Correlation functions for homogeneous, isotropic random classical electromagnetic radiation and the electromagnetic fields of a fluctuating classical electric dipole

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The two-point field correlation functions for homogeneous and isotropic random (Gaussian) classical electromagnetic radiation are shown to be related to the electromagnetic fields of a fluctuating electric dipole. The relationships derived between these two quantities are useful in calculations involving classical electric dipole oscillators bathed in classical electromagnetic radiation. Using these relationships, the van der Waals force is evaluated for a harmonic dipole oscillator that is a member of an arbitrary configuration of N oscillators, all of which are bathed in thermal plus zero-point classical electromagnetic radiation. Also, the expectation value of the Poynting vector is shown to be unchanged from its null value when a classical harmonic dipole oscillator is included within a classical electromagnetic isotropic and homogeneous random radiation field. A sketch is given as to how these calculations may be carried over to the case of a system of harmonic dipole oscillators uniformly accelerating through classical electromagnetic zero-point radiation. What enables these calculations to be extended to the situation of acceleration through zero-point radiation is a recent finding that the two-point field correlation functions, evaluated along trajectories described by uniform acceleration through classical electromagnetic zero-point radiation, are related to the electromagnetic fields of a uniformly accelerating electric dipole.

I. INTRODUCTION

A number of calculations have been performed within the context of classical electrodynamics that agree with the results of quantum electrodynamics. This agreement occurs provided a nonzero homogeneous solution to Maxwell's equations is assumed to exist in the form of random electromagnetic radiation that is present even when the temperature of the radiation equals zero. Assuming the stochastic properties of this classical electromagnetic zero-point radiation to be that of a Gaussian process in the fields, then the demands of isotropy, homogeneity, and Lorentz invariance result in the functional form of the radiation's spectrum being uniquely specified up to a multiplicative constant. Comparison with experiment then yields the numerical value for this multiplicative constant, which is found to agree with Planck's constant. Hence, it is in this manner that Planck's constant enters into this classical electrodynamical theory. The name frequently given to this classical theory is stochastic electrodynamics. (For reviews on this field of research, see Refs. 1–4.)

The electrodynamical system that has received the most attention within stochastic electrodynamics has been the charged harmonic oscillator. The equation of motion for this system is a linear stochastic differential equation; hence, the steady-state solution is readily obtained by the use of Fourier transforms. Although the physical interpretation for the behavior of such a system is markedly different from that of quantum electrodynamics, most of the physically observable statistical properties of this system agree between the two theories of stochastic and quantum electrodynamics. Perhaps the most remarkable agreement has been found in the case of the van der Waals force between two nonrelativistic charged harmonic oscillators, each taken in the electric dipole limit; the force expressions calculated within both theories at temperature $T=0$ agree for all distances between the two oscillators and to all orders in the electronic charge.[5]

Other electrodynamical systems, such as the classical hydrogen atom,[6] have not been so successfully tackled within stochastic electrodynamics; in most cases, the results obtained for these systems have not agreed with physical observation. Such systems are described by nonlinear stochastic differential equations. It remains unclear whether the basic theory of stochastic electrodynamics is an incorrect description of nature or whether the difficult mathematics of these nonlinear systems have simply not been solved with sufficient accuracy.

At this point in time, perhaps the most appropriate viewpoint of the theory of stochastic electrodynamics is that, at the very least, it offers alternative means for calculating certain quantities within quantum electrodynamics. In some instances, as in the case of van der Waals forces[5,7] or of the thermal effects of electromagnetic dipole systems accelerating through the so-called vacuum,[8–11] there exist calculational advantages of stochastic electrodynamics to the more traditional calculational methods of quantum electrodynamics.

In Sec. II of this article, the two-point correlation function of the electromagnetic radiation fields in stochastic electrodynamics are evaluated at fixed spatial points within an inertial reference frame. This calculation has certainly been done before; here, however, the correlation functions are shown to be expressible in terms of the electromagnetic fields radiated by a fluctuating electric dipole.[12] The use of this functional form for the correlation functions simplifies many of the calculations that have been performed previously within the context of stochastic electrodynamics; in particular, calculations involving the charged harmonic oscillator, taken in the electric dipole limit, become much more efficient and tractable. The calculations of Secs. III and IV illustrate this point.

