)-color gauge symmetry takes the form of the Becchi-Rouet-Stora (BRS) symmetry;[17] viz., we have $\delta S=0$, under the variation

$$\begin{aligned}\delta A^a_\mu(x)&=\mathcal{D}^{ab}_\mu\chi^b(x)\delta\omega ,\\ \delta\chi^a(x)&=\tfrac{1}{2}g_BZ^{1/2}f_{abc}\chi^b(x)\chi^c(x)\delta\omega ,\\ \delta\bar{\chi}^a(x)&=-\frac{1}{\alpha}Y^{-1}\partial\cdot A^a(x)\delta\omega ,\\ \delta\psi(x)&=ig_BZ^{1/2}\frac{\lambda^a}{2}\chi^a(x)\delta\omega\psi(x) ,\\ \delta\bar{\psi}(x)&=-ig_BZ^{1/2}\bar{\psi}(x)\frac{\lambda^a}{2}\chi^a(x)\delta\omega ,\end{aligned} \tag{2.10}$$

where $\delta\omega$ is a space-time-independent Grassmann number. In addition, we have following approximate global $\mathrm{SU}(N_f)_L\times\mathrm{SU}(N_f)_R\times\mathrm{U}_A(1)$ flavor symmetries:[18]

$$\begin{aligned}\delta S=\int d^4x(&-iZ_F\bar{\psi}[Z_MM,\boldsymbol{\alpha}\cdot\mathbf{T}]\psi\\ &+iZ_F\bar{\psi}\gamma_5\{Z_MM,\boldsymbol{\beta}\cdot\mathbf{T}\}\psi\\ &+2i\beta_0Z_F\bar{\psi}\gamma_5Z_MM\psi)\end{aligned} \tag{2.11}$$

under the infinitesimal variations

$$\begin{aligned}\delta\psi&=i\boldsymbol{\alpha}\cdot\mathbf{T}\psi-i\gamma_5\boldsymbol{\beta}\cdot\mathbf{T}\psi-i\beta_0\gamma_5\psi ,\\ \delta\bar{\psi}&=-i\bar{\psi}\boldsymbol{\alpha}\cdot\mathbf{T}-i\bar{\psi}\boldsymbol{\beta}\cdot\mathbf{T}\gamma_5-i\bar{\psi}\gamma_5\beta_0 .\end{aligned} \tag{2.12}$$

In Eq. (2.12), $(\boldsymbol{\alpha},\boldsymbol{\beta})$ are arbitrary infinitesimal $(N_f{}^2-1)$-component real vectors, T denote a set of $(N_f{}^2-1)$ traceless (and Hermitian) $N_f \times N_f$ flavor matrices, and the infinitesimal real constant $\beta_0$ generates $U_A(1)$. With $M$ set to zero, all of these global transformations correspond to exact symmetries of the action. Implications of these exact or approximate symmetries of the c-number action on Green's functions will be considered in the next section.

If we let $\Theta(x)$ approach a constant $\theta$ [in the sense of Eq. (2.3)], the $\Theta(x)$-dependent part of $S$ will turn into

$$\int d^4x\, \theta h \frac{g_B{}^2 Z}{32\pi^2} {}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x) = \int d^4x\, \theta h \frac{g_B{}^2 Z}{32\pi^2} \partial_\mu \mathcal{K}^\mu(A(x)) . \quad (2.13)$$

We have here made the assumption

$$\int d^4x\, \partial^\mu(\bar\psi_f \gamma^\mu \gamma_5 \psi_f) \equiv 0 ,$$

which should be reasonable since, unlike $\mathcal{K}(A)$, $\bar\psi_f \gamma^\mu\gamma_5\psi_f$ is manifestly gauge invariant. It is possible to "understand" why the $\theta$ term gets renormalized in the way shown in Eq. (2.13); in Euclidean space-time, $(g_B{}^2 Z/32\pi^2)\,{}^*\mathcal{F}^{\mu\nu a}\mathcal{F}^a_{\mu\nu}(x)$ may be actually identified with the renormalized Pontryagin density. This can be argued as follows. From the form of the renormalized covariant derivative $\mathcal{D}^{ac}_\mu = \partial_\mu \delta_{ac} + g_B Z^{1/2} f_{abc} A^b_\mu$ [see Eq. (2.7)], one can conclude that the so-called pure gauge states in renormalized theory correspond to classical configurations

$$A^a_\mu(x)\frac{\lambda^a}{2} = (g_B Z^{1/2})^{-1} U^{-1}(x) \frac{1}{i} \partial_\mu U(x) ,$$

with general SU(3)-color gauge transformation matrix $U(x)$. Then, under the restriction that gauge fields approach pure gauges at space-time infinity, one may naturally identify the Pontryagin index (in renormalized theory) with

$$\mathcal{W} = \int d^4x \frac{g_B{}^2 Z}{32\pi^2} {}^*F^{\mu\nu a}\mathcal{F}^a_{\mu\nu}(x) .$$

The index $\mathcal{W}$ is supposed to be a topological invariant and, since $\Pi_3(G) = Z$ for a compact non-Abelian group $G$, it may assume only integer values. Needless to say, it is then difficult to imagine that the quantity $\mathcal{W}$, quantized by a topological reason, is subject to any, infinite or finite, renormalization corrections.[19] This explains the expression (2.13). Furthermore, according to this argument, it is very reasonable to consider the parameter $\theta$, normalized as shown in Eq. (2.13) and without any further finite renormalization, as the true physical vacuum angle (i.e., that actually associated with relative phases between field configurations of distinct topological character under so-called "large" gauge transformations[4]).

Unfortunately, the above topological consideration to settle the $\theta$-term renormalization is difficult to implement in the standard renormalization theory framework based on order-by-order loop expansion. Many will regard the above topology-based reasoning just as a wishful thinking without real justification. In this paper a direct proof in order-by-order loop expansion will be given, taking full advantage of the anomalous $U_A(1)$ WT identity which is incorporated into the renormalization procedure. The fact that the $U_A(1)$ WT identity can be used to fix renormalization of the Pontryagin density should hardly come as a surprise; the $U_A(1)$ anomaly is related to the Pontryagin index through the Atiyah-Singer index theorem.[20]

## III. SYMMETRIES OF THE QCD GENERATING FUNCTIONALS (WT IDENTITIES)

In quantum field theories, various Green's functions with an arbitrary number of external legs can be represented collectively by appropriate generating functionals. In this section we are interested in knowing what kind of relationships exist for Green's functions, as consequences of the exact or approximate symmetries of the QCD action $S$. These are simply expressed through appropriate WT identities for generating functionals. They are very important for renormalization since these symmetry restrictions should be observed by a good renormalization procedure.

We may first define QCD generating functionals formally. The connected Green's function generating functional, $W(J^a_\mu, \eta, \bar\eta, \xi^a, \bar\xi^a, T^{\mu a}, L^a U, \bar U, \mathcal{J}_{5\mu}, \mathcal{J}_5; M, \Theta)$, is defined by the relation

$$\exp\left[\frac{i}{h} W\right] = N \int [dA^a_\mu][d\chi^a][d\bar\chi^a][d\psi][d\bar\psi] \exp\left[\frac{i}{h} S^*\right] , \quad (3.1)$$

with

$$S^* = S + \int d^4x \Big[ A^{\mu a}(x) J^a_\mu(x) + \bar\eta_f(x)\psi_f(x) + \bar\psi_f(x)\eta_f(x) + \bar\xi^a(x)\chi^a(x) + \bar\chi^a(x)\xi^a(x)$$
$$+ Y T^{\mu a}(x)\mathcal{D}^{ab}_\mu \chi^b(x) + Y\tfrac{1}{2} g_B Z^{1/2} L^a(x) f_{abc}\chi^b(x)\chi^c(x) - iY g_B Z^{1/2} \bar U_f(x)\frac{\lambda^a}{2}\chi^a(x)\psi_f(x)$$
$$- iY g_B Z^{1/2} \bar\psi_f(x) \frac{\lambda^a}{2}\chi^a(x) U_f(x) + Z_A Z_F \mathcal{J}_{5\mu}(x)\bar\psi_f(x)\gamma^\mu\gamma_5\psi_f(x)$$
$$+ Z_F Z_M \mathcal{J}_5(x)\bar\psi_f(x) i\gamma_5 M_{ff'}\psi_{f'}(x) \Big] . \quad (3.2)$$

Here, $N$ is the normalization constant, $S$ represents the renormalized QCD action given in Eq. (2.6), and $Z_A$ is not an independent renormalization counterterm but related to $\delta X$ [see Eq. (2.6)] by

$$Z_A = 1 - 2N_f h\delta X . \tag{3.3}$$

Note that, aside from external sources $(J^a, \eta_f, \bar{\eta}_f, \xi^a, \bar{\xi}^a)$ for elementary fields of the theory, we have also introduced additional source functions $(T^{\mu a}, L^a, U_f, \bar{U}_f, \mathscr{J}_{5\mu}, \mathscr{J}_5)$ associated with suitable composite fields to facilitate our WT identity analysis. Among the source functions, $[\eta_f(x), \bar{\eta}_f(x), \xi^a(x), \bar{\xi}^a(x), T^{\mu a}(x)]$ belong to elements of the Grassmann algebra and the rest are ordinary functions.

In studying renormalization structure, of more direct relevance is the generating functional for one-particle-irreducible (i.e., proper) vertex functions. Let us represent it by $\Gamma(\tilde{A}^a_\mu, \tilde{\psi}, \tilde{\bar{\psi}}, \tilde{\chi}^a, \tilde{\bar{\chi}}^a, T^{\mu a}, L^a, U, \bar{U}, \mathscr{J}_{5\mu}, \mathscr{J}_5; M, \Theta)$. The proper-vertex generating functional $\Gamma$ can be obtained from $W$ through the usual Legendre transform,[13] i.e.,

$$\Gamma = W - \int d^4x \,(J^a_\mu \tilde{A}^{\mu a} + \bar{\eta}_f \tilde{\psi}_f + \tilde{\bar{\psi}}\eta_f + \bar{\xi}^a \tilde{\chi}^a + \tilde{\bar{\chi}}^a \xi^a) , \tag{3.4}$$

where classical fields $(\tilde{A}^a_\mu, \tilde{\psi}_f, \tilde{\bar{\psi}}_f, \tilde{\chi}^a, \tilde{\bar{\chi}}^a)$ are defined by

$$\begin{aligned} &\tilde{A}^a_\mu(x) = \delta W/\delta J^{\mu a}(x), \quad \psi_f(x) = \delta W/\delta \bar{\eta}_f(x) , \\ &\tilde{\bar{\psi}}_f(x) = -\delta W/\delta \eta_f(x) , \\ &\tilde{\chi}^a(x) = \delta W/\delta \bar{\xi}^a(x), \quad \tilde{\bar{\chi}}^a(x) = -\delta W/\delta \xi^a(x) . \end{aligned} \tag{3.5}$$

To avoid cumbersome notations, we shall henceforth drop the tilde over classical fields (as arguments of the proper-vertex generating functional); viz., we shall just write $\Gamma(A^a_\mu, \psi, \bar{\psi}, \chi^a, \ldots)$ for $\Gamma(\tilde{A}^a_\mu, \tilde{\psi}, \tilde{\bar{\psi}}, \tilde{\chi}^a, \ldots)$. Also note that the WT identities in gauge theories take simpler forms in terms of the modified proper-vertex generating functional:[13]

$$\bar{\Gamma} \equiv \Gamma + \frac{1}{2\alpha} \int d^4x (\partial^\mu A^a_\mu)^2 . \tag{3.6}$$

Other than the two-point function of gauge fields, $\bar{\Gamma}$ serves the same role as $\Gamma$.

