Three-cocycle in quantum mechanics. II

Bernard Grossman
The Rockefeller University, 1230 York Avenue, New York, New York 10021
(Received 24 June 1985)

We offer a new topological interpretation of the three-cocycle in a consistent quantum mechanics. Our basis is homotopy theory. We show how higher cocycles may appear in quantum mechanics. Moreover, a three-cocycle gives rise to a nonassociative algebra describing defects in a quantum-mechanical system. We also show the relation of the three-cocycle to axions in string theory.

An example of a three-cocycle in Lie-algebra cohomology has recently been found for a consistent quantum mechanics with a background point magnetic monopole.[1] Clarification of our understanding of the three-cocycle is needed since the original description depended upon a δ-function singularity at the origin.[2] A new description should also resolve the question of what type of algebra can give rise to a violation of the Jacobi identity.[2,3] The issue can be resolved in terms of homotopy theory. Moreover, even though there are an infinite number of nonvanishing commutators for the velocities, only the second and the third are significant for the group cohmology if only one-particle wave functions are considered. If we consider the quantum mechanics of more than one particle or even of several dyons, then there are higher-order topological invariants that are relevant since the available configuration space is higher dimensional. We conclude the article with first a physical interpretation of a three-cocycle for a discrete group. [Discrete groups arise, from example, by approximating U(1) by torsion matrices.] Then we demonstrate the connection with string theory. A closed three-form in the string theory guarantees the existence of an axion and an associative orbit for axial U(1).

We assume that the magnetic monopole is located at the origin. There is no reason to exclude the origin if we allow δ functions to describe the singularity. We are interested in studying the wave function in the background field of a magnetic monopole. The Dirac quantization[4] condition is a relation between the electric change e of those particles whose wave function we are calculating and the magnetic charge g of the magnetic monopole. The condition states that the magnetic flux through any two-sphere S^2 enclosing the magnetic monopole is an integer, when appropriately normalized:

$$g=\int_{S^2}F=\frac{n}{2e}\hbar\ . \tag{1}$$

A two-sphere S^2 can be considered as the boundary ∂ of a three-dimensional disc D^3,

$$\int_{\partial D^3}F=\int_{D^3}dF=\frac{n}{2e}\hbar\ , \tag{2}$$

by Stokes's theorem, provided that dF exists.

Three linearly independent vectors a_1, a_2, a_3 determine a 3-chain $D^3(a_1,a_2,a_3)$ which we define as follows. Consider a point $\boldsymbol{\epsilon}$ of distance ϵ from the origin. There is a tetrahedron $D^3_{\boldsymbol{\epsilon}}$ determined by $\boldsymbol{\epsilon}$, $\boldsymbol{\epsilon}+\mathbf{a}_1$, $\boldsymbol{\epsilon}+\mathbf{a}_1+\mathbf{a}_2$, $\boldsymbol{\epsilon}+\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3$, such that the boundary of $D_\epsilon{}^3$ is homotopically a two-sphere S^2. By considering this S^2 homotopically, we avoid the problem of vectors puncturing it, except the ambiguity when they go through the origin. A Lie-algebra three-cochain α_3 is a mapping from three elements $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ of the additive group R^3 into an Abelian group (in this case the integers with a covering by the real numbers)

$$\alpha_2(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=\int_{D_\epsilon{}^3(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)}dF\ , \tag{3}$$

where we use the fact that R^3 also has a Lie-algebra structure induced by the cross product.

α_3 is a three-cocycle[5] for the Lie algebra since

$$\delta\alpha_3=\int_{\partial D^4}dF \tag{4a}$$

$$=\int_{D^4}d^2F=0\ , \tag{4b}$$

where D^4 is a four-dimensional manifold of which D^3 is a constant time slice.

In general an n-cochain is defined by replacing 3 by n where

$$\delta\alpha_n\equiv\sum_{i=1}^{n+1}(-1)^i\alpha_n(a_1,\ldots,\hat{a}_i,\ldots,a_{n+1})\ ,$$

where a caret denotes omission. α_n is an n-cocycle if $\delta\alpha_n=0$. Given a triangulation[9] of a surface S, we obtain an n-chain over which we can integrate. The existence of a magnetic flux in space whose source is enclosed by an incontractible surface S defines an n-cocycle for the translation group. One just replaces S^2 above by S and D^3 by the n-chain. This produces a flux conservation condition in the example where a circular flux tube produces a nontrivial three-cochain and a trivial four-cocycle.

Notice that all elements of R^3 are considered for a_1,a_2,a_3, but zero elements determine degenerate D^3 since they do not leave the vector ϵ. For any ϵ, we see an incontractible two-sphere surrounding the origin.