Recently, the correlation functions of the classical electromagnetic zero-point fields were calculated along spatially separated trajectories described by relativistic uniform acceleration.[11] These correlation functions were shown to be related to the electromagnetic fields of a uniformly accelerated electric dipole. Hence, this relationship may be viewed as an extension of the unaccelerated case discussed in the present article. Thus, there exists a second advantage, besides greater simplicity, for recasting previous calculations that have been done in stochastic electrodynamics in such a way that the functional form of the correlation functions in Sec. II are utilized: namely, in many cases, without too much additional work, the calculations for nonaccelerating electromagnetic systems may then be extended to the corresponding situation of electromagnetic systems uniformly accelerating through classical electromagnetic zero-point radiation. Such is the case for the calculations presented in Secs. III and IV.

In Sec. III, the expressions obtained for the correlation functions in Sec. II are used to compute the van der Waals force acting on a single harmonic dipole oscillator that is a member of an arbitrary configuration of N oscillators. This calculation for N oscillators generalizes the work of Refs. 5 and 7 for two oscillators. The case where the temperature of the electromagnetic radiation equals zero has previously been carried out for an N-oscillator system in quantum electrodynamics (see Ref. 13); the results found in the present article, via the means of stochastic electrodynamics, agree exactly with the results of this particular work.

By using the functional form found in Sec. II for the correlation functions of the electromagnetic radiation fields, the force calculation of the N-oscillator system in Sec. II may be extended to the case of a system of N transversely positioned oscillators that are uniformly accelerated through classical electromagnetic zero-point radiation. This extension may be carried out by combining the work of Ref. 11, which calculates the expectation value of the force between a pair of transversely positioned accelerating charged oscillators, along with the calculation of Sec. III, which gives the expectation value of the force acting on one charged oscillator of an N-oscillator nonaccelerating system.

Section IV of this article repeats a calculation presented in Appendix B of Ref. 1 that proves the expectation value of the Poynting vector equals zero at a point in space near a stationary harmonic dipole oscillator. The difference between the proof given in Ref. 1 and the proof presented here, however, is that the proof of the present article uses the functional form obtained in Sec. II for the correlation functions of the classical electromagnetic radiation fields. Explicitly using this functional form demonstrates that the proof can be carried out due to the relationships found in Sec. II between the correlation functions of electromagnetic radiation and the electromagnetic fields of a fluctuating electric dipole. The original proof did not identify this fact. Moreover, the explicit use of these relationships then enables this proof to be extended to the case of a single accelerating charged harmonic oscillator. This extension is briefly discussed in Sec. IV and sketched more fully in the Appendix.

II. CORRELATION FUNCTIONS OF CLASSICAL ELECTROMAGNETIC RADIATION FIELDS

The functional form for the electromagnetic radiation fields that is often used for performing explicit calculations within stochastic electrodynamics consists of the following expressions:[14]

$$\mathbf{E}^{\text{in}}(\mathbf{x},t)=\sum_{\lambda=1}^{2}\int d^3k\, h_{\text{in}}(\omega)\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)\times\cos[\mathbf{k}\cdot\mathbf{x}-\omega t+\theta(\mathbf{k},\lambda)]\ , \tag{1}$$

$$\mathbf{B}^{\text{in}}(\mathbf{x},t)=\sum_{\lambda=1}^{2}\int d^3k\, h_{\text{in}}(\omega)[\hat{\mathbf{k}}\otimes\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)]\times\cos[\mathbf{k}\cdot\mathbf{x}-\omega t+\theta(\mathbf{k},\lambda)]\ . \tag{2}$$

Thus, the radiation fields are expressed here as a sum of plane waves; hence, they satisfy Maxwell's equations in free space. The phase angle $\theta(\mathbf{k},\lambda)$ is treated here as a random variable that takes on values between 0 and 2π with uniform probability density. For each value of $\mathbf{k}$ and λ, $\theta(\mathbf{k},\lambda)$ is independently distributed. The polarization vectors $\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)$ satisfy the relationships of

$$\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)\cdot\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda')=\delta_{\lambda\lambda'}\ , \tag{3}$$

$$\mathbf{k}\cdot\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)=0\ , \tag{4}$$