For our purpose it is convenient to regard the function $\Theta(x)$ and the mass matrix $M$ also as kinds of external sources; we may consider (functional) derivatives of generating functionals with respect to $\Theta(x)$ or matrix elements of $M^{(S)}$ and $M^{(P)}$ [see Eq. (1.2)]. Usefulness of mass derivatives (or expansion in powers of $M$) with generating functionals is closely tied up with the fact that all the renormalization counterterms in our case are taken to be $M$ independent. Clearly, the relation such as

$$\int d^4x \frac{\delta W}{\delta \mathscr{J}_5(x)} = M^{(P)}_{ff'} \frac{\delta W}{\delta M^{(S)}_{ff'}} - M^{(S)}_{ff'} \frac{\delta W}{\delta M^{(P)}_{ff'}} , \tag{3.7}$$

or, using the generating functional $\bar{\Gamma}$,

$$\int d^4x \frac{\delta \bar{\Gamma}}{\delta \mathscr{J}_5(x)} = M^{(P)}_{ff'} \frac{\delta \bar{\Gamma}}{\delta M^{(S)}_{ff'}} - M^{(S)}_{ff'} \frac{\delta \bar{\Gamma}}{\delta M^{(P)}_{ff'}} \tag{3.8}$$

will be valid only when all renormalization constants are chosen to be $M$ independent. Also, at this point, we want to make it clear that we are interested only up to linear terms in $\mathscr{J}_{5\mu}(x)$ or $\mathscr{J}_5(x)$ with our generating functionals. Hence, possible problems (e.g., renormalization) associated with two or more insertions of the externally introduced vertex $\bar{\psi}_f \gamma^\mu \gamma_5 \psi_f(x)$ or $\bar{\psi}_f i\gamma_5 M_{ff'} \psi_{f'}(x)$ shall be simply ignored. With regards to all the other sources [including the mass matrix $M$ and $\Theta(x)$], no such restriction is imposed; we consider their arbitrary powers.

The generating functionals above should be interpreted as those defined in a suitably *regularized theory.* If we characterize regularization by a certain large parameter $\Lambda$ (e.g., $1/4-n$ in the dimensional regularization scheme or large mass parameters appearing in the Pauli-Villars regularization), the renormalization constants $(Z, Y, g_B, Z_F, Z_M, \delta X)$ will be appropriately chosen functions of $\Lambda$. Symbolically, removing regularization from the theory may be represented by the limit $\Lambda \to \infty$ and this is the topic of the next section. We shall not specify a particular regularization scheme; for our purpose, it suffices to specify conditions which a good *regularization* of QCD should meet. First, in the limit $\Lambda \to \infty$, all ambiguities in the theory should be restricted to local Lagrangian counterterms of dimension not larger than four. Second, the generating functionals defined with the help of a good regularization should realize various symmetries of QCD in a proper way—it must obey various WT identities given below, at least for sufficiently large $\Lambda$ with discrepancies vanishing like powers of $1/\Lambda$ in each order of loop expansion. (In this section, however, we do not assume finiteness of our generating functional themselves in the limit $\Lambda \to \infty$.) Note that most commonly used regularization procedures can be made consistent with these conditions by incorporating, if necessary, suitable "improvement" terms[21] in each loop order.

The WT identities for suitably regularized QCD generating functionals can be inferred by considering change of path integration variables corresponding to infinitesimal transformations (2.10) and (2.12) with our path integral representation (3.1). The procedure is quite straightforward when there is no anomaly, viz., for the BRS symmetry and the global $SU(N_f)_L \times SU(N_f)_R$ symmetries. In terms of the modified proper-vertex generating functional $\bar{\Gamma}$, the BRS WT identity reads[22]

$$\int d^4x \left[ -\frac{\delta\bar{\Gamma}}{\delta A^{\mu a}(x)} \frac{\delta\bar{\Gamma}}{\delta T^a_\mu(x)} + \frac{\delta\bar{\Gamma}}{\delta \chi^a(x)} \frac{\delta\bar{\Gamma}}{\delta L^a(x)} + \frac{\delta\bar{\Gamma}}{\delta \psi_f(x)} \frac{\delta\bar{\Gamma}}{\delta \bar{U}_f(x)} + \frac{\delta\bar{\Gamma}}{\delta U_f(x)} \frac{\delta\bar{\Gamma}}{\delta \bar{\psi}_f(x)} \right] = 0 . \tag{3.9}$$

The ghost equation of motion

$$\partial_\mu \frac{\delta\bar{\Gamma}}{\delta T^a_\mu(x)} - \frac{\delta\bar{\Gamma}}{\delta \bar{\chi}^a(x)} = 0 \tag{3.10}$$

is also useful in studying renormalization. We may express the WT identities associated with the global $SU(N_f)_L \times SU(N_f)_R$ symmetries in terms of two separate relations. The WT identity associated with the diagonal subgroup $SU(N_f)_V$ of $SU(N_f)_L \times SU(N_f)_R$ [i.e., related to the $\alpha$ variation in Eq. (2.12)] reads

$$\int d^4x \left[ \frac{\delta\overline{\Gamma}}{\delta\psi_f(x)}(T^l)_{ff'}\psi_{f'}(x)+\overline{\psi}_f(x)(T^l)_{ff'}\frac{\delta\overline{\Gamma}}{\delta\overline{\psi}_{f'}(x)}+\overline{U}_f(x)(T^l)_{ff'}\frac{\delta\overline{\Gamma}}{\delta\overline{U}_{f'}(x)}-\frac{\delta\overline{\Gamma}}{\delta U_f(x)}(T^l)_{ff'}U_{f'}(x)\right] + ([M^{(S)},T^l])_{ff'}\frac{\delta\overline{\Gamma}}{\delta M^{(S)}_{ff'}}+([M^{(P)},T^l])_{ff'}\frac{\delta\overline{\Gamma}}{\delta M^{(P)}_{ff'}}=0 , \quad (3.11)$$

where $l=1,2,\ldots,N_f^2-1$. On the other hand, the WT identity associated with the group actions along the coset space $SU(N_f)\times SU(N_f)/SU(N_f)_V$ [i.e., related to the $\beta$ variation in Eq. (2.12)] reads

$$\int d^4x \left[ -\frac{\delta\overline{\Gamma}}{\delta\psi_f(x)}\gamma_5(T^l)_{ff'}\psi_{f'}(x)+\overline{\psi}_f(x)\gamma_5(T^l)_{ff'}\frac{\delta\overline{\Gamma}}{\delta\overline{\psi}_{f'}(x)}-\overline{U}_f(x)\gamma_5(T^l)_{ff'}\frac{\delta\overline{\Gamma}}{\delta\overline{U}_{f'}(x)} -\frac{\delta\overline{\Gamma}}{\delta U_f(x)}\gamma_5(T^l)_{ff'}U_{f'}(x)\right] -i(\{M^{(P)},T^l\})_{ff'}\frac{\delta\overline{\Gamma}}{\delta M^{(S)}_{ff'}}+i(\{M^{(S)},T^l\})_{ff'}\frac{\delta\overline{\Gamma}}{\delta M^{(P)}_{ff'}}=0 . \quad (3.12)$$

Note that, to have Eqs. (3.11) and (3.12), the condition that all renormalization constants be $M$ independent is absolutely necessary.

In Eq. (2.12) the $\beta_0$ variation describes global infinitesimal $U_A(1)$ transformation, and it suffers from an anomaly in the quantum theory.[7] As for this anomalous $U_A(1)$, it suits our purpose better to have at hand the WT identity corresponding to a general *local* $U_A(1)$ variation:

$$\delta\psi(x)=-i\beta_0(x)\gamma_5\psi(x),\quad \delta\overline{\psi}(x)=-i\overline{\psi}(x)\gamma_5\beta_0(x) . \quad (3.13)$$

The anomaly arises because the Fermi field integration measure defined in the presence of a background-gauge field does not remain invariant (i.e., the Jacobian is not equal to 1)[23] under the change of variables corresponding to Eq. (3.13). Explicitly, for arbitrary given functions $A^a_\mu(x)$ [and $\eta_f(x),\overline{\eta}_f(x)$], take the quantity

$$N'\int[d\psi][d\overline{\psi}]\exp\left[\frac{i}{h}\int d^4x'\left\{Z_F\overline{\psi}_f\left[i\gamma^\mu\left[\partial'_\mu-ig_BZ^{1/2}A^a_\mu\frac{\lambda^a}{2}\right]\delta_{ff'}-Z_MM_{ff'}\right]\psi_{f'}+\overline{\eta}_f\psi_f+\overline{\psi}_f\eta_f\right\}\right] \quad (3.14)$$

with $N'$ representing a suitable normalization constant needed to make the given path integral well defined. We shall assume that this path integral is defined with the help of a *SU(3)-color gauge-invariant* (and preserving locality of the theory in the limit $\Lambda\to\infty$) *regularization.* Then, as is well known, considering the change of field variables corresponding to Eq. (3.13) with the path integral (3.14) leads to the following relationship (up to terms vanishing like powers of $1/\Lambda$):

$$0=N'\int[d\psi][d\overline{\psi}]\left[-Z_F\partial_\mu[\overline{\psi}_f\gamma^\mu\gamma_5\psi_f(x)]+2Z_FZ_M\overline{\psi}_fi\gamma_5M_{ff'}\psi_{f'}(x) -i\overline{\eta}_f(x)\gamma_5\psi_f(x)-i\overline{\psi}_f(x)\gamma_5\eta_f(x)+h\frac{N_f}{16\pi^2}g_B{}^2Z\,{}^*\mathcal{F}^{\mu\nu a}\mathcal{F}^a_{\mu\nu}(x)\right] \times\exp\left[\frac{i}{h}\int d^4x'\left\{Z_F\overline{\psi}_f\left[i\gamma^\mu\left[\partial'_\mu-ig_BZ^{1/2}A^a_\mu\frac{\lambda^a}{2}\right]\delta_{ff'}-Z_MM_{ff'}\right]\psi_{f'}+\overline{\eta}_f\psi_f+\overline{\psi}_f\eta_f\right\}\right] . \quad (3.15)$$

Under the condition on regularization which we have just stated, this relation can be shown to be unique and we may here identify the term proportional to ${}^*\mathcal{F}^{\mu\nu a}\mathcal{F}^a_{\mu\nu}$ as the $U_A(1)$ current anomaly.[7]

Based on Eq. (3.15), we can formulate the correct $U_A(1)$ WT identity which should be satisfied by the properly regularized QCD generating functional. In Eq. (3.1), imagine performing quark-field integrations first. (This should be permissible for a suitably regularized expression.) Separating $S^*$ into two parts, namely, parts $\mathcal{A}_1$ and $\mathcal{A}_2$ with all quark-field-dependent terms in $S^*$ included in $\mathcal{A}_2$, we may then cast Eq. (3.1) in the form

$$\exp\left[\frac{i}{h}W\right]=N''\int[dA^a_\mu][d\chi^a][d\overline{\chi}^a]\exp\left[\frac{i}{h}\mathcal{A}_1(A_\mu,\chi,\overline{\chi},J_\mu,\xi,\overline{\xi},T^\mu,L)\right] \times N'\int[d\psi][d\overline{\psi}]\exp\left[\frac{i}{h}\mathcal{A}_2(A_\mu,\chi,\overline{\chi},\psi,\overline{\psi},\eta,\overline{\eta},U,\overline{U},\mathcal{J}_{5\mu},\mathcal{J}_5)\right] , \quad (3.16)$$

where we have set $N=N''N'$ and

$$\mathcal{A}_2=\int d^4x\left\{Z_F\overline{\psi}_f\left[i\gamma^\mu\left[\partial_\mu-ig_BZ^{1/2}A^a_\mu\frac{\lambda^a}{2}\right]\delta_{ff'}-Z_MM_{ff'}\right]\psi_{f'}+\left[\overline{\eta}_f-iYg_BZ^{1/2}\overline{U}_f\frac{\lambda^a}{2}\chi^a\right]\psi_f +\overline{\psi}_f\left[\eta_f-iYg_BZ^{1/2}\frac{\lambda^a}{2}\chi^aU_f\right]+Z_FZ_A\mathcal{J}_{5\mu}\overline{\psi}_f\gamma^\mu\gamma_5\psi_f+Z_FZ_M\overline{\psi}_fi\gamma_5M_{ff'}\psi_{f'}\right\} . \quad (3.17)$$

With the amplitude $N' \int [d\psi][d\bar\psi] e^{(i/\hbar)\mathscr{A}_2}$, a relation similar to Eq. (3.15) can be immediately written down and that in turn will lead to the following relation for $W$:

$$\left[ -\partial_\mu \frac{\delta W}{\delta \mathscr{J}_{5\mu}(x)} + 2\frac{\delta W}{\delta \mathscr{J}_5(x)} - i\left[ \bar\eta_f(x)\gamma_5 \frac{\delta W}{\delta \bar\eta_f(x)} - \frac{\delta W}{\delta \eta_f(x)}\gamma_5 \eta_f(x)\right] - i\left[ \bar U_f(x)\gamma_5 \frac{\delta W}{\delta \bar U_f(x)} + \frac{\delta W}{\delta U_f(x)}\gamma_5 U_f(x)\right] + 2N_f \frac{\delta W}{\delta \Theta(x)}\right]\Bigg|_{\mathscr{J}_{5\mu}=\mathscr{J}_5=0} = 0 \,. \tag{3.18}$$