One can formally consider the distribution-valued form as a source of the Abelian magnetic monopole

$$-e\,dF=\frac{eg}{h}\delta(\mathbf{r})r^2dr\sin\theta\,d\theta\,d\theta\ . \tag{5a}$$

The δ function is to be considered in the sense of a distribution. Therefore, we consider it as a limit of a nice function. For example,

$$\delta(r)dr=\lim_{\delta\to 0}\frac{1}{\pi}\frac{\delta}{\delta^2+r^2}dr \tag{5b}$$

which we can rescale by δ for any finite δ to become

$$\frac{1}{\pi}\frac{dr}{r^2+1}=\frac{1}{\pi}d\chi \tag{5c}$$

by using $\tan\chi$ in place of the rescaled r. This describes the possible virtual motion for finite δ, when the monopole is not strictly located at the origin.[10] Moreover, integrating over χ defines a type of dimensional reduction to the surface of an incontractible two-sphere surrounding the origin.[11]

Notice that the right-hand side of (5a) is invariant under rotations, gauge transformations, and conformal transformations. However, it is not invariant under translations, which would lead to a measure for χ that is not concentrated at the origin. Since the gauge group is U(1), one can look for an extension of the gauge group as a circle to the real line of translations as in the exact sequence $0\to Z\to R\to S^1\to 1$. Moreover, the integers can naturally be considered as the excitations to the different states of different magnetic charges. These give rise to the noncompact extension of the circle to the real line. However, there is an arbitrariness in the way U(1) can be extended in the translations corresponding to the arbitrariness of the classical value for $2eg$. The gauge group should be mapped into the translations around closed loops. These closed loops can then be naturally embedded in the space of paths determined by the translations. Moreover, since there is an incontractible two-sphere surrounding the magnetic monopole, one can construct a nontrivial U(1) bundle over this two-sphere. Transport around a closed loop, which we identify as the equator of the two-sphere, defines a winding number. Transport around a longitudinal closed loop requires a nontrivial gauge transformation since it intersects the equator. This confirms with the two-patch description of the magnetic monopole, where one divided the incontractible two-sphere into two patches: an upper hemisphere and an overlapping lower hemisphere. The overlap is a region that is contractible to the equator. The map of the equator into the gauged U(1) defines a winding number.

There is a nontrivial three-form associated with the Hopf map from $S^3\to S^2$ that determines the nontrivial U(1) bundle over S^2. We would like to identify this three-form with the three-form determining the three-cocycle for the translation group. In order to understand the additional information contained in the fact that $dF=0$ we first consider a one-form ω,

$$\omega=f\,d\phi\ , \tag{6a}$$

where f is a function of the coordinates to be determined. At each point, ω determines a plane perpendicular to the ϕ direction, except at the origin where ω becomes singular:

$$d\omega=\eta\omega \tag{6b}$$

for some η; however, there is an ambiguity in η since any one-form $fh\,d\phi$ can be added to η. In particular

$$\eta=d\ln f+fh\,d\phi\ , \tag{7a}$$

$$d\eta=h\,df\,d\phi+f\,dh\,d\phi\ , \tag{7b}$$

$$\eta\,d\eta=df\,dh\,d\phi\ . \tag{7c}$$

The integral of $\eta\,d\eta$ over S^3 is arbitrary. However, to identify with the magnetic-monopole source, we choose

$$h=\frac{1}{\pi}\frac{\epsilon}{\epsilon^2+\phi^2}\Theta(\theta-\pi/2)\ , \tag{8a}$$

$$f=\frac{eg}{\pi h}\arctan r/\delta\ , \tag{8b}$$

where Θ is the Heaviside Θ function that is 1 for positive arguments and zero otherwise. Then

$$df=\frac{eg}{\pi h}\frac{\delta dr}{\delta^2+r^2}\ , \tag{8c}$$

$$\eta\,d\eta=\frac{eg}{\pi^2 h}\frac{\delta}{\delta^2+r^2}\frac{\epsilon}{\epsilon^2+r^2}\delta(\theta-\pi/2)d\theta\,d\phi\,dr\ . \tag{8d}$$

For nonzero ϵ and δ this projects onto the plane $\theta=\pi/2$ that is the stereographic projection of S^2. Notice that in the limit $\delta,\ \epsilon\to 0$, we recover $-e\,dF$ and

$$\eta\to\delta(r)dr+\frac{eg}{h}\delta(\phi)\theta(\theta-\pi/2)d\phi\ . \tag{9}$$

The ambiguity in η corresponds to the fact that θ and ϕ are not well defined at the origin. One can take the limit of δ approaching zero along any orbit, like a spiral, instead of a straight line. This leads to an undefined phase in the solution

$$d\omega=\eta\omega\ . \tag{10}$$

In order to patch the local solutions together, $2eg/h$ must be an integer times $\sqrt{-1}$, since each time we cross $\phi=0$ in the $\theta=\pi/2$ plane, we mut pick up an integral phase. This gives an integral Hopf invariant when $\eta\,d\eta$ is integrated over R^3 so that $\eta\,d\eta$ covers the plane a finite number of times. Notice that even for nonzero δ, when the monopole charge is spread out through space, one must still quantize according to the Dirac condition in order to have integral Hopf invariant.

In the Hopf map, one can also consider letting the radius of S^2 go to zero. In this limit case, one has a circle sitting over one point and the limiting case makes sense. If eg is not integrally quantized, one can start with the pointlike limit but there is no way of expanding to a two-sphere.

The existence of the three-cocycle for the additive group R^3 depends crucially on the existence of a cross product in three dimensions. This product is nonassociative:

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\ . \tag{11}$$

However, there exists a nonlinear product using both the scalar and the cross product which is associative:

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\ . \tag{12}$$

Classically, there are three independent velocity vectors at each point and they commute since they are c numbers. One can solve the equation of motion

$$\dot{\mathbf{v}}=e\mathbf{v}\times\mathbf{B} \tag{13a}$$

for any B by solving v into components parallel to B and perpendicular to B. The component parallel to B (in this case the radial component) remains constant, while the component perpendicular to B rotates and determines a one-parameter group. Notice that because of the associative product

$$\dot{\mathbf{v}}\cdot\mathbf{v}=0 \tag{13b}$$

so $v^2=\text{const}$. Hence, there are classical ambiguities in the angular velocity (it becomes infinite) at the origin, related to the ambiguity we encountered in (7a).