which can be used to show that

$$\sum_{\lambda=1}^{2}\epsilon_i(\mathbf{k},\lambda)\epsilon_j(\mathbf{k},\lambda)=\sum_{\lambda=1}^{2}[\hat{\mathbf{k}}\otimes\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)]_i[\hat{\mathbf{k}}\otimes\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)]_j=\delta_{ij}-\frac{k_ik_j}{k^2}\ , \tag{5}$$

$$\sum_{\lambda=1}^{2}\epsilon_i(\mathbf{k},\lambda)[\hat{\mathbf{k}}\otimes\hat{\boldsymbol{\epsilon}}(\mathbf{k},\lambda)]_j=\sum_{l=1}^{3}\epsilon_{ijl}\frac{k_l}{k}\ . \tag{6}$$

The frequency ω in Eqs. (1) and (2) is defined by $\omega=c\,|\mathbf{k}|$. When the electromagnetic fields of Eqs. (1) and (2) belong to a thermal radiation field described by the temperature T, then the quantity $h_{\text{in}}(\omega)$ will be denoted by $h_T(\omega)$, where

$$h_T{}^2(\omega)=\frac{\hbar\omega}{2\pi^2}\coth\left[\frac{\hbar\omega}{2kT}\right]$$
$$=\frac{1}{\pi^2}\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\left[\exp\left[\frac{\hbar\omega}{kT}\right]-1\right]}\right] . \quad (7)$$

When $T=0$, then Eqs. (1) and (2) constitute the classical electromagnetic zero-point fields. The quantity $h_{T=0}(\omega)$ will be abbreviated by $h(\omega)$. From Eq. (7),

$$h_{T=0}{}^2(\omega)=h^2(\omega)=\frac{\hbar\omega}{2\pi^2} . \quad (8)$$

The two-point correlation functions of the fields in Eqs. (1) and (2) will now be evaluated. Using the probability distribution described earlier for the random variables $\theta(\mathbf{k},\lambda)$, one can show that

$$\langle \cos[A+\theta(\mathbf{k}',\lambda')]\cos[B+\theta(\mathbf{k}'',\lambda'')]\rangle$$
$$=\tfrac{1}{2}\delta_{\lambda'\lambda''}\delta^3(\mathbf{k}''-\mathbf{k}')\cos(B-A) , \quad (9)$$

where angular brackets are used here to indicate that the expectation value is to be taken for the quantity within the brackets. Using Eqs. (1), (2), (5), and (9), the following two-point correlation functions may be expressed by

$$\langle E_i^{\text{in}}(\mathbf{x}_1,t_1)E_j^{\text{in}}(\mathbf{x}_2,t_2)\rangle$$
$$=\langle B_i^{\text{in}}(\mathbf{x}_1,t_1)B_j^{\text{in}}(\mathbf{x}_2,t_2)\rangle$$
$$=\tfrac{1}{2}\int d^3k'\,h_{\text{in}}^2(\omega')\left[\delta_{ij}-\frac{k_i'k_j'}{k'^2}\right]$$
$$\times\cos[\mathbf{k}'\cdot(\mathbf{x}_2-\mathbf{x}_1)-\omega'(t_2-t_1)] . \quad (10)$$

The rotation matrix $[O]$ will now be introduced such that

$$\mathbf{R}=(\mathbf{x}_2-\mathbf{x}_1)=[O]\hat{\mathbf{z}}R , \quad (11)$$

where $R=|\mathbf{x}_2-\mathbf{x}_1|$. Let θ and ϕ be the polar and azimuthal angles of $(\mathbf{x}_2-\mathbf{x}_1)$. The transformation of Eq. (11) can be accomplished as shown in Fig. 1, which results in an explicit form for $[O]$:

$$[O]=\begin{bmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{bmatrix} . \quad (12)$$

Substituting $\mathbf{k}'=[O]\mathbf{k}$ and Eq. (11) into Eq. (10), then yields

$$\langle E_i^{\text{in}}(\mathbf{x}_1,t_1)E_j^{\text{in}}(\mathbf{x}_2,t_2)\rangle=\tfrac{1}{2}\int d^3k\,h_{\text{in}}{}^2(\omega)\left[\delta_{ij}-\sum_{m,n=1}^{3}O_{im}O_{jn}\frac{k_mk_n}{k^2}\right]\cos[k_3R-\omega(t_2-t_1)] . \quad (13)$$