We have used the relation (3.3) here, and also made an assumption that path integrations over variables $(A_\mu^a, \chi^a, \bar\chi^a)$ in Eq. (3.16) do not generate new $U_A(1)$ anomalies other than the one already identified in Eq. (3.15). This latter assumption can be justified[24] in the regularization scheme using higher covariant derivatives.[25] Equation (3.18) is the desired $U_A(1)$ current WT identity. In terms of the functions $\bar\Gamma$, this $U_A(1)$ current WT identity will read

$$\left[ -\partial_\mu \frac{\delta \bar\Gamma}{\delta \mathscr{J}_{5\mu}(x)} + 2\frac{\delta \bar\Gamma}{\delta \mathscr{J}_5(x)} - i\left[ \frac{\delta \bar\Gamma}{\delta \psi_f(x)}\gamma_5 \psi_f(x) - \bar\psi_f(x)\gamma_5 \frac{\delta \bar\Gamma}{\delta \bar\psi_f(x)}\right] - i\left[ \bar U_f(x)\gamma_5 \frac{\delta \bar\Gamma}{\delta \bar U_f(x)} + \frac{\delta \bar\Gamma}{\delta U_f(x)}\gamma_5 U_f(x)\right] + 2N_f \frac{\delta \bar\Gamma}{\delta \Theta(x)}\right]\Bigg|_{\mathscr{J}_{5\mu}=\mathscr{J}_5=0} = 0 \,. \tag{3.19}$$

Note that we are accepting the Adler-Bardeen theorem[24] for *regularized* QCD generating functionals. To prove the corresponding result for *renormalized* QCD generating functionals,[26] one must also establish finiteness of amplitudes involved in the limit $\Lambda \to \infty$. That will be considered in Sec. IV. We can also give the integrated form of the $U_A(1)$ current WT identity. Especially, in our $(M,\theta)$-independent renormalization scheme, we may combine the integrated (over the space-time) version of Eq. (3.19) with Eq. (3.8) to deduce

$$\int d^4x \left[ -\frac{\delta \bar\Gamma}{\delta \psi_f(x)}\gamma_5 \psi_f(x) + \bar\psi_f(x)\gamma_5 \frac{\delta \bar\Gamma}{\delta \bar\psi_f(x)} - \bar U_f(x)\gamma_5 \frac{\delta \bar\Gamma}{\delta \bar U_f(x)} - \frac{\delta \bar\Gamma}{\delta U_f(x)}\gamma_5 U_f(x)\right] - 2iM_{ff'}^{(P)} \frac{\delta \bar\Gamma}{\delta M_{ff'}^{(S)}} + 2iM_{ff'}^{(S)} \frac{\delta \bar\Gamma}{\delta M_{ff'}^{(P)}} - 2iN_f \int d^4x \frac{\delta \bar\Gamma}{\delta \Theta(x)} = 0 \,. \tag{3.20}$$

This is the WT identity associated with global $U_A(1)$ transformations and the piece, $-2iN_f \int d^4x\, \delta\bar\Gamma/\delta\Theta(x)$, here corresponds to the global $U_A(1)$ anomaly.

The regularized proper-vertex generating functional of QCD should be consistent with various WT identities given above. Among them, note that the WT identities (3.11), (3.12), and (3.20), which are connected with *global* $SU(N_f)_L \times SU(N_f)_R \times U_A(1)$ symmetries, have been given under the premise that all renormalization constants are to be chosen to be $M$ independent. Actually, the meaning of these three WT identities is quite simple. Including also the trivial U(1) symmetry related to the fermion number, they imply that the regularized QCD generating functional $\bar\Gamma(A_\mu^a, \psi, \bar\psi, \chi^a, \bar\chi^a, T^{\mu a}, L^a, U, \bar U, \mathscr{J}_{5\mu}, \mathscr{J}_5; M, \Theta)$ or $\Gamma$ should be invariant under transformations of the form[27]

$$\begin{aligned}
&\psi(x) \to \psi'(x) = C\psi(x) \,, \\
&\bar\psi(x) \to \bar\psi'(x) = \bar\psi(x)\tilde C \,, \\
&U(x) \to U'(x) = \tilde C^{-1} U(x) \,, \\
&\bar U(x) \to \bar U'(x) = \bar U(x) C^{-1} \,, \\
&M \to M' = \tilde C^{-1} M C^{-1} \,, \\
&\Theta(x) \to \Theta'(x) = \Theta(x) - 2N_f \beta_0 \,, \\
&(A_\mu^a, \chi, \bar\chi, T^{\mu a}, L^a, \mathscr{J}_{5\mu}, \mathscr{J}_5\text{: unchanged})
\end{aligned} \tag{3.21}$$

with general global flavor rotation matrices $C, \tilde C \equiv \gamma_0 C^\dagger \gamma_0$ parametrized as

$$\begin{aligned}
&C = \exp(i\alpha_0 - i\gamma_5\beta_0 + i\boldsymbol{\alpha}\cdot\mathbf{T} - i\gamma_5\boldsymbol{\beta}\cdot\mathbf{T}) \,, \\
&\tilde C = \exp(-i\alpha_0 - i\gamma_5\beta_0 - i\boldsymbol{\alpha}\cdot\mathbf{T} - i\gamma_5\boldsymbol{\beta}\cdot\mathbf{T}) \,.
\end{aligned} \tag{3.22}$$

[Here, real numbers $(\alpha_0, \beta_0, \boldsymbol{\alpha}, \boldsymbol{\beta})$ need not be infinitesimal.] This information will be able to tell us precisely what changes are necessary for parameters of the theory when we make a global rotation of quark field basis in the flavor space.

Finally, we wish to say something about the behavior of our regularized functional $\bar\Gamma$ under the parity $(P)$ transformation. We do not wish the regularization procedure to be a source of $P$ violation in QCD—given $M=0$ and $\Theta(x)=0$, we want $P$ to be an exact symmetry of QCD. In the $(M,\theta)$-independent renormalization scheme this can be also easily implemented with the generating functional $\bar\Gamma(A_\mu^a, \psi, \bar\psi, \chi^a, \bar\chi^a, T^{\mu a}, L^a, U, \bar U, \mathscr{J}_{5\mu}, \mathscr{J}_5; M, \Theta)$; we may require that the functional $\bar\Gamma$ (or $\Gamma$) be a scalar under the parity transformation

$$\begin{aligned}
&M \to M' = \gamma_0 M \gamma_0, \quad \Theta(x) \to \Theta'(x') = -\Theta(x) \,, \\
&\psi(x) \to \psi'(x') = \gamma_0 \psi(x), \\
&\bar\psi(x) \to \bar\psi'(x') = \bar\psi(x)\gamma_0 \,, \\
&A^{\mu a}(x) \to A'^{\mu a}(x') = (A^{0a}(x), -A^{ia}(x)) \,, \\
&\mathscr{J}_5(x) \to \mathscr{J}'_5(x') = -\mathscr{J}_5(x) \,, \\
&\mathscr{J}_5^\mu(x) \to \mathscr{J}'^\mu_5(x') = (-\mathscr{J}_5^0(x), \mathscr{J}_5^i(x)), \text{ etc.} \,,
\end{aligned} \tag{3.23}$$

where $x'^\mu = (x^0, -x^i)$. [Ghost fields $(\chi^a, \bar\chi^a)$ may be regarded as either scalar fields or pseudoscalar fields as long as parity transformation rules for $(T^{\mu a}, U, \bar{U})$ are assigned consistently with those.]

## IV. PROOF OF FINITENESS TO ALL ORDERS AND THE RENORMALIZED $U_A(1)$ WT IDENTITY

In the previous section, symmetries of properly regularized generating functionals in QCD have been identified. While observing those WT identities, we now wish to remove regularization, i.e., consider the limit $\Lambda \to \infty$ in order-by-order loop expansion. The freedom at our disposal is the choice of $\Lambda$-dependent renormalization constants $(Z, Y, g_B, Z_F, Z_M, \delta X)$. When the finite $\Lambda \to \infty$ limits for the regularized generating functionals can be secured by exploiting this freedom, the resulting well-defined limiting expressions are renormalized generating functionals of QCD. Within order-by-order loop expansion, we shall demonstrate below that well-defined renormalized QCD generating functionals can indeed be constructed to all orders if the six renormalization constants are chosen judiciously.

For this demonstration it is convenient to work with the functional $\bar{\Gamma}$. This is because, with suitable care regarding subgraph divergences, possible ultraviolet divergences for the proper-vertex generating functional can be always isolated as the space-time integral of local polynomials in variables $(A^a_\mu, \psi, \bar{\psi}, \chi^a, \bar{\chi}^a, T^{\mu a}, L^a, U, \bar{U}, \mathscr{J}_{5\mu}, \mathscr{J}_5, M, \Theta)$ with net dimension not exceeding four. Our dimension assignments will be based on the following rule:

$$
\begin{array}{llllll}
\text{Variable} & \mathscr{J}_5 & A^a_\mu, \chi^a, \bar{\chi}^a, M, R_\mu \equiv \partial_\mu \Theta, \mathscr{J}_{5\mu} & \psi, \bar{\psi}, U, \bar{U} & T^{\mu a}, L^a \\
\hline
\text{Dimension} & 0 & 1 & \frac{3}{2} & 2
\end{array} . \tag{4.1}
$$

These are dimensions to be used by us for ultraviolet-divergence power counting. A noteworthy point here is that we have assigned dimension 1 (rather than 0) to the mass matrix $M$ and also to $R_\mu(x) \equiv \partial_\mu \Theta(x)$ [see Eq. (2.4) and ensuing comments]; these dimension assignments in fact naturally lead to the $(M,\theta)$-independent renormalization scheme. The presence of the dimension-0 quantity $\mathscr{J}_5(x)$ will not complicate the ultraviolet power counting; as should be evident from the form (3.2) for $S^*$, $\mathscr{J}_5(x)$ enters $\bar{\Gamma}$ always in the form of the dimension-1 quantity $\mathscr{J}_5(x)M$. Another useful quantity is the ghost number. We will assign the ghost number $-1$ to the field $\chi$, $+1$ to $(\bar{\chi}, T^{\mu a}, U, \bar{U})$, $+2$ to $L^a$, and zero to the rest of variables. Then the functional $\bar{\Gamma}$ should clearly have the ghost number 0.

The regularized functional $\bar{\Gamma}$ is expanded as a power series in the loop-expansion parameter $h$ (which may be identified with 1 after making the expansion):

$$
\bar{\Gamma} = \sum_{k=0}^{\infty} h^k \bar{\Gamma}_k . \tag{4.2}
$$

Here one should not forget that renormalization constants are also some power series in $h$ and, in writing Eq. (4.2), the $h$ dependences resulting from renormalization constants are included. To calculate $\bar{\Gamma}$ up to $l$-loop order, it will be sufficient to have the renormalization constants $(Z, Y, g_B, Z_F, Z_M)$ expanded up to $O(h^l)$ and $\delta X$ up to $O(h^{l-1})$ (for $l \geq 1$), i.e.,

$$
\begin{gathered}
Z \to Z^{(l)} \equiv \sum_{k=0}^{l} h^k Z_k, \quad Y \to Y^{(l)} \equiv \sum_{k=0}^{l} h^k Y_k , \\
g_B \to g_B^{(l)} \equiv \sum_{k=0}^{l} h^k g_{B,k}, \ldots, \delta X \to \delta X^{(l)} \equiv \sum_{k=0}^{l-1} h^k \delta X_k
\end{gathered} \tag{4.3}
$$

with the lowest-order values identified to

$$
\begin{aligned}
& Z_0 = \rho_A, \quad Y_0 = \rho_\chi , \\
& Z_{F,0} = \rho_\psi, \quad Z_{M,0} = 1 , \\
& g_{B,0} = g, \quad \delta X_0 = \frac{g^2}{32\pi^2} t .
\end{aligned} \tag{4.4}
$$