Quantum mechanically, the velocities must be considered as operators. Of course the standard momenta and positions, p_i and x_i, respectively, satisfy the commutation relations

$$[p_i,p_j]=0=[x_i,x_j]\ , \tag{14a}$$

$$[p_i,x_j]=-i\delta_{ij}\hbar\ . \tag{14b}$$

However, these are not gauge invariant. The gauge-invariant velocities $v_i=p_i+eA_i$ (where A_i is defined only locally) satisfy the same commutation relations with the positions

$$[v_i,x_j]=-i\delta_{ij}\hbar \tag{15}$$

but they do not commute with themselves. The algebra for an operator solution of the equation of motion is complicated. Since we have that

$$[v_i,v_j]=-i\,e\hbar F_{ij} \tag{16}$$

one must extend the algebra to include F_{ij}. If F_{ij} were constant, then this would determine a central extension insofar as all higher-order commutators would vanish. In the language of cohmology, there is a one-to-one correspondence between central extensions of the three one-parameter groups generated by the classical velocities and the two-cocycles determined by the constant curvatures F_{ij} (Ref. 8). In the case of the point magnetic monopole $\nabla\cdot\mathbf{B}\neq 0$. The nonzero value gives the alternating sum of the triple commutators.[12] These are obstructions to F_{ij} being a central extension. Moreover, one cannot avoid the δ-function singularity at the origin by restricting the wave function to vanish at the origin since one cannot also assume all derivatives of the wave function also vanish at the origin. There is, however, a power associative algebra[13] with an alternating trilinear form for vector fields v, w, and z such that

$$\mathbf{v}\cdot(\mathbf{w}\times\mathbf{z})=\sum_{\substack{i,j,k\\ \text{cyclic}}}[v_i,[w_j,z_k]]=(\mathbf{v}\times\mathbf{w})\cdot\mathbf{z}\ , \tag{17a}$$

$$(\mathbf{v}\times\mathbf{v})\cdot\mathbf{z}=0=\mathbf{z}\cdot(\mathbf{v}\times\mathbf{v})\ , \tag{17b}$$

(13b) is crucial for consistency of the equation of motion when v, w, and z are velocities because we want $\mathbf{v}\times\mathbf{B}=-\mathbf{B}\times\mathbf{v}$. This triple product shows that the velocities themselves may be difficult to define, but powers of the velocity, like the Hamiltonian, $\frac{1}{2}\mathbf{v}^2$, and the curvature, F_{ij}, are well defined.

One can consider extensions by other sources or by higher tensor fields G_{ijk} obtained from a two-form potential. For $\mathbf{B}=\mathbf{r}$, $\nabla\cdot\mathbf{B}$ is a constant so the fourth-order commutation vanishes.[2,14] However, when $\mathbf{B}=r^n\mathbf{r}, n>0$, there is a nonvanishing fourth-order as well as a higher-order commutator. Not much is known algebraically about these extensions. However, homotopically, there is a very beautiful framework for discussing them.[15]

We shall consider as an example of the formalism, the point magnetic monopole. We have a triple product h of operators that is a nonvanishing associative product. However, there may be obstructions from higher-order commutators to associativity. (We will see that on a group level, only the triple product is relevant.) The first obstructions are seen by considering the five possible products of four elements determined by the commutators:

$$(1)\ \ [[v_i,v_j],[v_k,v_l]]=0\ \text{ for classical background}\ , \tag{18a}$$

$$(2)\ \ [[[v_i,v_j],v_k],v_l]\ , \tag{18b}$$

$$(3)\ \ [[v_i,[v_j,v_k]],v_l]\ , \tag{18c}$$

$$(4)\ \ [v_i,[[v_j,v_k],v_l]]\ , \tag{18d}$$

$$(5)\ \ [v_i,[v_j,[v_k,v_l]]]\ . \tag{18e}$$

Adjacent products and (1) and (5) represent the five homotopics $H_t^{(i)}$ that can exist such that

$$H_0^{(i)}(x,y,z)=\sum_{\substack{\text{cyclic permutation}\\ \text{of components}}}[[x,y],z]\ , \tag{19}$$

$$H_1^{(i)}(x,y,z)=-\sum_{\substack{\text{cyclic permutation}\\ \text{of components}}}[x,[y,z]]\ , \tag{20}$$

where the minus sign comes from the commutator.

For (1) and (2) $x=[v_i,v_j], y=v_k, z=v_l$. When $x=v_i$, $y=v_j, z=v_k$, we see that $H_0=H_1$. Even though (18b)–(18e) do not vanish, they do not present any new obstruction. Suppose we rescale $v_i\rightarrow tv_i$ and consider the possible obstructions to letting $t\rightarrow 0$. First there is the commutator $[v_i,v_j]=-i\,ehF_{ij}$ whose integral over a two-sphere enclosing the origin must be constant. One must therefore rescale S^2 by t^{-2}. The next obstruction is the triple commutator $\epsilon^{ijk}[v_i,[v_j,v_k]]$. If we rescale volume by t^{-3}, this will yield a nonvanishing integral. However, higher commutators do not give rise to new obstructions since space is three dimensional. Moreover, in order to specify the algebra uniquely, we would of course write down all higher-order commutators and see that the obstructions remain in two and three dimensions. One can even note that in an abstract setting the algebra has a parameter e which we can vary to zero and obtain an associative, a commutative algebra. Of course there is an obstruction to such a variation in quantum mechanics. If one has more complicated flux-tube configurations, then dimensional parameters will enter and there is an obstruction to the homotopy argument. Finally, we note interesting possibilities when one is in four dimensions with a non-Abelian group or if the background field commute and higher-order relations are possible.