The cosine term in Eq. (13) can be expanded into two terms using the cosine sum of angles formula. Only

$$\cos(k_3R)\cos[\omega(t_2-t_1)]$$

will remain, since

$$\sin(k_3R)\sin[\omega(t_2-t_1)]$$

results in an integrand odd in k_3 with regard to the δ_{ij} term in the large parentheses and an integrand odd in k_1, k_2, or k_3 for the second term in the parentheses. For a similar reason, the second term in the large parentheses results in a nonzero contribution only when $m=n$. Performing angular integrations, substituting in

$$\sum_{m=1}^{2}O_{im}O_{jm}=\delta_{ij}-O_{i3}O_{j3} ,$$

and realizing from either Eq. (11) or Eq. (12) that $O_{i3}=R_i/R$, yields[15]

$$\langle E_i^{\text{in}}(\mathbf{x}_1,t_1)E_j^{\text{in}}(\mathbf{x}_2,t_2)\rangle=\langle B_i^{\text{in}}(\mathbf{x}_1,t_1)B_j^{\text{in}}(\mathbf{x}_2,t_2)\rangle$$
$$=\int_0^\infty d\omega\cos[\omega(t_2-t_1)]$$
$$\times f_{ij}^{\text{in}}(\mathbf{x}_2-\mathbf{x}_1,\omega) , \quad (14)$$

where

$$f_{ij}^{\text{in}}(\mathbf{R},\omega)=\frac{2\pi h_{\text{in}}{}^2(\omega)}{\omega}\,\text{Im}[\eta_{ij}^D(\mathbf{R},\omega)] , \quad (15)$$

$$\eta_{ij}^D(\mathbf{R},\omega)=k^3\left[\frac{(\delta_{ij}-R_iR_j/R^2)}{kR}+\frac{i(\delta_{ij}-3R_iR_j/R^2)}{(kR)^2}-\frac{(\delta_{ij}-3R_iR_j/R^2)}{(kR)^3}\right]e^{ikR} . \quad (16)$$

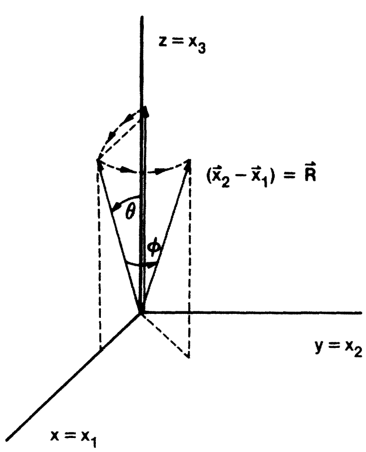

FIG. 1. Transformation that results in the explicit construction of $[O]$ given by Eq. (12).

The quantity $\eta_{ij}^D(\mathbf{R},\omega)$ is intimately related to the electric field $\mathbf{E}^{D\alpha}(\mathbf{R},t)$ of a fluctuating electric dipole $\mathbf{p}_\alpha(t)$. Here, the superscript D on η_{ij}^D and $\mathbf{E}^{D\alpha}$ stands for "dipole"; the index α on $\mathbf{E}^{D\alpha}$ and $\mathbf{p}_\alpha$ will be used to label a particular electric dipole under discussion. Let $\mathbf{R}_\alpha$ represent the position of $\mathbf{p}_\alpha(t)$. From the formulas of standard textbooks,[16] one can readily show that

$$E_i^{D\alpha}(\mathbf{R},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\times\left[\sum_{j=1}^{3}\eta_{ij}^{D}(\mathbf{R}-\mathbf{R}_\alpha,\omega)\tilde{p}_{\alpha j}(\omega)\right], \quad (17)$$

where

$$p_{\alpha i}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\tilde{p}_{\alpha i}(\omega) . \quad (18)$$