Here, $(\rho_A, \rho_\chi, \rho_\psi, g, t)$ are renormalized (and thus finite) parameters of the theory and their explicit values depend on normalization conditions chosen. [Only $g$, $M$, and $\theta$ are genuine free physical parameters of QCD. Additional free parameters $(\rho_A, \rho_\chi, \rho_\psi, t)$ introduced here will allow one to treat finite-renormalization effects systematically. Note that the unrenormalized action (2.1) is obtained if we set $\rho_A = \rho_\chi = \rho_\psi = 1$ and $t = 0$.] Formally we have

$$
Z = \sum_{k=1}^{\infty} h^k Z_k, \quad g_B = \sum_{k=1}^{\infty} h^k g_{B,k}, \ldots, \text{etc.} ,
$$

and general $k$-loop renormalization constants $(Z_k, Y_k, g_{B,k}, Z_{F,k}, Z_{M,k}, \delta X_{k-1})$ are supposed to be independent of $M$ and $\theta$. Note that, because of the relation (3.3), the $U_A(1)$ current renormalization constant $Z_A$ is fixed in loop expansion as

$$
\begin{aligned}
& Z_A = \sum_{k=0}^{\infty} h^k Z_{A,k} , \\
& Z_A^{(l)} \equiv \sum_{k=0}^{l} h^k Z_{A,k} = 1 - 2N_f h \delta X^{(l-1)} , \\
& Z_{A,0} = 1, \quad Z_{A,k} = -2N_f \delta X_{k-1} \quad (\text{for } k \geq 1) .
\end{aligned} \tag{4.5}
$$

Let $S^{*(l)}$ represent the expression obtained from $S^*$ [see Eq. (3.2)] simply via the substitution (4.3) for renormalization constants. In $S^{*(l)}$, one should not truncate the expression up to the $l$-loop order but keep every term obtained through the substitution (4.3) with $S^*$. (We shall need $S^{*(l)}|_{(l+1)\,\text{loop}}$ below.) We may then use the notation $\bar{\Gamma}^{(l)}$ to indicate the suitably regularized modified proper-vertex generating functional which is obtained by using the action $S^{*(l)}$ in place of $S^*$ in defining generating functionals. The functional $\bar{\Gamma}^{(l)}$ may also be expanded in powers of $h$, viz.,

$$\bar{\Gamma}^{(l)}=\bar{\Gamma}_0^{(l)}+h\bar{\Gamma}_1^{(l)}+\cdots+h^l\bar{\Gamma}_l^{(l)}+h^{l+1}\bar{\Gamma}_{l+1}^{(l)}+\cdots$$

[illegible]

(4.6)

Clearly, $\bar{\Gamma}_k^{(l)}=\bar{\Gamma}_k$ for $k\le l$. Here, we note that none of the renormalization constants appeared explicitly in our WT identities formulated in Sec. III—Eqs. (3.9), (3.11), (3.12), (3.19), and (3.20)—or in the relations (3.8) and (3.10). Then, naturally, those relations should remain true even if $\bar{\Gamma}^{(l)}$ takes the position of $\bar{\Gamma}$. Below we will thus assume that, with $\bar{\Gamma}$ replaced by $\bar{\Gamma}^{(l)}$, various WT identities of Sec. III and Eqs. (3.8) and (3.10) are satisfied at least up to the $(l+1)$-loop order in loop expansion; these are our requirements for the "suitably regularized" generating functional $\bar{\Gamma}^{(l)}$. Also for any given $l$, the functional $\bar{\Gamma}^{(l)}$ is supposed to be a scalar under the parity transformation (3.23).

We now proceed to the construction of the renormalized QCD generating functional. The functional $\bar{\Gamma}$ in the tree approximation, $\bar{\Gamma}_0$ $(=\bar{\Gamma}_0^{(0)})$, of course requires no renormalization and is simply given by

$$\bar{\Gamma}_0=S^{*(0)}\,|_{\Theta(x)=0}+\frac{1}{2\alpha}\int d^4x\,(\partial^\mu A_\mu^a)^2\ . \tag{4.7}$$

Here, in accordance with the above definition, $S^{*(0)}$ is obtained from $S^*$ via the substitution $(Z,Y,g_B,Z_F,Z_M,\delta X,Z_A)\to(\rho_A,\rho_\chi,g,\rho_\psi,1,(g^2/32\pi^2)t,1)$. Beyond the tree approximation, $\bar{\Gamma}$ becomes highly nonlocal and is very difficult to calculate explicitly. Nevertheless, the ultraviolet-divergence structure and renormalizability of the theory can be studied systematically on the basis of power counting and various WT identities. Below we shall employ the usual iterative procedure.

This means that, with a judicious choice of $(M,\theta)$-independent renormalization constants,

$$(Z_k,Y_k,g_{B,k},Z_{F,k},\delta X_{k-1},Z_{A,k}=-2N_f\delta X_{k-1})\qquad (k=1,\ldots,l) \tag{4.8}$$

the amplitudes $\bar{\Gamma}$ $(=\bar{\Gamma}_k^{(l)})$ for $k$ ranging from 1 to $l$ have been established to have finite $\Lambda\to\infty$ limits. We then wish to extend the statement to $(l+1)$-loop order. First, we note that the $(l+1)$-loop amplitude $\bar{\Gamma}_{l+1}$ may be writ-

There should be no subgraph divergences with $\bar{\Gamma}_{l+1}^{(l)}$ since it is based on the $l$-loop renormalized action $S^{*(l)}$

[illegible]

nite $\Lambda\to\infty$ limits; viz., $\bar{\Gamma}_{l+1}^{(l)}$ may have only whole graph divergences. From this fact and power counting, we may conclude that divergent pieces of $\bar{\Gamma}_{l+1}^{(l)}$ (denoted by $\bar{\Gamma}_{l+1,\mathrm{div}}^{(l)}$) are restricted to the space-time integral of a *local* polynomial in $(A_\mu^a,\chi^a,\bar{\chi}^a,\psi,\bar{\psi},T^{\mu a},L^a,U,\bar{U},\mathscr{J}_{5\mu},\mathscr{J}_5,M,\partial_\mu\Theta)$ with the dimension of each monomial not exceeding four. [See our dimension assignments (4.1).] We may then use the BRS WT identity (3.9), with $\bar{\Gamma}$ in the equation replaced by $\bar{\Gamma}^{(l)}$. Especially, using the loop expansion (4.6) for $\bar{\Gamma}^{(l)}$, look at the terms proportional to $h^{l+1}$ from the relation. Since $\bar{\Gamma}_k^{(l)}$ for $k\le l$ have finite $\Lambda\to\infty$ limits, it gives us the following constraint equation for $\bar{\Gamma}_{l+1,\mathrm{div}}^{(l)}$:

$$\mathscr{S}\bar{\Gamma}_{l+1,\mathrm{div}}^{(l)}=0\ , \tag{4.10}$$

where

$$\mathscr{S}\equiv\int d^4x\left[-\frac{\delta\bar{\Gamma}_0}{\delta T_\mu^a}\frac{\delta}{\delta A^{\mu a}}+\frac{\delta\bar{\Gamma}_0}{\delta L^a}\frac{\delta}{\delta\chi^a}+\frac{\delta\bar{\Gamma}_0}{\delta\bar{U}_f}\frac{\delta}{\delta\psi_f}+\frac{\delta\bar{\Gamma}_0}{\delta U_f}\frac{\delta}{\delta\bar{\psi}_f}-\frac{\delta\bar{\Gamma}_0}{\delta A_\mu^a}\frac{\delta}{\delta T_\mu^a}+\frac{\delta\bar{\Gamma}_0}{\delta\chi^a}\frac{\delta}{\delta L^a}+\frac{\delta\bar{\Gamma}_0}{\delta\psi_f}\frac{\delta}{\delta\bar{U}_f}+\frac{\delta\bar{\Gamma}_0}{\delta\bar{\psi}_f}\frac{\delta}{\delta U_f}\right]. \tag{4.11}$$

As is well known,[17,22] the functional differential operator $\mathscr{S}$ obeys the nilpotency relation $\mathscr{S}\mathscr{S}\equiv 0$. Exploiting this nilpotency, one can show that the local functional $\bar{\Gamma}_{l+1,\mathrm{div}}^{(l)}$ obeying Eq. (4.10) is necessarily of the form[22]

$$\bar{\Gamma}_{l+1,\mathrm{div}}^{(l)}=G(A_\mu^a,\psi,\bar{\psi},\mathscr{J}_{5\mu},\mathscr{J}_5,M,\partial_\mu\Theta)+\mathscr{S}\Omega\ . \tag{4.12}$$

Here, $G(A_\mu^a,\psi,\bar{\psi},\mathscr{J}_{5\mu},\mathscr{J}_5,M,\partial_\mu\Theta)$ is a general, local, Lorentz scalar functional (involving only variables $A_\mu^a$, $\psi$, $\bar{\psi}$, $\mathscr{J}_{5\mu}$, $\mathscr{J}_5$, $M$, and $\partial_\mu\Theta$) which is invariant under the (tree level) SU(3)-color gauge transformation

$$\delta A_\mu^a(x)=[\partial_\mu\delta_{ac}+g\rho_A{}^{1/2}f_{abc}A_\mu^b(x)]\Lambda^c(x)\ ,$$

$$\delta\mathscr{J}_{5\mu}(x)=\delta\mathscr{J}_5(x)=\delta M=\delta\Theta(x)=0\ ,$$

(4.13)

function $\mathscr{J}_{5\mu}(x)$ or $\mathscr{J}_5(x)$ with our generating functional, and the dimensionless quantity $\mathscr{J}_5(x)$ should be always accompanied by $M$. On top of these conditions, we may impose the global $SU(N_f)_L \times SU(N_f)_R$ symmetries, i.e., use the WT identities (3.11) and (3.12) with $\overline{\Gamma}$ replaced by $\overline{\Gamma}^{(l)}$. This will lead to the conclusion that Eqs. (3.11) and (3.12) should hold even if $\overline{\Gamma}$ there is replaced by the local functional $\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}$. [The $U_A(1)$ will be considered separately below.] Also, $\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}$ should be a scalar under the parity transformation (3.23). The most general form for the functionals $G$ and $\Omega$, consistent with all these requirements, can be easily recognized as[28]

$$G=\int d^4x\left[-\tfrac{1}{4}a_{l+1}F^{\mu\nu a}(\rho_A{}^{1/2}A)F^a_{\mu\nu}(\rho_A{}^{1/2}A)+b_{l+1}\rho_\psi\overline{\psi}_f i\gamma^\mu\left[\partial_\mu-ig\rho_A{}^{1/2}A^a_\mu\frac{\lambda^a}{2}\right]\psi_f(x)\right.$$
$$-c_{l+1}\rho_\psi\overline{\psi}_f M_{ff'}\psi_{f'}(x)+d_{l+1}\rho_\psi\mathscr{J}_{5\mu}(x)\overline{\psi}_f\gamma^\mu\gamma_5\psi_f(x)+c'_{l+1}\rho_\psi\mathscr{J}_5(x)\overline{\psi}_f i\gamma_5 M_{ff'}\psi_{f'}(x)$$
$$\left.+r_{l+1}\partial_\mu\Theta(x)K^\mu(\rho_A{}^{1/2}A)+s_{l+1}\rho_\psi(\partial_\mu\Theta)(x)\overline{\psi}_f\gamma^\mu\gamma_5\psi_f(x)\right], \tag{4.14}$$

$$\Omega=\int d^4x\{e_{l+1}L^a(x)\chi^a(x)+f_{l+1}(T^{\mu a}-\partial^\mu\overline{\chi}^a)(x)A^a_\mu(x)+h_{l+1}[\overline{U}_f(x)\psi_f(x)+\overline{\psi}_f(x)U_f(x)]\}\ , \tag{4.15}$$

where various constants $(a_{l+1},b_{l+1},c_{l+1},d_{l+1},c'_{l+1},r_{l+1},s_{l+1},e_{l+1},f_{l+1},h_{l+1})$, which may diverge as $\Lambda\to\infty$, do not depend on the mass matrix $M$ or $\theta$. Power counting alone indicates that all these constants can be logarithmically divergent as $\Lambda\to\infty$. Note that, in the expression (4.14) for $G$, pure quark bilinears without involving the mass matrix $M$ explicitly are forbidden by the $SU(N_f)_L\times SU(N_f)_R$ consideration.