The fact that the individual velocities are not well defined gives rise to an ambiguous phase in $\nabla\cdot\mathbf{B}$ corresponding to the fact that $-i\,d/dr$, the radial component of the velocity, is not well defined. This is related to the fact that the triple product is really a determinant. Just as with the determinant of the Dirac operator, nonvanishing phase gives rise to an anomaly or ambiguity. We can get rid of this phase by restricting the domain. Moreover, just as in the families index theorem, one can consider this phase as varying as the parameter e varies, even when eA does not define a connection as in Sec. II.

The key formula for translating from the Lie algebra to the group is the Campbell-Baker-Hausdorff formula:

$$e^{A}e^{B}=\exp(A+B+\tfrac{1}{2}[A,B]+\tfrac{1}{12}[A,[A,B]] + \tfrac{1}{12}[B,[B,A]]+\cdots) \tag{21}$$

(Ref. 16). Classically, the one-parameter groups determined by the solutions to the equations of motion are additive since the velocities commute. However, quantum mechanically, all the nonvanishing commutators contribute to the product. All the higher commutators are determined by differentials of the commutator given by the curvature. There is a possible obstruction to the extension as one might choose a coefficient incompatible with the given extension. However, if we are in three spatial dimensions then the only forms that matter are the two-form F and the three-form dF.

Consider another description of the origin of the three-cocycle in the group cohmology in connection with the fiber degree of freedom. Namely, one can perform a unitary transformation on the momenta p_i to w_i that also commute and satisfy the Jacobi identity

$$w_i=U^{-1}p_iU\ , \tag{22}$$

$$U(x)=\exp\left[i\int_{\mathbf{a}}^{\mathbf{x}}\frac{e}{\hbar}A_i dx^i\right]\ , \tag{23}$$

where the integral is over a straight line from $\mathbf{a}$ to $\mathbf{x}$.

The representation of the Heisenberg algebra generated by p^i, x^j, and $\hbar$ depends upon the potential about which one perturbs by the magnetic monopole. If we consider a background three-dimensional, quantum-mechanical harmonic oscillator then the creation $a_i^\dagger$ and annihilation a_j operators generate a U(3) algebra, $a_i^\dagger a_j$. The Hamiltonian is the diagonal $H=\sum a_i^\dagger a_i$, the angular momentum is $a_i^\dagger a_j - a_j^\dagger a_i$, and the remaining generators are given by the symmetric quadrupole tensor $a_i^\dagger a_j + a_j^\dagger a_i - \frac{1}{3}H\delta_{ij}$. The unitary transformation to the monopole state leaves invariant the rotation group SO(3) that now includes a piece from $eg\hat{\mathbf{r}}$ and U(1) electromagnetism. The remaining algebra forms an induced representation of $\mathrm{SO(3)}\times\mathrm{U(1)}$. In the case of the point monopole, the representation is generated by the velocities and all their products. In any case, for constant energy, we have level sets S^5 in the total phase space, R^6, of momenta and positions. The Hamiltonian generates a circle action whose quotient is CP^2. However, there is also a circle action by the gauge group under which the Hamiltonian and the velocities, but not the momenta are invariant. The latter circle action can be identified with the nontrivial U(1) bundle which was described simplicially in Sec. I. Note that one can also consider the Hamiltonian for a free particle in the background magnetic monopole. This gives level sets $\mathrm{CP}^1=S^2$ which appear as the zero oscillator frequency limit. The additional terms are apparent when we consider the products of the U's as in

$$U(\mathbf{x})U(\mathbf{y})=\exp\left[i\int_{S_{\mathbf{x},\mathbf{y}}}e/\hbar B_j dS^j\right]U(\mathbf{x}+\mathbf{y})\ , \tag{24}$$

where S is the plane surface determined by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x}+\mathbf{y}$. If the U's determine a ray representation of the translation group, they certainly associate

$$U(\mathbf{x})[U(\mathbf{y})U(\mathbf{z})] =[U(\mathbf{x})U(\mathbf{y})]U(\mathbf{z}) \tag{25}$$

$$= U(\mathbf{x})\left[\exp\left[i\int_{S_{\mathbf{y},\mathbf{z}}}\frac{e}{\hbar}\mathbf{B}\cdot d\mathbf{S}\right]\right]U(\mathbf{y}+\mathbf{z})$$

$$=\exp\left[i\int_{S_{\mathbf{x},y}}\frac{e}{\hbar}\mathbf{B}\cdot d\mathbf{S}\right]U(\mathbf{x}+\mathbf{y})U(\mathbf{z}) \tag{26a}$$

$$=\exp\left[i\int_{S_{\mathbf{y},\mathbf{z}}}+\int_{S_{\mathbf{x},\mathbf{y}+\mathbf{z}}}\frac{e}{\hbar}\mathbf{B}\cdot d\mathbf{S}\right]U(\mathbf{x}+\mathbf{y}+\mathbf{z}) \tag{26b}$$