A similar relationship can be obtained between the magnetic field of a fluctuating electric dipole and the two-point correlation function of the radiation fields listed below. The first line in Eq. (19) follows from Eqs. (1), (2), (6), and (9). The second line may be obtained by again introducing the rotation matrix $[O]$, using the cosine sum of angles formula, and recognizing that $k_m\cos(k_3R)$ results in an integrand that is odd in k_m:

$$\langle B_i^{\rm in}(\mathbf{x}_1,t_1)E_j^{\rm in}(\mathbf{x}_2,t_2)\rangle=\tfrac{1}{2}\int d^3k'h_{\rm in}{}^2(\omega')\sum_{l=1}^{3}\epsilon_{jil}\frac{k_l'}{k'}\cos[\mathbf{k}'\cdot(\mathbf{x}_2-\mathbf{x}_1)-\omega'(t_2-t_1)]$$
$$=\tfrac{1}{2}\int d^3k\,h_{\rm in}{}^2(\omega)\sum_{l=1}^{3}\epsilon_{jil}\sum_{m=1}^{3}O_{lm}\frac{k_m}{k}\sin(k_3R)\sin[\omega(t_2-t_1)] . \quad (19)$$

Only the $m=3$ term in Eq. (19) yields an integrand that is even in k_3. After performing angular integrations, one obtains

$$\langle B_i^{\rm in}(\mathbf{x}_1,t_1)E_j^{\rm in}(\mathbf{x}_2,t_2)\rangle=\int_0^{\infty}d\omega\sin[\omega(t_2-t_1)]g_{ij}^{\rm in}(\mathbf{x}_2-\mathbf{x}_1,\omega) , \quad (20)$$

where

$$g_{ij}^{\rm in}(\mathbf{R},\omega)=\frac{2\pi h_{\rm in}{}^2(\omega)}{\omega}\,{\rm Re}[\rho_{ji}^{D}(\mathbf{R},\omega)] , \quad (21)$$

$$\rho_{ij}^{D}(\mathbf{R},\omega)=-k^3\sum_{l=1}^{3}\epsilon_{ijl}\frac{R_l}{R}\left[\frac{1}{kR}+\frac{i}{(kR)^2}\right]e^{ikR} . \quad (22)$$

It should be noted that the order of the i,j indices are reversed on the right- and left-hand sides of Eq. (21). This convention then agrees with results found in Ref. 11, where the two-point field correlation functions were evaluated in a plane uniformly accelerating through classical electromagnetic zero-point radiation.

The quantity $\rho_{ij}^{D}(\mathbf{R},\omega)$ is intimately related to the magnetic field $\mathbf{B}^{D\alpha}(\mathbf{R},t)$ of a fluctuating electric dipole $\mathbf{p}_\alpha(t)$. Again, from the formulas of standard textbooks,[16] one can readily show that

$$B_i^{D\alpha}(\mathbf{R},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\times\left[\sum_{j=1}^{3}\rho_{ij}^{D}(\mathbf{R}-\mathbf{R}_\alpha,\omega)\tilde{p}_{\alpha j}(\omega)\right], \quad (23)$$

where $\tilde{p}_{\alpha j}(\omega)$ and ρ_{ij}^{D} are given in Eqs. (18) and (22).

Thus, from Eqs. (14), (15), (20), and (21), the two-point correlation functions of the electromagnetic radiation fields in Eqs. (1) and (2) have been related to the functions η_{ij}^{D} and ρ_{ij}^{D} that appear in the expressions of Eqs. (17) and (23) for the electric and magnetic fields of a fluctuating electric dipole. These relationships will be found useful in performing calculations in stochastic electrodynamics that involve the forced stochastic behavior of an electric dipole due to the electromagnetic radiation fields of Eqs. (1) and (2). Sections III and IV of the present article contain such calculations.

Various symmetry properties can be identified for the two-point correlation functions of Eqs. (14) and (20). Both correlation functions depend only upon the difference in time (t_2-t_1) and the difference in spatial position $(\mathbf{x}_2-\mathbf{x}_1)$. Because of the cosine and sine expansions in Eqs. (14) and (20), the correlation functions of Eqs. (14) and (20) are even and odd functions of (t_2-t_1), respectively. Other properties of the correlation functions that may readily be deduced from earlier equations are

$$f_{ij}^{\rm in}(\mathbf{R},\omega)=f_{ij}^{\rm in}(-\mathbf{R},\omega) , \quad (24)$$

$$f_{ij}^{\rm in}(\mathbf{R},\omega)=f_{ji}^{\rm in}(\mathbf{R},\omega) , \quad (25)$$

$$g_{ij}^{\rm in}(\mathbf{R},\omega)=-g_{ij}^{\rm in}(-\mathbf{R},\omega) , \quad (26)$$

$$g_{ij}^{\rm in}(\mathbf{R},\omega)=-g_{ji}^{\rm in}(\mathbf{R},\omega) . \quad (27)$$