Without any loss of generality, we may set $h_{l+1}=0$ in Eq. (4.15) since the terms proportional to $h_{l+1}$ from $\Omega$ are already represented fully in the expression (4.14) for $G$. Now, using Eq. (4.12), we have

$$\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}=G+\int d^4x\left[f_{l+1}\rho_\chi(T^{\mu a}-\partial^\mu\overline{\chi}^a)(x)(\partial_\mu\delta_{ac}+g\rho_A{}^{1/2}f_{abc}A^b_\mu)\chi_c(x)\right.$$
$$\left.-f_{l+1}\frac{\delta\overline{\Gamma}_0}{\delta A^{\mu a}(x)}A^{\mu a}(x)+e_{l+1}\tfrac{1}{2}\rho_\chi g\rho_A{}^{1/2}L^a(x)f_{abc}\chi^b(x)\chi^c(x)+e_{l+1}\frac{\delta\overline{\Gamma}_0}{\delta\chi^a(x)}\chi^a(x)\right]. \tag{4.16}$$

With this expression, we may now use our last constraint—the $U_A(1)$ current WT identity (3.19), with $\overline{\Gamma}$ replaced by $\overline{\Gamma}^{(l)}$. This will lead to the conclusion that Eq. (3.19) holds with $\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}$ in place of $\overline{\Gamma}$. If we use the expression (4.16) for $\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}$, that equation implies that we may in fact identify

$$d_{l+1}=b_{l+1}-2N_f s_{l+1},\quad c'_{l+1}=c_{l+1},\quad r_{l+1}=0\ . \tag{4.17}$$

Note that, for the one-loop case where $\delta\overline{\Gamma}^{(0)}_1/\delta\Theta(x)$ is exactly equal to the finite expression

$$\frac{g^2}{32\pi^2}[{}^*F^{\mu\nu a}(\rho_A{}^{1/2}A)F^a_{\mu\nu}(\rho_A{}^{1/2}A)-t\rho_\psi\partial^\mu(\overline{\psi}_f\gamma_\mu\gamma_5\psi)]\ ,$$

we obviously have $\delta\overline{\Gamma}^{(0)}_{1,\mathrm{div}}/\delta\Theta(x)=0$ and $r_1=s_1=0$. Then, Eq. (4.17) says that $d_1=b_1$ and $c'_1=c_1$. Beyond the one-loop order, $s_l(l\geq 2)$ cannot be set to zero (i.e., divergent as $\Lambda\to\infty$) and hence $d_l$ for $l\geq 2$ will be in general different from $b_l$.

The last relation in Eq. (4.17), $r_{l+1}=0$, is especially remarkable since it will immediately lead to the conclusion

$$\left.\frac{\delta\overline{\Gamma}^{(l)}_{l+1}}{\delta\Theta(x)}\right|_{\psi=\overline{\psi}=0}:\ \text{finite as }\Lambda\to\infty\quad\text{for all } l=0,1,2,\ldots\ . \tag{4.18}$$

Equation (4.18) indicates that, as far as $\Theta(x)$-dependent bosonic counterterms are concerned, the $(l+1)$-loop counterterms coming from the $l$-loop renormalized action $S^{*(l)}$, i.e., the $(l+1)$-loop terms of the expression

$$\int d^4x\,\Theta(x)h\frac{(g_B^{(l)})^2Z^{(l)}}{32\pi^2}[\partial^\mu A^{\nu a}-\partial^\nu A^{\mu a}+g_B^{(l)}(Z^{(l)})^{1/2}f_{abc}A^{\mu b}A^{\nu c}][\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g_B^{(l)}(Z^{(l)})^{1/2}f_{ade}A^d_\mu A^e_\nu] \tag{4.19}$$

are in fact sufficient for the $(l+1)$-loop renormalization consideration. (For $l=1$, this has been checked explicitly in Ref. 29. Note that, up to the one-loop order at least, $g_B Z^{1/2}$ is independent of the number of quark flavors although $g_B$ and $Z^{1/2}$ separately do depend on $N_f$ in a nontrivial way[29].)

Having applied all symmetry restrictions we have on $\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}$, let us now go back to Eq. (4.9). As we let $\Lambda\to\infty$, a finite functional $\overline{\Gamma}_{l+1}$ will be secured if we can set

$$\overline{\Gamma}^{(l)}_{l+1,\mathrm{div}}+(S^{*(l+1)}-S^{*(l)})\,|_{(l+1)\,\mathrm{loop}}=0\ . \tag{4.20}$$

This may be regarded as the *renormalizability condition.* With the form of $\overline{\Gamma}^{(l)}_{l+1,\text{div}}$ restricted as above, Eq. (4.20) is indeed satisfied if we let our $(l+1)$-loop renormalization constants $(Z_{l+1},Y_{l+1},g_{B,l+1},Z_{F,l+1},Z_{M,l+1},\delta X_l,Z_{A,l+1})$ be determined according to

$$\begin{aligned}
&Z_{l+1}=\rho_A(-a_{l+1}+2f_{l+1}),\;\; g_{B,l+1}=\tfrac{1}{2}ga_{l+1}\,,\\
&Z_{F,l+1}=-\rho_\psi b_{l+1},\;\; Z_{M,l+1}=b_{l+1}-c_{l+1}\,,\\
&Y_{l+1}=\rho_\chi(-f_{l+1}+e_{l+1}),\\
&\delta X_l=-s_{l+1}+\delta_{l0}\frac{g^2}{32\pi^2}t\,,\\
&Z_{A,l+1}=-2N_f\delta X_l\,.
\end{aligned} \tag{4.21}$$

By construction, these $(l+1)$-loop renormalization constants will be independent of $M$ and $\theta$. On the basis of the renormalized action (3.2), we have thus shown that the existence of a well-defined $\Lambda\to\infty$ limit for the functional $\overline{\Gamma}$ can be extended to $(l+1)$-loop order.

Assuming loop expansion, we have so far presented an iterative proof of renormalizability on the basis of our renormalized action $S^*$. The renormalized QCD generating functional can be set to the expression

$$\overline{\Gamma}^{\text{ren}}=\sum_{k=0}^{\infty}h^k\overline{\Gamma}^{\text{ren}}_k \tag{4.22}$$

$$(\overline{\Gamma}^{\text{ren}}_0=\overline{\Gamma}_0,\;\overline{\Gamma}^{\text{ren}}_k\equiv\lim_{\Lambda\to\infty}\overline{\Gamma}^{(k)}_k\;\text{ for } k\geq 1)\,.$$

Clearly, $\overline{\Gamma}^{\text{ren}}$ will possess the same symmetry properties as our suitably regularized functional $\overline{\Gamma}$; viz., Eqs. (3.9), (3.11), (3.12), and (3.19) hold with $\overline{\Gamma}$ there replaced by $\overline{\Gamma}^{\text{ren}}$. They are appropriate *WT identities for the renormalized proper-vertex generating functional* in QCD. The six $(M,\theta)$-independent renormalization constants $(Z,Y,g_B,Z_F,Z_M,\delta X)$ may be chosen such that the following normalization conditions (in momentum space) may hold:

$$\text{(i)}\;\; \overline{\Gamma}^{\text{ren}}_{(AA)}(p,-p)^{ab}_{\mu\nu}\,|_{M=0}=i\delta_{ab}\rho_A(\eta_{\mu\nu}p^2-p_\mu p_\nu)\;\text{ at } p^2=-\mu^2\,, \tag{4.23a}$$

$$\text{(ii)}\;\; \frac{\partial}{\partial p^2}\overline{\Gamma}^{\text{ren}}_{(\chi\bar\chi)}(p,-p)\,|_{M=0}=i\delta_{ab}\rho_\chi\;\text{ at } p^2=-\mu^2\,, \tag{4.23b}$$

$$\begin{aligned}\text{(iii)}\;\; \overline{\Gamma}^{\text{ren}}_{(AAA)}(p,q,r=-p-q)^{abc}_{\mu\nu\lambda}\,|_{M=0}=&\,i\rho_A{}^{3/2}gf_{abc}[\eta_{\mu\lambda}(r-q)_\mu+\eta_{\lambda\mu}(p-r)_\nu+\eta_{\mu\nu}(q-p)_\lambda]\\
&+(\text{different tensor structure})\;\text{ at } p^2=q^2=r^2=-\mu^2\,,\end{aligned} \tag{4.23c}$$

$$\text{(iv)}\;\; \overline{\Gamma}^{\text{ren}}_{(\psi\bar\psi)}(p,-p)\underset{M\to 0}{\sim}i\rho_\psi \not p-i\rho_\psi M+O(M^2)\;\text{ at } p^2=-\mu^2\,, \tag{4.23d}$$

$$\text{(v)}\;\; \frac{\partial}{\partial\mathscr{F}_{5\mu}(-q)}\overline{\Gamma}^{\text{ren}}_{(\psi\bar\psi)}(q;p,p'=-p-q)\,|_{M=0}=i\rho_\psi\rho_5\gamma^\mu\gamma_5\;\left[\text{with } \rho_5\equiv 1-h\frac{N_fg^2}{16\pi^2}t\right]\;\text{ at } p^2=-\mu^2\,. \tag{4.23e}$$

(Here, for quantities given on the left-hand sides, setting $T^{\mu a}=L^a=U=\overline{U}=\mathscr{F}_{5\mu}=\mathscr{F}_5=\Theta=0$ is implicitly assumed.) Note that we have chosen a (arbitrary) spacelike momentum scale, $p^2=-\mu^2$, as our normalization point.

If one changes the value of $\mu^2$, the values of renormalized parameters $(\rho_A,\rho_\chi,g,\rho_\psi,M,\rho_5)$ should be adjusted appropriately to have vertex functions unchanged. In this sense, renormalized parameters $(\rho_A,\rho_\chi,g,\rho_\psi,M,\rho_5)$ are really functions of $\mu^2$ and the precise functional dependences are expressed through renormalization-group equations:

$$\begin{aligned}
&\mu\frac{d\rho_A}{d\mu}=\gamma_A(g)\rho_A,\;\; \mu\frac{d\rho_\chi}{d\mu}=\gamma_\chi(g)\rho_\chi\,,\\
&\mu\frac{dg}{d\mu}=\beta(g),\;\; \mu\frac{d\rho_\psi}{d\mu}=\gamma_\psi(g)\rho_\psi\,,\\
&\mu\frac{dM}{d\mu}=\gamma_m(g)M,\;\; \mu\frac{d\rho_5}{d\mu}=\delta(g)\rho_5\,.
\end{aligned} \tag{4.24}$$

Here, $[\gamma_A(g),\ \gamma_\chi(g),\ \beta(g),\ \gamma_\psi(g),\ \gamma_m(g),\ \delta(g)]$ are renormalization-group coefficients which become independent of $M$ and $\theta$ in the $(M,\theta)$-independent renormalization scheme. Note that the value of *the vacuum angle* $\theta$, fixed by the action (2.6) [with the substitution (2.3)], *remains unchanged for different values of* $\mu^2$.

$\overline{\Gamma}^{\text{ren}}$ is guaranteed to be invariant under the general flavor transformation (3.21). Based on this, we can now establish the formula (1.3) as a relation valid to all orders. Here, $M$ [see Eq. (1.2)] should be taken as the renormalized mass matrix which appears in our renormalized action (2.6) and also in the normalization condition (4.23d). Also, we emphasize that the parameter $\theta$ should enter the renormalized action precisely in the form shown in Eq. (2.13), with no additional finite (multiplicative) renormalization with the composite field

$$(g_B{}^2Z/32\pi^2)\,{}^*\mathscr{F}^{\mu\nu a}\mathscr{F}^a_{\mu\nu}(x)$$

allowed. Given a quark mass matrix $M$ of the general form (1.2), one can always make a suitable $SU(N_f)_L\times SU(N_f)_R\times U_A(1)$ rotation on quark-field basis such that the mass matrix in the new basis may become $\gamma_5$-free and diagonal. If one does not wish to end up with a different theory, such flavor rotation must be made according to our transformation law (3.21). An elementary analysis tells us that, to obtain a $\gamma_5$-free and diagonal mass matrix $M'=\tilde{C}^{-1}MC^{-1}$ with flavor matrices $(C,\tilde{C})$ specified by Eq. (3.22), we must assume $\beta_0=-(1/2N_f)\arg(\det\mathcal{M})$. Then the transformation law $\Theta'(x)=\Theta(x)-2N_f\beta_0$, upon making the substitution (2.3), immediately leads to the formula (1.3). The formula (1.3) should be thus exact to all orders since it is a consequence of our symmetry transformation law (3.21). In case a strictly massless quark exists in the theory, the vacuum angle $\theta$ is not physically meaningful (and may be set to zero if one wishes) since the value depends on the (physically undistinguishable) chiral phase convention for that massless quark-field variable.[4]