$$=\exp\left[i\int_{S_{\mathbf{x},\mathbf{y}}}+\int_{\mathbf{x}+\mathbf{y},z}\frac{e}{\hbar}\mathbf{B}\cdot d\mathbf{S}\right]U(\mathbf{x}+\mathbf{y}+\mathbf{z}) \tag{26c}$$

which is equivalent to the statement that

$$e\int_V \nabla\cdot\mathbf{B}\,dV=2\pi n\hbar\ , \tag{27}$$

V a volume enclosing the magnetic monopole. [For flux tube, $\nabla(\nabla\cdot\mathbf{B})\neq 0$, so one can integrate over a loop in R^3 as well as over the volume since there is a length scale. Higher gradients correspond to more complicated fluxes like cylinders.]

However, when we consider the algebra of velocities, we are truncating w_i. Therefore, the fact that the velocities have an infinite number of nonvanishing commutators is equivalent to the fact that in order to relate p_i to w_i by a unitary transformation, one must include an infinite number of terms. If, however, one approximates U(1) by torsion matrices, i.e., matrices such that $A^N=1$ for some w, then the series will terminate. This is an interesting case because one might consider U(1) as a limit as N goes to infinity of Z_N. For finite N, only a finite number of relations appear as infinite group theory where one can define certain large groups as automorphisms of algebras. As an example, one might have an algebra where the Jacobi identity is not satisfied, but some multiple of the triple product does vanish (mod$N$, for example).

There are interesting complications one can consider. First, one might look at the angular momentum algebra. Classically, one knows that the algebra for $L_i=\epsilon_{ijk}r_jv_k$ does not close. However, $J_i=L_i-(1/r)egr_i$ does. It is once more interesting to consider a projection onto the algebra, generated by L_i, that does not close. One has

$$[L_i,L_j]=i\epsilon_{ijk}L_k-ie\epsilon_{ilm}\epsilon_{jrs}r_lr_jF_{ms}\ . \tag{28a}$$

But

$$\epsilon_{ilm}\epsilon_{jrs}r_lr_jF_{ms}=F_{ij}r^2 \tag{28b}$$

for the magnetic monopole, so we must extend by r^2F_{ij}. The triple commutator one checks to satisfy the Jacobi identity fails by a term which vanishes classically, but does not vanish when the positions and velocities are considered as operators. In particular if one perturbs by additional interactions, the wave function may depend upon the velocity vectors. The possible failure of the Jacobi identity is thus once more determined by considering the motion in the fiber (dependence on the gauge group) but now inconsistent with the equations of motion:

$$\epsilon^{ijk}[L_i,[L_j,L_k]]=-ie(r_i\partial_jF_{ij}-r_j\partial_iF_{ij})r^2\ . \tag{29}$$

As long as the motion in the fiber is trivial topologically, there are no further obstructions to form a group of rotations and the Jacobi identity is satisfied.

Additional interactions may cause the Jacobi identity to fail as an operator statement because $r_i=-i\partial/\partial v_i$. One could even consider F_{ij} as a dynamical variable which can be excited.[17] One is then no longer in the realm of quantum mechanics. The algebra one considers is now closer to the algebra for the angular momenta in the string theories where there are an infinite number of oscillator states corresponding to the excitations of the magnetic monopole. In terms of the group, one now must consider not only straight lines, but lines with attached loops which bound surfaces through which flux flows. For this reason, it is not clear whether this algebra represents a string theory or a membrane theory. The latter is plausible because the excitations are stringlike solitons wrapped n times around the equator of a two-dimensional disc. Such a membrane may explain why space is three dimensional.

We have stated that for a one-particle system, the third cohomology of the group is the highest-dimensional cohomology of significance. However, when we consider systems with a large number of particles, there are higher-dimensional cells that are possibly incontractible since the available coordinate space is $3N$ dimensional for an N-dimensional system. In particular, if we consider the quantum-mechanical interaction of monopoles or dyons among themselves, then there are additional possible structures. This is related to the surfaces in which motion can occur. The simplest question one can ask is whether volumes determined by motion of two particles separately, each encloses a different unit of flux. This is determined by a cohomological product of three forms which could possibly determine a six-cocycle. This product is antisymmetric so the number obtained from the six-cocycle depends upon the order in which one measures the flux. Of course, if the flux satisfies the Dirac quantization condition, the difference is unobservable at the group level. There is the suggestion that cohomology will be useful for considering quantum-mechanical correlations in general.

In the product space of motion, there is a well-known five-form that determines the angular momentum of a pair of dyons

$$\mathbf{L}=\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\ . \tag{30}$$

If one integrates this form over the product space of motion of the two dyons as well as the one-dimensional space of angular motion, one obtains a number. If the angular motion is to determine a finite-dimensional representation of the rotation group, this number fixes the quantization of angular momentum for dyons of electric and magnetic charges (e_i,g_i) $i=1,2$ to satisfy

$$e_1g_2-e_2g_1=\frac{n}{2}\hbar\ . \tag{31}$$

There are interesting geometries that can arise by considering several dyons,[18] especially when the total magnetic charge is zero.