Similar relationships to those of Eqs. (14), (15), (17), (18), (20), (21), and (23) have recently been obtained between the correlation functions of the classical electromagnetic zero-point radiation fields, as calculated along trajectories described by relativistic uniform acceleration, and the electromagnetic fields of a uniformly accelerated electric dipole.[11] The configuration assumed for this calculation consisted of two points located in a plane, where the plane followed a trajectory of uniform acceleration along the normal to the plane. Because of the correspondence between the functional forms of the field correlation functions found in the present section and those found for an accelerating system, many of the calculations performed in an unaccelerated-thermal system can be carried over to a system uniformly accelerating through classical electromagnetic zero-point radiation. This point will be mentioned again briefly in Sec. III and illustrated a bit more clearly in Sec. IV. (Reading Ref. 11 should greatly clarify this point.)

Because of a connection discussed in Ref. 2 between the electromagnetic radiation field correlation functions in stochastic electrodynamics and the corresponding expectation value of electric and magnetic field operators in quantum electrodynamics, the relationships of Eqs. (14) and (20) may be immediately carried over to quantum electrodynamics. Let

$$[\underline{a},\underline{b}]_{+}=\underline{a}\,\underline{b}+\underline{b}\,\underline{a}\ , \tag{28}$$

where $\underline{a}$ and $\underline{b}$ are underlined to denote quantum-mechanical operators. Then, from Ref. 2 and Eqs. (14) and (20),

$$\begin{aligned}\langle 0|\tfrac{1}{2}[\underline{E}_i(\mathbf{x}_1,t_1),\underline{E}_j(\mathbf{x}_2,t_2)]_{+}|0\rangle &= \langle 0|\tfrac{1}{2}[\underline{B}_i(\mathbf{x}_1,t_1),\underline{B}_j(\mathbf{x}_2,t_2)]_{+}|0\rangle\\ &=\int_0^\infty d\omega\cos[\omega(t_2-t_1)]f_{ij}^{\text{zp}}(\mathbf{x}_2-\mathbf{x}_1,\omega)\ ,\end{aligned} \tag{29}$$

$$\langle 0|\tfrac{1}{2}[\underline{B}_i(\mathbf{x}_1,t_1),\underline{E}_j(\mathbf{x}_2,t_2)]_{+}|0\rangle =\int_0^\infty d\omega\sin[\omega(t_2-t_1)]g_{ij}^{\text{zp}}(\mathbf{x}_2-\mathbf{x}_1,\omega)\ . \tag{30}$$

Here, $\underline{E}_i(\mathbf{x},t)$ and $\underline{B}_i(\mathbf{x},t)$ are the electric and magnetic field operators in quantum electrodynamics; Eqs. (29) and (30) represent the vacuum expectation value of the symmetrized products of these operators. The functions f_{ij}^{zp} and g_{ij}^{zp} in Eqs. (29) and (30) are given by Eqs. (15), (16), (21), and (22), with $h_{\text{in}}{}^2(\omega)$ replaced by the function of Eq. (8) that is appropriate for the zero-point radiation field situation. Equations (29) and (30) may be generalized to the situation of a thermal radiation spectrum by replacing the vacuum state $|0\rangle$ on the left-hand sides of Eqs. (29) and (30) by the appropriate incoherent superposition of photon states at temperatures T; in correspondence with this change, the function $h_{\text{in}}{}^2(\omega)$, which occurs in the expressions for f_{ij}^{in} and g_{ij}^{in} of Eqs. (15) and (21), should be replaced by the thermal expression of Eq. (7). (Referring to Ref. 2 should clarify these points.)

III. RETARDED VAN DER WAALS FORCE FOR A SYSTEM OF N CLASSICAL HARMONIC OSCILLATORS

In this section an arbitrary configuration of N-charged classical harmonic oscillators will be considered, where each oscillator will be taken in the electric dipole limit. A thermal plus zero-point electromagnetic radiation field will be assumed to exist, corresponding to the choice of $h_T(\omega)$ for $h_{\text{in}}(\omega)$ in Eq. (7). This radiation field provides the mechanism for the forced steady-state behavior of each oscillator. All oscillators interact with each other via the electromagnetic radiation they emit due to their forced harmonic motion.