Note that the $U_A(1)$ WT identity plays a very important role in our discussion. It may be thus useful to have our properly renormalized expression for it translated into the operator language. Looking at Eq. (3.18), it is easy to see that our renormalized $U_A(1)$ current WT identity corresponds to the operator relation[30]

$$\partial^\mu \hat{J}^{\,\mathrm{ren}}_{5\mu}(x)=2\hat{J}^{\,\mathrm{ren}}_{5}(x)+2N_f h\,\hat{\mathscr{A}}^{\,\mathrm{ren}}(x) - i\bar{\eta}_f(x)\gamma_5\psi_f(x)-i\bar{\psi}_f(x)\gamma_5\eta_f(x)\,, \tag{4.25}$$

where

$$\hat{J}^{\,\mathrm{ren}}_{5\mu}(x)=Z_F(1-2N_f h\,\delta X)\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)\,, \tag{4.26a}$$

$$\hat{J}^{\,\mathrm{ren}}_{5}(x)=Z_F Z_M\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)\,, \tag{4.26b}$$

$$\hat{\mathscr{A}}^{\,\mathrm{ren}}(x)=\frac{g_B{}^2 Z}{32\pi^2}\,{}^*\mathscr{F}^{\mu\nu a}(A(x))\mathscr{F}^a_{\mu\nu}(A(x)) - Z_f(\delta X)\partial^\mu[\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)]\,. \tag{4.26c}$$

Formally, Eq. (4.25) is just a rearrangement of the relation

$$\begin{aligned}\partial^\mu[Z_F\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)] &=2Z_M Z_F\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)\\ &\quad+2N_f h\frac{g_B{}^2 Z}{32\pi^2}\,{}^*\mathscr{F}^{\mu\nu a}(A(x))\mathscr{F}^a_{\mu\nu}(A(x))\\ &\quad-i\bar{\eta}_f(x)\gamma_5\psi_f(x)-i\bar{\psi}_f(x)\gamma_5\eta_f(x)\,.\end{aligned} \tag{4.27}$$

Note that Eq. (4.27) can be regarded as the *regularized form* of the $U_A(1)$ current divergence relation (with the Adler-Bardeen theorem[24]), and in Eq. (4.25) certain rearrangement has been made to write the relation in terms of finite (i.e., suitably renormalized) operators. Equation (4.25) is the generalization of the similar result obtained in spinor electrodynamics.[26,31] In non-Abelian gauge theories, people have been uneasy in making a definite statement (in other than the background gauge[9]) because the finiteness property for the operator representing the anomaly term has not been clearly established.[32] In this paper we have shown that in the generalized Landau gauge, the set of operators $[\hat{J}^{\,\mathrm{ren}}_{5\mu}(x),\hat{J}^{\,\mathrm{ren}}_{5}(x),\hat{\mathscr{A}}^{\,\mathrm{ren}}(x)]$ defined in Eqs. (4.26a)–(4.26c) are finite operators and Eq. (4.25) indeed corresponds to the properly renormalized form. [We must warn readers here that, in the context of supersymmetric Yang-Mills theory, there exist some controversies concerning the renormalized $U_A(1)$ current divergence relation and its compatibility with supersymmetry transformations.[33] In this paper, we shall remain in (nonsupersymmetric) QCD and thus not worry about supersymmetry.]

Using the operator mixing language, we shall here give a quick reaccount on why $[\hat{J}^{\,\mathrm{ren}}_{5\mu}(x),\hat{J}^{\,\mathrm{ren}}_{5}(x),\hat{\mathscr{A}}^{\,\mathrm{ren}}(x)]$ define finite operators. (In our iterative proof, this has been implicitly incorporated.) First, taking into account dimensions of operators and various symmetry restrictions, it is easy to see that the three dimension-4 operators

$$[\partial^\mu(\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)),\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)\,,\ {}^*\mathscr{F}^{\mu\nu a}(x)\mathscr{F}^a_{\mu\nu}(x)]\,, \tag{4.28}$$

may mix only among them under renormalization. Actually, the first two operators in Eq. (4.28) are multiplicatively renormalizable (i.e., no mixing with other operators). The reason is as follows. The first in Eq. (4.28) is multiplicatively renormalizable since the operator $\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)$ is so; $\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)$ is the only dimension-3, gauge-invariant, flavor-singlet, pseudovector operator. The operator $\bar{\psi}(x)i\gamma_5 M_{ff'}\psi_{f'}(x)$ is multiplicatively renormalizable since (i) $M$ is an external factor [with dimension 1 according to Eq. (4.1)] and thus the counterterms must be also proportional to $M$ and (ii) $\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)$ is the only dimension-4, gauge-invariant, flavor-singlet [in the sense of Eq. (3.21)], pseudoscalar [in the sense of Eq. (3.23)] operator which is proportional to $M$. On the other hand, the operator ${}^*\mathscr{F}^{\mu\nu a}(x)\mathscr{F}^a_{\mu\nu}(x)\equiv\partial^\mu\mathscr{K}_\mu(A(x))$ may mix with $\partial^\mu(\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x))$ but not with $\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)$. This is because (i) $\partial^\mu$ of $\partial^\mu\mathscr{K}_\mu(A(x))$ behaves also as an external factor (just like $M$) and thus the counterterms must be also of the form $\partial^\mu(\cdots)$ and (ii) the set $\{\partial^\mu\mathscr{K}_\mu(A(x)),\partial^\mu(\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x))\}$ encompass all dimension-4, gauge-invariant, flavor-singlet, pseudoscalar operators which are of the form $\partial^\mu(\cdots)$. Counterterms for the three operators in Eq. (4.28) are further controlled by the regularized $U_A(1)$ current divergence relation (4.27) (which is just the Adler-Bardeen theorem). Equation (4.27) indicates that the dimension-4 operators,

$$\left[\hat{J}^{\,\mathrm{ren}}_{5}\equiv Z_F Z_M\bar{\psi}_f(x)i\gamma_5 M_{ff'}\psi_{f'}(x)\,,\ \partial^\mu\left[Z_F\bar{\psi}_f(x)\gamma_\mu\gamma_5\psi_f(x)-2N_f h\frac{g_B{}^2 Z}{32\pi^2}\mathscr{K}_\mu(A(x))\right]\right]\,, \tag{4.29}$$

having different structures [i.e., one proportional to $M$ and the other having the form $\partial^\mu(\cdots)$], should define finite operators *separately*. Then the structures of finite operators $[\hat{J}^{\,\mathrm{ren}}_{5}(x),\hat{\mathscr{A}}^{\,\mathrm{ren}}(x)]$ shown in Eqs. (4.26a) and (4.26c) follow immediately from the three observations:

(i) $\partial^\mu(\bar\psi_f(x)\gamma_\mu\gamma_5\psi_f(x))$ is multiplicatively renormalizable; (ii) ${}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ mixes only with $\partial^\mu(\bar\psi_f(x)\gamma_\mu\gamma_5\psi_f(x))$ under renormalization; and (iii)

$$\partial^\mu(Z_F\bar\psi_f(x)\gamma_\mu\gamma_5\psi_f(x))-2N_f h\frac{g_B{}^2Z}{32\pi^2}{}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$$

corresponds to a finite operator.

In the renormalized $U_A(1)$ current divergence relation there still remains the connection to the Atiyah-Singer index theorem since, as we argued at the end of Sec. II, the quantity $\int d^4x\,\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)$ may be identified with the Pontryagin index operator in renormalized theory. Note that this fixes the overall normalization of the term proportional to ${}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ in $\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)$. Once the renormalized action is fixed to the form (2.6), the only ambiguity inherent with Eq. (4.25) is the finite renormalization associated with the renormalization constant $\delta X$; this will affect the definitions of the operators $[J_5^{\mathrm{ren}}(x),\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)]$ according to Eqs. (4.26a) and (4.26c), but not the quantity $\int d^4x\,\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)$. If some of the quarks were massless and could appear as asymptotic states, this freedom in choosing $\delta X$ could be used in securing the correct $U_A(1)$ charge for isolated massless quarks [cf. the condition (4.23e)].

Before closing this section, we may briefly comment on the finiteness of the Pontryagin density $(g_B{}^2Z/32\pi^2){}^*F^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ in the "zero flavor" QCD, i.e., in pure SU(3) Yang-Mills theory. We have no such thing like the $U_A(1)$ WT identity to help us here. But the operator $(g_B{}^2Z/32\pi^2){}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ must be still a finite operator, given the facts that in QCD with $N_f$ quark flavors, (i) the operator $\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)$ defined in Eq. (4.26c) corresponds to a *finite* operator for any $N_f=1,2,3,\ldots$, (ii) with boson external lines only and operator insertions involving only boson fields, Feynman amplitudes for any given loop order are polynomials in $N_f$ (so that the expressions can be safely extended to the $N_f=0$ case), and (iii) our operator $\hat{\mathcal{A}}^{\,\mathrm{ren}}(x)$ can be identified with $(g_B{}^2Z/32\pi^2){}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ in the zero flavor (i.e., $N_f=0$) case.

At the moment, within the generalized Landau gauge, we do not have a *direct* order-by-order proof demonstrating the finiteness of $(g_B{}^2Z/32\pi^2){}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x)$ in pure Yang-Mills theory. (But, read our topology-based reasoning at the end of Sec. II.) Writing the Pontryagin density as the divergence of the Chern-Simons form $\mathcal{K}^\mu(A)$, one may here think that a suitable BRS WT identity involving the operator $\mathcal{K}^\mu(A)$ may provide the additional information necessary for the proof.[32] (Note that the Cher-Simons form possesses very simple gauge transformation property.[31]) But it is not sufficient to establish the finiteness (for an arbitrary gauge-fixing parameter $\alpha$)[32]; further nontrivial properties unique to the Pontryagin density have to be incorporated somehow.

## V. THE DECOUPLING THEOREM AND LOW-ENERGY EFFECTIVE $\theta$ PARAMETER

The total number of quarks in nature are not known at present. (By quarks we mean any elementary spin-$\frac{1}{2}$ fermions to which color gluons couple.) This is because there is always a possibility that some quarks, with very large Lagrangian masses (compared to the presently accessible energy scale), have not been seen yet in experiments. It is also conceivable that certain quarks, even with relatively light Lagrangian masses, have not been identified yet because they are strongly bound by some superstrong gauge interactions. Theoretically, the decoupling theorem[12] of renormalizable field theories is relevant here. Below we shall explain how this theorem applies for QCD with nonvanishing $\theta$ (in the generalized Landau gauge) in case the full theory includes some quarks with very large Lagrangian masses. In this discussion, weak-interaction effects will be ignored completely and no attempt is made to resolve the strong $CP$ problem itself.

First, let us suppose that the full theory, with total $N_f$ quark flavors and some quarks quite heavy, is described by the renormalized action $S$ shown in Eq. (2.6) [with the understanding that a substitution of the form (2.3) will be made for $\Theta(x)$ at the end]. Since we can freely make global flavor rotations of variables in accord with the transformation (3.21), we may assume without any loss of generality the quark mass matrix of the following form:

$$M=\begin{bmatrix} M_L & 0 & & \\ & m_{L_f+1} & 0 & \\ 0 & & \ddots & \\ & 0 & & m_{N_f}\end{bmatrix}\begin{array}{l}\updownarrow L_f(\text{light})\\ \\ \updownarrow N_f-L_f(\text{heavy})\end{array}\ . \quad (5.1)$$

Here, $(m_{L_f+1},\ldots,m_{N_f})$ are real and $\gamma_5$-free although the mass matrix for light quarks, $M_L$, may still have the general form as assumed in Eq. (1.2). Relative to the external energy scale ($E_0$) we are interested in, it will be assumed that eigenvalues of the matrix $M_L$ are smaller or at most comparable to $E_0$ while $m_h \gg E_0$ for $h=L_f+1,\ldots,N_f$ (:the heavy-quark flavors). When the quark mass matrix takes the form (5.1), let us denote the vacuum angle of the theory by $\theta$ [i.e., when we make the substitution (2.3), this $\theta$ value enters]. Other free parameters of the theory, $(\rho_A,\rho_\chi,g,\rho_\psi,M,t)$, may be defined (as suitable functions of the normalization scale $\mu^2$) through the normalization condition (4.23a)–(4.23e).