The analysis of the defect structures of a simple system in nature leads to a physical interpretation of a nonassociative extension of an algebra. Consider biaxial nematics[19] with isometry group $G=\mathrm{SU}(2)$ and isotropy group $H=Q$, the quaternion group. The defects are classified by the conjugacy classes of $\pi_1(G/H)=Q$, which is non-Abelian. Given two classes, α and β of $\pi_1(G/H)$, we can consider the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$. For the quaternion group, if α and β are noncommuting 180° disinclinations of distinct types represented by $i\sigma_x$ and $i\sigma_y$, for example, then the commutator is -1, a 360° disinclination. The commutator is considered in order to see the effect of trying to cross one defect with another. We see that there is an obstruction to crossing of the defects $i\sigma_x$ and $i\sigma_y$ as represented by the 360° defect -1. Just as in the case of the magnetic monopole, we can consider an extension of the group represented by the Z_2 obstruction. The new representation considers defects as pairs with one term an element of Q and the other an element of Z_2. However, this determines a nonassociative extension.

Consider (α,a), (β,b) for $\alpha,\beta\in Q, ab\in Z_2$,

$$(\alpha,a)\cdot(\beta,b)=(\alpha\beta,a+b+C(\alpha,\beta))\ , \tag{32}$$

where C is determined by the rule $C(\alpha,\beta)=0$ if $\alpha\beta$ is real and 1 otherwise (the algebra is associative if $C=1$ for $\beta=1$ or $\alpha=1$):

$$((\alpha,a)\cdot(\beta,b))\cdot(\gamma,d)$$
$$=(\alpha\beta\gamma,a+b+d+C(\alpha,\beta)+C(\alpha\beta,\gamma))\ , \tag{33a}$$

$$(\alpha,a)\cdot((\beta,b)\cdot(\gamma,d))$$
$$=(\alpha\beta\gamma,a+b+d+C(\beta,\gamma)+C(\alpha,\beta\gamma))\ . \tag{33b}$$

But

$$C(\alpha,\beta)+C(\alpha\beta,\gamma)-C(\beta,\gamma)-C(\alpha,\beta\gamma)\neq 0\ . \tag{34}$$

Consider $\alpha=i\sigma_x$, $\beta=i\sigma_x$, $\gamma=i\sigma_y$, for example. We therefore have a nonassociative extension. This example suggests a physical interpretation of a nonassociative representation. One is considering an extension of the symmetry group for an ordered phase by a disorder variable. The ordered phase is not stable to all defect structures. Therefore, nonassociativity of the representation may mean that we are using order variables for a disordered phase. It is crucial in this consideration that the extension was by a central element, i.e., one that commutes with all the elements in a non-Abelian algebra. One could make the algebra Abelian and restore associativity by modding out a Z_2 in the quaternion part of the algebra.

The dynamics of the movement of defects can be reversed in time. Start with the defect described by $(-1,0)=(i\sigma_y,0)(i\sigma_y,0)$. In the presence of an 180° disinclination represented by $i\sigma_x$, we can transport $(i\sigma_y,0)$ to $(-i\sigma_y,0)$ since

$$(i\sigma_x,0)(i\sigma_y,0)(-i\sigma_x,0)=(-i\sigma_y,0) . \tag{35}$$

However,

$$(-i\sigma_y,0)(i\sigma_y,0)=(1,0)\neq(-1,0) . \tag{36}$$

Defects can catalyze changes in the obstruction. This is related to the fact that the extended algebra has zero divisors. It is suggestive of considering the new variables as fermionic.

We can also consider the higher-order relations for the C's:

$$(1)\quad C(\alpha,\beta)+C(\gamma,\delta)+C(\alpha\beta,\gamma\delta) , \tag{37a}$$

$$(2)\quad C(\alpha,\beta)+C(\alpha\beta,\gamma)+C(\alpha\beta\gamma,\delta) , \tag{37b}$$

$$(3)\quad C(\beta,\gamma)+C(\alpha,\beta\gamma)+C(\alpha\beta\gamma,\delta) , \tag{37c}$$

$$(4)\quad C(\beta,\gamma)+C(\beta\gamma,\delta)+C(\alpha,\beta\gamma\delta) , \tag{37d}$$

$$(5)\quad C(\gamma,\delta)+C(\beta,\gamma\delta)+C(\alpha,\beta\gamma\delta) . \tag{37e}$$

We therefore can satisfy a fourth-order condition for this algebra since C is an element of Z_2.

We can also consider an SU(3) example by looking at the subgroup generated by

$$A=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & e^{i\theta}\\ 0 & -e^{-i\theta} & 0\end{bmatrix} ,\quad B=\begin{bmatrix}e^{i\phi} & 0 & 0\\ 0 & e^{i\phi} & 0\\ 0 & 0 & e^{-2i\phi}\end{bmatrix} , \tag{38a}$$

$$AB=CBA,\quad A^2B=BA^2 , \tag{38b}$$

$$C=\begin{bmatrix}1 & 0 & 0\\ 0 & e^{-3i\phi} & 0\\ 0 & 0 & e^{3i\phi}\end{bmatrix} . \tag{38c}$$

In general $C\neq1$ unless $\phi=2\pi N/3$ when A and B commute. However, if $\theta=2\pi N/3K$ one has

$$(BAB^{-1}A^{-1})^K=1 . \tag{38d}$$

[One can obtain a U(1) in the limit $K\rightarrow\infty$. If one also restricts the range of $\theta=2\pi N/M$, one obtains a discrete subgroup of SU(3) with A torsion, $A^{2M}=1$.] Therefore, if A and B represent two-cochains, we can extend by Z_K to obtain a nonassociative algebra just as we extend by Z_2 with Z_K represented by C. This is a consequence of the relation (38d).

In the new versions of string theory, there is a two-form potential which determines a closed three-form H (Ref. 20). This three-form is crucial in order to cancel the anomalies in the theory. H has an equation of motion

$$\partial_\mu H_{\mu\alpha\beta}=0 . \tag{39a}$$

This leads to an axion in four dimensions:[21]

$$Y^\mu=\epsilon^{\mu\alpha\beta\gamma}H_{\alpha\beta\gamma} , \tag{39b}$$

$$\partial^\mu Y^\nu-\partial^\nu Y^\mu=0 , \tag{39c}$$

$$Y^\mu=\frac{1}{M}\partial^\mu\phi . \tag{39d}$$

However, we see that the above equation of motion does not have to be satisfied exactly, but may be true up to homotopy. For example, there may be a source term for $H_{\mu\alpha\beta}$ (Ref. 22) if there is dependence on the extra dimensions:

$$\partial_\mu H^{\mu\alpha\beta}=\rho^{\alpha\beta} , \tag{40a}$$

$$\partial_\alpha Y_\beta-\partial_\beta Y_\alpha=\epsilon_{\alpha\beta\mu\nu}\rho^{\mu\nu} . \tag{40b}$$

Hence Y^μ cannot be the gradient of a scalar field ϕ. However, Y^μ can be considered as a vector potential for a connection that determines a translation. This is a connection for an anomalous axial (by definition of Y_μ) U(1) gauge symmetry. Even though H has a source, there is still a consistency condition on the derivatives. We find that the consistency condition for H is equivalent to the vanishing of a magnetic source for the anomalous U(1):

$$\partial_\alpha\partial_\beta H^{\alpha\beta\gamma}=\partial_\alpha\rho^{\alpha\gamma}=\epsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\beta\delta} , \tag{41a}$$

$$F_{\alpha\beta}=\partial_\alpha Y_\beta-\partial_\beta Y_\alpha . \tag{41b}$$

However, the U(1) field has a conserved current which is not gauge invariant. The variation of this current under gauge transformations Λ that do not vanish at infinity are related to this same magnetic source:

$$\delta J^0\propto\epsilon^{oijk}\partial_i\Lambda F_{jk} , \tag{42}$$

yielding a flux at infinity proportional to the magnetic charge of a monopole. Since there is no restriction on the value of the large gauge transformation at infinity, there is an obstruction to obtaining an associative representation of the translation group with a background nontrivial field in the anomalous U(1) theory. In the case of the magnetic monopole in quantum mechanics, the Dirac quantization condition imposed a restriction of all real values of $2eg/\hbar$ to the integers with quotient space the circle of topologically trivial gauge transformations. For axial U(1) there is an analogous quantization condition only if the large gauge transformations are restricted. We also note that (18a) is not zero for the translation group in four dimensions with quantum operators. A fifth-order relation may then be responsible for restricting the large gauge transformations parameterized by a fifth coordinate.

Finally, we note that quantum field theories and string theories have quantization conditions which can appear in the form of anomaly cancellation conditions. Perturbation theory is usually calculated about a fixed background field in such a way that the perturbative corrections are assumed not to change the topology of the background field, whether this is trivial or nontrivial. Therefore, the most the perturbative corrections can do to change the topology is add an exact form to a closed form. If one integrates about a chain and not a cycle, one must be careful to choose the exact form consistent with the quantization condition. A failure to do so can lead to problems like infinities or nonassociativity of a representation of an alleged symmetry. Sometimes the consistency must be checked a finite number of times, sometimes an infinite number. One can violate the consistency condition for any particular number of commutators only by adding an additional cell that is related to the original space by a topologically nontrivial map.

The author is grateful to Professor James Stasheff for numerous encouraging and illuminating conversations. The author is also grateful to George Chapline for encouraging him to continue to understand the three-cocycle. Conversations with S. Deser, R. Jackiw, L. Alvarez-Gaumé, A. Niemi, and B. Zumino are also gratefully acknowledged. Finally, I thank J. Cuozzo, M. Evans, M. Grossi, N. N. Khuri, C. Rhodes, and A. I. Sanda for their immeasurable contributions to this work in the face of the possible failure of associativity. This work was supported in part under Department of Energy Contract Grant No. De AC02-81ER400338.

---

[1]B. Grossman, Phys. Lett. **152B**, 93 (1985); R. Jackiw, Phys. Rev. Lett. **54**, 159 (1985); Phys. Lett. **154B**, 303 (1985); Y. S. Wu and A. Zee, *ibid.* **152B**, 98 (1985); B. Y. Hou and B. Y. Hou, Chin. Phys. Lett. (to be published).

[2]D. Boulware, S. Deser, and B. Zumino, Phys. Lett. **153B**, 307 (1985). These authors were the first to point out that our understanding of the three-cocycle was incomplete. See also J. Mickelson, Phys. Rev. Lett. **54**, 2379 (1985). For a review, see R. Jackiw, Report No. CTP 1268 MIT, 1985 (unpublished). L. Baulieu and B. Grossman, Nucl. Phys. **B264**, 317 (1986).

[3]M. Günaydin and B. Zumino, LBL Report No. LBL-19200 UCB-PTH 85/8, 1985 (unpublished).