For studying very high-energy physics satisfying the criterion $E_0 \gtrsim m_h$'s, we could conveniently choose our normalization scale at $\mu^2 \sim E_0{}^2$. Here it would be quite suitable to express various physical amplitudes using the set of renormalized parameters defined through the normalization conditions (4.23a)–(4.23e)—no large logarithms, assuming cancellation of mass singularities, would appear in the expressions. But, for physics at energy scale $E_0 \ll m_h$'s, setting the normalization scale at $\mu^2 \sim E_0{}^2$ alone does not make our renormalized parameters especially good choices; uncontrolled powers of $\ln(m_h{}^2/\mu^2)$ may still show up in the expressions for physical amplitudes. The situation can be remedied by working with an *effective theory* which is obtained by integrating out heavy quark fields.[34] This will be considered below.

Under the restriction $\eta_h=\bar\eta_h=0$ for $h=L_f+1,\ldots,N_f$ (i.e., no external heavy quark legs), we may write the connected Green's-function generating functional $W$ of the full theory as

$$\exp\left[\frac{i}{h}W\right]=N\int[dA_\mu^a][d\chi^a][d\bar{\chi}^a]\prod_{l=1}^{L_f}[d\psi_l][d\bar{\psi}_l]\prod_{h=L_f+1}^{N_f}[d\psi_h][d\bar{\psi}_h]$$
$$\times\exp\left[\frac{i}{h}\left\{S'(A_\mu,\chi,\bar{\chi},\psi_l,\bar{\psi}_l)+\sum_{h=L_f+1}^{N_f}\int d^4x\left[Z_F\bar{\psi}_h i\gamma^\mu\left[\partial_\mu-ig_BZ^{1/2}A_\mu^a\frac{\lambda^a}{2}\right]\psi_h(x)-Z_FZ_M\bar{\psi}_h m_h\psi_h(x)-\Theta(x)hZ_F(\delta X)\partial^\mu(\bar{\psi}_h\gamma_\mu\gamma_5\psi_h)(x)\right]+\int d^4x\left[A^{\mu a}J_\mu^a+\sum_{l=1}^{L_f}(\bar{\eta}_l\psi_l+\bar{\psi}_l\eta_l)+\bar{\xi}^a\chi^a+\bar{\chi}^a\xi^a\right]\right\}\right] \tag{5.2}$$

where $S'(A_\mu,\chi,\bar{\chi},\psi_l,\bar{\psi}_l)$ denotes the expression left after deleting all terms depending on $\psi_h$ or $\bar{\psi}_h$ $(h=L_f+1,\ldots,N_f)$ from the full renormalized action $S$ [see Eq. (2.6)]. For $E_0$ (the typical external energy scale) $\ll m_h$'s, there exists in general,[34,35] to all orders in renormalized perturbation theory, an effective purely light-particle field theory action $\tilde{S}$ such that the above generating functional $W$ of the full theory may be expressed as

$$\exp\left[\frac{i}{h}W\right]=\tilde{N}\int[dA_\mu^a][d\chi^a][d\bar{\chi}^a]\prod_{l=1}^{L_f}[d\psi_l][d\bar{\psi}_l]\times\exp\left\{\frac{i}{h}\left[\tilde{S}(A_\mu,\chi,\bar{\chi},\psi_l,\bar{\psi}_l)+\int d^4x\left[A^{\mu a}J_\mu^a+\sum_{l=1}^{L_f}(\bar{\eta}_l\psi_l+\bar{\psi}_l\eta_l)+\bar{\xi}^a\chi^a+\bar{\chi}^a\xi^a\right]\right]\right\}. \tag{5.3}$$

To provide the precise meaning to the right-hand side of Eq. (5.3), one should specify suitable regularization and renormalization procedures with the effective theory. If one compares expression (5.3) with Eq. (5.2), the meaning of "integrating out heavy-quark fields," to obtain an effective theory (which is described by the action $\tilde{S}$), should be obvious. Note that, in a sense, heavy-quark fields behave like extra regulator fields. Starting from the full theory described by the right-hand side of Eq. (5.2), $\tilde{S}$ can be constructed systematically by replacing heavy-quark loop effects with *effective local interactions* involving only light particle fields.[35] As emphasized in Ref. 35, this procedure can be really looked upon from the viewpoint of *factorization* and, because of that, we may here sensibly talk about the low-energy effective field theory without explicitly mentioning nonperturbative aspects of QCD.

In considering the effective field theory we shall not worry about terms suppressed by inverse powers of heavy-quark masses. Then, as a result of simple heavy-mass power counting,[12,35] $\tilde{S}$ may be restricted to the space-time integral of a local polynomial in $(A_\mu^a,\chi^a,\bar{\chi}^a,\psi_l,\bar{\psi}_l,M_L,\partial_\mu\Theta)$ with dimension not exceeding four. In this power counting, dimension 1 may be assigned to the light-quark mass matrix $M_L$ and to $\partial_\mu\Theta(x)$. These dimension assignments can be easily justified by studying the change in large-$m_h$ asymptotic behaviors of Feynman amplitudes (including heavy-quark propagators as internal lines) as one takes (functional) derivatives with respect to $M_L$ or $R_\mu(x)\equiv\partial_\mu\Theta(x)$. (See Ref. 35.) Furthermore, there are restrictions on the structure of $\tilde{S}$ from symmetry considerations, i.e., from consistency with various WT identities of the full theory. Especially, set $\psi_h=\bar{\psi}_h=U_h=\bar{U}_h=0$ $(h=L_f+1,\ldots,N_f)$ in the BRS WT identity (3.9) and restrict one's attention to the $SU(L_f)_L\times SU(L_f)_R$ flavor transformations (involving light quarks) with Eqs. (3.11) and (3.12)—those relations, which coincide with the BRS and $SU(L_f)_L\times SU(L_f)_R$ WT identities for QCD with $L_f$ flavors, should be observed by generating functionals of the effective theory. From gauge invariance and $SU(L_f)_L\times SU(L_f)_R$-symmetry considerations, we can then only conclude that the effective theory action $\tilde{S}$ should take precisely the form of renormalized action for QCD with $L_f$ quark flavors. Here, an implicit assumption is that regularization procedures for the effective theory do not spoil the WT identities. This is the content of the decoupling theorem.[12]

Ignoring terms down by inverse powers of heavy-quark masses, we may thus write[36]

$$\tilde{S}=\int d^4x\left\{-\tfrac{1}{4}\tilde{Z}\,\tilde{\mathcal{F}}^{\mu\nu a}(A)\tilde{\mathcal{F}}^{a}_{\mu\nu}(A)-\frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2-\tilde{Y}\partial^\mu\bar{\chi}^a\tilde{\mathcal{D}}^{ac}_\mu\chi^c\right.$$
$$+\tilde{Z}_F\sum_{l=1}^{L_f}\bar{\psi}_l\left[i\gamma^\mu\left[\partial_\mu-i\tilde{g}_B\tilde{Z}^{1/2}A^a_\mu\frac{\lambda^a}{2}\right]\delta_{ll'}-\tilde{Z}_M(M_L)_{ll'}\right]\psi_{l'}$$
$$\left.+\tilde{\Theta}(x)h\left[\frac{\tilde{g}_B{}^2\tilde{Z}}{32\pi^2}\,{}^*\tilde{\mathcal{F}}^{\mu\nu a}\tilde{\mathcal{F}}^{a}_{\mu\nu}(x)-\tilde{Z}_F(\delta\tilde{X})\partial^\mu\left[\sum_{l=1}^{L_f}\bar{\psi}_l\gamma_\mu\gamma_5\psi_l\right](x)\right]\right\}, \tag{5.4}$$

where

$$\tilde{\mathcal{F}}^{a}_{\mu\nu}(A)\equiv\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+\tilde{g}_B\tilde{Z}^{1/2}f_{abc}A^b_\mu A^c_\nu\,,$$
$$\tilde{\mathcal{D}}^{ac}_\mu\equiv\partial_\mu\delta_{ac}+\tilde{g}_B\tilde{Z}^{1/2}f_{abc}A^b_\mu\,,$$

and we may set $\tilde{\Theta}(x)=\tilde{\theta}e^{-\epsilon(|x|+|y|+|z|+|t|)}$ ($\tilde{\theta}$ is the low-energy effective vacuum angle). According to our reasoning given earlier (near the end of Sec. II), the expression

$$\int d^4x\frac{\tilde{g}_B{}^2\tilde{Z}}{32\pi^2}\,{}^*\tilde{\mathcal{F}}^{\mu\nu a}\tilde{\mathcal{F}}^{a}_{\mu\nu}(x)$$

may be identified with the appropriately renormalized Pontryagin index in the effective theory. The effective theory has different renormalization counterterms, denoted as $(\tilde{Z},\tilde{Y},\tilde{g}_B,\tilde{Z}_F,\tilde{Z}_M,\delta\tilde{X})$, and possibly a new vacuum angle $\tilde{\theta}$; virtual heavy-quark effects of the full theory have gone into them. Here, the heavy-mass power counting guarantees that the effective theory counterterms $(\tilde{Z},\tilde{Y},\tilde{g}_B,\tilde{Z}_F,\tilde{Z}_M,\delta\tilde{X})$ can be taken to be $(M_L,\theta)$ independent and we may also set

$$\tilde{\Theta}(x)=(1+\tilde{C}_1)\Theta(x)+\tilde{C}_2$$
or $$\tag{5.5}$$
$$\tilde{\theta}=(1+\tilde{C}_1)\theta+\tilde{C}_2$$

with certain $(M_L,\theta)$-independent constants $\tilde{C}_1,\tilde{C}_2$, In Eq. (5.5), sources of having possibly nonzero values for $\tilde{C}_1$ or $\tilde{C}_2$ are diagrams with some internal heavy-quark lines.

To specify the counterterms $(\tilde{Z},\tilde{Y},\tilde{g}_B,\tilde{Z}_F,\tilde{Z}_M,\delta\tilde{X})$ precisely, it would be natural to adopt the same type of normalization conditions as used for the fully theory, i.e., conditions (4.23a)–(4.23e). For renormalized vertex functions of the effective theory, let us thus assume that the normalization conditions (4.23a)–(4.23e) hold with the following replacements:

$$N_f\to L_f\,, \tag{5.6a}$$
$$(\rho_A,\rho_\chi,g,\rho_\psi,\rho_5)\to(\tilde{\rho}_A,\tilde{\rho}_\chi,\tilde{g},\tilde{\rho}_\psi,\tilde{\rho}_5)\,, \tag{5.6b}$$
$$M\to\tilde{M}_L=\tilde{r}M_L$$
$$[\tilde{r}\text{ is a finite, }(M_L,\theta)\text{-independent constant}]\,, \tag{5.6c}$$
$$[\text{at }M=0,\Theta(x)=0]\to[\text{at }\tilde{M}_L=0,\tilde{\Theta}(x)=0]\,. \tag{5.6d}$$

An arbitrary spacelike momentum value, $p^2=-\mu^2$, may be again assumed for the normalization point. Of course, precise values of $(\tilde{\rho}_A,\tilde{\rho}_\chi,\tilde{g},\tilde{\rho}_\psi,\tilde{\rho}_5,\tilde{M}_L,\tilde{\theta})$ here should be chosen such that, at energy scale sufficiently below the heavy-quark threshold, the effective theory may reproduce the predictions of the full theory within our approximation. In general, $(\tilde{\rho}_A,\tilde{\rho}_\chi,\tilde{g},\tilde{\rho}_\psi,\tilde{\rho}_5,\tilde{r},\tilde{C}_1,\tilde{C}_2)$ will be given as suitable functions of $\rho_A$, $\rho_\chi$, $g$, $\rho_\psi$, $\rho_5$, $m_h$'s, and $\mu^2$. In Ref. 34, the explicit formula expressing $\tilde{g}$ in terms of $g$, $m_h$'s, and $\mu^2$ is given in the one-loop approximation. The parameters $\tilde{C}_1,\tilde{C}_2$ will be fixed to all orders in a short while.

We may regard $(\tilde{\rho}_A,\tilde{\rho}_\chi,\tilde{g},\tilde{\rho}_\psi,\tilde{\rho}_5,\tilde{M}_L,\tilde{\theta})$ as free parameters of the effective theory, obtained after integrating out heavy-quark-field freedoms. As functions of $\mu^2$, they will be governed by suitable renormalization-group equations. Here, without much deliberation, one will recognize that Eq. (4.24) will hold exactly under the replacements (5.6a)–(5.6c). Also the parameter $\tilde{\theta}$ should remain unchanged as one varies $\mu^2$; viz., the properly defined effective vacuum angle does not run. The reduction of the quark flavor number, $N_f\to L_f$, in renormalization-group coefficients has an obvious meaning—heavy-quark-field freedoms decouple in low-energy physics. For processes happening at typical energy scale $E_0\ll m_h$'s, physics will be best described in terms of the parameters $(\tilde{\rho}_A,\tilde{\rho}_\chi,\tilde{g},\tilde{\rho}_\psi,\tilde{\rho}_5,\tilde{M}_L,\tilde{\theta})$ normalized at $\mu^2\sim E_0{}^2$. [Here, needless to say, only $(\tilde{g},\tilde{M}_L,\tilde{\theta})$ have real physical significance.] For the description of physics at energy scale $E_0\gtrsim m_h$'s, the effective-theory parameters lose physical meaning and one must go to our original set of parameters $(\rho_A,\rho_\chi,g,\rho_\psi,\rho_5,M,\theta)$ obeying the renormalization-group equations (4.24) (with $N_f$ quark flavors).