[4]P. A. M. Dirac, Proc. R. Soc. London **A133**, 60 (1981).

[5]An analysis similar to that in Eqs. (2)—(4) applies to the third cocycle in the group cohomology of non-Abelian gauge theories (Ref. 6). The third cocycle is determined by $\omega^{2,3}$, which is a two-form in space with differential d and a three-form on the gauge group with Becchi-Rouet-Stora (BRS) differential s:

$$\int_{S^2}\omega^{2,3}=\int_{D^3}d\omega^{2,3}=-\int_{D^3}s\omega^{3,2}\ .$$

The first term is evaluated over the two-sphere at infinity for the open space D^3. This determines a flux, which suggests the use of the η invariant to measure it (Ref. 7). The last term determines the group cohomological cocycle, as in (4a), after integration over the group. For more complicated configurations, like a circular flux tube, one integrates over more complicated simplices, like a four-simplex to obtain a trivial four-cocycle. See Ref. 8, Baulieu and Grossman in Refs. 2 and 6.

[6]B. Grossman, Phys. Lett. **160B**, 94 (1985).

[7]M. F. Atiyah, V. Patodi, and I. Singer, Math. Proc. Cambridge Philos. Soc. **79**, 71 (1976); E. Witten, Phys. Lett. **117B**, 324 (1982); A. Niemi and G. Semenoff, Phys. Rep. (to be published); B. Grossman, Phys. Rev. Lett. **50**, 664 (1983); **51**, 959 (1983); H. Yamagishi, Phys. Rev. D **27**, 2383 (1983).

[8]K. S. Brown, *Cohomology of Groups* (Springer, Berlin, 1982); L. Faddeev, Phys. Lett. **145B**, 81 (1984); B. Zumino, Nucl. Phys. **B253**, 477 (1985); O. Alvarez, Commun. Math. Phys. **100**, 279 (1985).

[9]R. Bott and L. Tu, *Differential Forms in Algebraic Topology* (Springer, Berlin, 1982).

[10]An alternate regularization procedure for the δ function indicates a resolution of the problem of the domain for the quantum-mechanical velocities. Namely, consider

$$\delta(r)dr=\lim_{\sigma\to 0}\frac{1}{\sqrt{\pi\sigma}}e^{-r^2/2\sigma}dr$$
$$=e^{-y^2/2}dy$$
$$=\frac{1}{\pi}d\chi$$

for $y=1/\sqrt{\sigma}r$. Therefore, we see contributions to the variation in χ coming from the quantum-mechanical oscillators represented by the Gaussian. If we define the magnetic monopole as a perturbation of the oscillator system, we will effectively obtain such a Gaussian measure. Moreover, we note that $-i\,d/dr$ is not a self-adjoint operator on $[0,\infty]$ with Lebesgue measure; there is only one normalizable eigenvector, e^{-r}, with imaginary eigenvalue. However, if we restrict the domain to the harmonic-oscillator wave functions that have Gaussian falloff, it is self-adjoint. Moreover, the Hamiltonian is now symmetric in x and v:

$$H=\tfrac{1}{2}\mathbf{v}^2+\tfrac{1}{2}\mathbf{x}^2\ .$$

Note that the symmetry of the unperturbed oscillator is U(3). The symmetry unbroken by the magnetic monopole is SO(3) of rotations and U(1) gauge invariance. Therefore, there are five broken generators determined by the associated bundle $S^3\times R^3/\mathrm{U}(1)\times Z_2$. The U(1) acts diagonally as the fiber for the Hopf fibration of S^3 and as a noncompact translation in R/Z_2. This allows the identification of the radial coordinate with the fiber and is the origin of the noncompact covering of S^1 by R.

[11]P. Forgacs and N. Manton, Commun. Math. Phys. **72**, 1 (1980).

[12]H. Lipkin, W. Weisberger, and M. Peshkin, Ann. Phys. (N.Y.) **53**, 203 (1969); N. Cabibbo and E. Ferrari, Nuovo Cimento **23**, 241 (1962); J. Schwinger, Phys. Rev. **144**, 1087 (1966).

[13]R. Schafer, *An Introduction to Non-associative Algebras* (Academic, New York, 1966).

[14]Günaydin and Zumino in Ref. 3 point out that Malcev algebras satisfy a fourth-order relation equivalent to $\nabla(\nabla\cdot\mathbf{B})=0$.

There is also a fourth-order relation in finite group theory called Teichmüller condition for nonassociative algebras. See Brown in Ref. 8.

[15]J. D. Stasheff, Trans. Am. Math. Soc. **108**, 275 (1963); J. F. Adams, *Infinite Loop Spaces,* Annals of Mathematics Studies (Princeton University Press, Princeton, New Jersey, 1978).

[16]A. Hochschild, *The Structure of Lie Groups* (Holden-Day, San Francisco, 1965).

[17]Also considered by B. Y. Hou and B. Y. Hou, Chin. Phys. Lett. (to be published).

[18]B. Grossman (in preparation).

[19]V. Poenaru and G. Toulouse, J. Phys. (Paris) **8**, 887 (1977); D. Mermin, Rev. Mod. Phys. **51**, 591 (1977).

[20]M. Green and J. Schwarz, Phys. Lett. **149B**, 117 (1984).

[21]E. Witten, Phys. Lett. **149B**, 351 (1984).

[22]R. Nepomechie, University of Seattle report (unpublished).