We now wish to show that our low-energy effective vacuum angle $\tilde{\theta}$ should have the *same* value as the original vacuum angle (or the value relevant at energy scale larger than $m_h$'s) $\theta$, viz., $\tilde{C}_1=0$ and $\tilde{C}_2=0$ in Eq. (5.5). Technically it implies that given the piece

$$\int d^4x\,\Theta(x)h\frac{g_B{}^2Z}{32\pi^2}\,{}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^{a}_{\mu\nu}(x)$$

in the full renormalized action [assuming the $(M,\theta)$-independent renormalization scheme], we have the piece

$$\int d^4x\,\Theta(x)h\frac{\tilde{g}_B{}^2\tilde{Z}}{32\pi^2}\,{}^*\tilde{\mathcal{F}}^{\mu\nu a}(x)\tilde{\mathcal{F}}^{a}_{\mu\nu}(x)$$

in the effective theory [based on the $(M_L,\theta)$-independent renormalization scheme]. The proof is quite simple. We may fix the constant $\widetilde{C}_2$ first by examining parity. The proper-vertex generating functional based on the action (5.4), say $\widetilde{\Gamma}(A_\mu,\psi_l,\overline{\psi}_l,\chi^a,\overline{\chi}^a;M_L,\widetilde{\Theta})$, will be invariant under the parity transformation of the form [see Eq. (3.23)]

$$\begin{aligned} &M_L \to M_L' = \gamma_0 M_L \gamma_0, \quad \widetilde{\Theta}(x) \to \widetilde{\Theta}'(x') = -\widetilde{\Theta}(x) , \\ &\psi_l(x) \to \psi_l'(x') = \gamma_0 \psi_l(x) , \\ &\overline{\psi}_l(x) \to \overline{\psi}_l'(x') = \overline{\psi}(x)\gamma_0, \ldots . \end{aligned} \tag{5.7}$$

Up to terms down by inverse powers of $m_h$'s, $\widetilde{\Gamma}$ is supposed to yield the same low-energy light-particle vertex functions as the functional $\Gamma(A_\mu,\psi_f,\overline{\psi}_f,\chi^a,\overline{\chi}^a{:}M, \Theta)|_{\psi_h=\overline{\psi}_h=0}$ of the full theory. But, with the full renormalized quark mass matrix given by the form (5.1), it is an immediate consequence of Eq. (3.23) that the latter functional is invariant under

$$\begin{aligned} &M_L \to M_L' = \gamma_0 M_L \gamma_0, \quad \Theta(x) \to \Theta'(x') = -\Theta(x) , \\ &\psi_l(x) \to \psi_l'(x') = \gamma_0 \psi_l(x) , \\ &\overline{\psi}_l(x) \to \overline{\psi}_l'(x') = \overline{\psi}(x)\gamma_0, \ldots . \end{aligned} \tag{5.8}$$

Now, compare this with the transformation (5.7). Clearly, only with $\widetilde{C}_2=0$ [and hence $\widetilde{\Theta}(x)=\widetilde{C}_1\Theta(x)$], $\widetilde{\Gamma}(A_\mu,\psi_l,\overline{\psi}_l,\chi^a,\overline{\chi}^a;M_L,\widetilde{\Theta})$ can be invariant under the transformation (5.8). To fix the constant $\widetilde{C}_1$, we may check the $U_A(1)$ symmetry involving $L_f$ light quarks. By construction, $\widetilde{\Gamma}(A_\mu,\psi_l,\overline{\psi}_l,\chi^a,\overline{\chi}^a;M_L,\widetilde{\Theta})$ will be invariant under [see Eq. (3.21)]

$$\begin{aligned} &\psi_l(x) \to \psi_l'(x) = e^{-i\gamma_5\beta_0}\psi_l(x) , \\ &\overline{\psi}_l(x) \to \overline{\psi}_l'(x) = \overline{\psi}_l(x) e^{-i\gamma_5\beta_0} , \\ &M_L \to M_L' = e^{i\gamma_5\beta_0} M_L e^{i\gamma_5\beta_0} , \\ &\widetilde{\Theta}(x) \to \widetilde{\Theta}'(x) = \widetilde{\Theta}(x) - 2L_f\beta_0 \quad (l=1,\ldots,N_f) . \end{aligned} \tag{5.9}$$

But, to yield the same low-energy light-particle vertex functions as the functional $\Gamma(A_\mu,\psi_f,\overline{\psi}_f,\chi^a,\overline{\chi}^a, M,\Theta)|_{\psi_h=\overline{\psi}_h=0}$, we know [as a direct consequence of Eq. (3.21)] that $\widetilde{\Gamma}$ should be also invariant under

$$\begin{aligned} &\{\psi_l(x),\overline{\psi}_l(x),M_L\}\text{: transform according to Eq. (5.9)} , \\ &\Theta(x) \to \Theta'(x) = \Theta(x) - 2L_f\beta_0 . \end{aligned} \tag{5.10}$$

Clearly, the two transformations will be compatible only with $\widetilde{C}_1=0$.

We have so far shown that no threshold effects show up[37] in the observed value of the QCD vacuum angle $\theta$ with regards to the presence of heavy quarks, *assuming the quark mass matrix of the form (5.1)* (i.e., diagonal and $\gamma_5$-free in the heavy-quark sector). Stated differently, integration out quark fields with large *Dirac mass* term does not change the vacuum angle. If large scales enter the quark mass matrix in a different way, one should first make a suitable flavor rotation in accord with our transformation (3.21) to bring the quark mass matrix to the form (5.1) and then, with the resulting vacuum angle, apply the above conclusion. Explicitly, suppose that the Lagrangian mass term for heavy quarks is given as

$$-Z_F Z_M \sum_{h=L_f+1}^{N_f} \overline{\psi}_h(x)\left[(\mathcal{M}_H)_{hh'}\frac{1+\gamma_5}{2} + (\mathcal{M}_H^\dagger)_{hh'}\frac{1-\gamma_5}{2}\right]\psi_{h'}(x) \tag{5.11}$$

with certain nondiagonal mass matrix $\mathcal{M}_H$, and let the given vacuum angle in this quark basis be $\theta$. Then, after integrating out all heavy-quark-field freedoms, we will find the effective vacuum angle $\widetilde{\theta}=\theta+\arg(\det\mathcal{M}_H)$.

## VI. DISCUSSIONS

In this paper we have established a renormalization structure for QCD with nonvanishing vacuum angle $\theta$ (to all orders in loop expansion), assuming the generalized Landau gauge. We have identified the renormalized Pontryagin density and then proved nonrenormalizability of $\theta$. Full global symmetries of QCD, including $U_A(1)$, are incorporated in the renormalization procedure and corresponding WT identities valid for renormalized vertex functions are given explicitly. A simple consequence of our WT identities is the well-known formula (1.3) which relates the quark mass matrix phase with the vacuum angle (in a quark field basis leading to a $\gamma_5$-free mass matrix); hence, the formula is established as a relation valid to all orders. We have then studied how the Appelquist-Carazzone decoupling theorem applies to QCD with nonvanishing $\theta$. Here, through a careful examination of various symmetry constraints [including that from the $U_A(1)$], we have been able to determine unambiguously the low-energy effective QCD vacuum angle in terms of parameters of the full theory (relevant at sufficiently high energy).

Further extensions of our work are desirable. Especially, with QCD, quark masses appearing in the Lagrangian should be considered just as externally given parameters. But, in more extended theories including weak interactions, Yukawa-type interactions between quarks and Higgs fields can be a source of such quark masses. In the context of such "larger" theories, at this moment we cannot make definite statements concerning the $\theta$-term renormalization and the global $U_A(1)$ flavor rotation with quark field variables. The situation may actually change depending on the status of $U_A(1)$ (the Peccei-Quinn symmetry[38]) and *CP* in the given larger theory. We shall elaborate below on this point to a certain extent.

Let us concentrate here on renormalization of the QCD vacuum angle term in such a larger theory[39] (assuming, say, the generalized Landau gauges with color gluon fields). [Note that, without settling this, one cannot make a definite statement on the global $U_A(1)$ phase change with quark field variables, either.] As we explained in Sec. II, the $\theta$ term in the action can be always regarded as a superrenormalizable term. Hence, regardless of the theory in consideration, it should be possible to represent its renormalized form such as [cf. Eq. (2.13)]

$$\int d^4x\, \theta_B h \frac{g_B{}^2 Z}{32\pi^2} {}^*\mathcal{F}^{\mu\nu a}(x)\mathcal{F}^a_{\mu\nu}(x) , \tag{6.1}$$

where

$$\theta_B = Z_1^{(\theta)}\theta + Z_2^{(\theta)} \text{ ($\theta$ is the renormalized vacuum angle)} . \tag{6.2}$$

Here, $Z_1^{(\theta)}$ and $Z_2^{(\theta)}$ are possible renormalization counterterms which do not depend on parameters associated solely with superrenormalizable interactions in the theory (e.g., bare masses, $\theta$). Now, the point is that (i) if (aside from the vacuum angle term) *CP* happens to be a good or softly broken symmetry[40] of the Lagrangian, we may consistently set $Z_2^{(\theta)}=0$ in Eq. (6.2) and (ii) if (ignoring the anomaly contribution) the global $U_A(1)$ quark flavor rotation corresponds to an exact[38] or softly broken symmetry of the Lagrangian, we may consistently set $Z_1^{(\theta)}=1$.

Spontaneous symmetry breaking, by having an asymmetric vacuum, whether that is for *CP* (Ref. 41) or for $U_A(1)$ (Ref. 8), will not affect these assertions.

For QCD [with a general quark mass matrix of the form (1.2)] we have in fact both criteria satisfied and consequently $\theta_B=\theta$, viz., $\theta$ is not renormalized. [In our main text we have invoked *P* (instead of *CP*) in setting $Z_2^{(\theta)}=0$; this is allowed since QCD is a vectorlike gauge theory.] Point (i) above is self-evident. On the other hand, point (ii) may be established generally by following the more-or-less same steps as we have taken for the QCD case. The above two criteria should apply separately, viz., *CP* with the subtractive renormalization counterterms $Z_2^{(\theta)}$ and $U_A(1)$ with the multiplicative renormalization counterterm $Z_1^{(\theta)}$. When a given larger theory does not meet these criteria, the situation—including physical interpretation for the vacuum angle (in case $\theta_B \neq \theta$)—is very uncertain. With *CP* explicitly broken by dimension-4 terms in the Lagrangian,[42] we do not have any reason not to expect any infinite subtractive renormalization for the vacuum angle.[43] At present, for this case, we do not have a well-defined calculation scheme to verify whether such infinite subtractive renormalization is really necessary or not. Similarly, when the $U_A(1)$ is explicitly broken by dimension-4 terms in the Lagrangian [e.g., the standard $SU(2)\times U(1)\times SU(3)$ model with one Higgs doublet], we do not yet know whether we can still consistently set $Z_1^{(\theta)}=1$. [The situation here is better in the sense that we can here at least resolve the issue by calculation—one may use our trick of replacing $\theta$ by an arbitrary externally given function $\Theta(x)$ in intermediate steps.]

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$$\int d^4x\, \theta \frac{g^2}{32\pi^2}\, {}^{*}F^{\mu\nu a}(A) F^a_{\mu\nu}(A)$$

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$$\Theta'(x)=\Theta(x)-2N_f\beta_0 e^{-\epsilon(|x|+|y|+|z|+|t|)}\,.$$

[28]In writing the form for $G$, we have ignored terms depending on external variables ($T^{\mu a}, U, \bar{U}, \mathcal{J}_{5\mu}, \mathcal{J}_5, M, \Theta$) only. Also, in writing the form for $\Omega$, we have demanded proper behaviors under the *global* part of SU(3)-color gauge transformations.

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