The expectation value of the Lorentz force on one of the oscillators will be calculated in this section. From the viewpoint of stochastic electrodynamics, this quantity is simply the van der Waals force. In the process of carrying this calculation through, frequent use will be made of the relationships found in Sec. II of the present article.

The model chosen for each charged harmonic oscillator will be that of a point mass m with charge $+e$ that oscillates under the action of a simple harmonic potential. A convenient model for providing the mechanism for such a potential consists of a spherical uniform distribution of charge, with net value $-e$. If a $+e$ point charge is located within this sphere at a position $\mathbf{x}_\alpha(t)$ from the center of this charge distribution, then the particle will experience a force proportional to the displacement $\mathbf{x}_\alpha(t)$. For a sufficiently small spherical volume of charge distribution and for sufficiently small amplitudes of oscillation of the point particle, the net charge configuration then approximates an electric dipole of value $+e\mathbf{x}_\alpha(t)$.

Under the small oscillator approximation (see Refs. 8 and 10), the equation of motion for one of the N oscillators is given by[17]

$$m\ddot{x}_{\alpha i}=-m\omega_0{}^2x_{\alpha i}+m\Gamma\dddot{x}_{\alpha i}+eE_i^T(\mathbf{R}_\alpha,t)+e\sum_{\beta\neq\alpha}E_i^{D\beta}(\mathbf{R}_\alpha,t)\ , \tag{31}$$

where $i=1,2,3$ and $\alpha=1,2,\ldots,N$. Here, the index α serves as a label to distinguish each of the N oscillators. The quantity $\Gamma=\frac{2}{3}e^2/mc^3$ is the radiation reaction damping constant. The force constant of the harmonic potential is denoted by $m\omega_0{}^2$. The equilibrium position of the αth oscillator is given by $\mathbf{R}_\alpha$ and the displacement from equilibrium by $\mathbf{x}_\alpha$. The electric field $\mathbf{E}^T$ stands for the field of Eq. (1), where $h_{\text{in}}(\omega)$ is replaced by $h_T(\omega)$ in Eq. (7). Finally, $\mathbf{E}^{D\beta}$ represents the electric dipole field of the βth oscillator; hence, the expression for this quantity is given by Eqs. (16)–(18), where $p_{\alpha i}(t)$ in Eq. (18) is approximated by $ex_{\alpha i}(t)$.

In order to solve the linear stochastic differential set of equations indicated by Eq. (31), the following Fourier transforms will be introduced:

$$x_{\alpha i}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\tilde{x}_{\alpha i}(\omega)\ , \tag{32}$$

$$E_i^T(\mathbf{R},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\tilde{E}_i^T(\mathbf{R},\omega)\ . \tag{33}$$

Also, let

$$\eta_{ij}(\mathbf{R},\omega)=-\frac{e^2}{m}\eta_{ij}^D(\mathbf{R},\omega)\ , \tag{34}$$

$$\rho_{ij}(\mathbf{R},\omega)=-\frac{e^2}{m}\rho_{ij}^D(\mathbf{R},\omega)\ . \tag{35}$$

Replacing $\tilde{p}_{\alpha i}(\omega)$ in Eq. (17) by $e\tilde{x}_{\alpha i}(\omega)$, then, from Eqs. (17) and (31)–(34), one obtains

$$C(\omega)\tilde{x}_{\alpha i}(\omega)+\sum_{\beta\neq\alpha}\sum_{j=1}^{3}\eta_{ij}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)\tilde{x}_{\beta j}(\omega)=\frac{e}{m}\tilde{E}_i^T(\mathbf{R}_\alpha,\omega)\ , \tag{36}$$

where

$$C(\omega)=-\omega^2+\omega_0{}^2-i\Gamma\omega^3\ . \tag{37}$$

Equation (36) can be expressed in the form

$$\sum_{\beta=1}^{N}\sum_{j=1}^{3} M_{\alpha i;\beta j}(\omega)\tilde{x}_{\beta j}(\omega)=\frac{e}{m}\frac{\tilde{E}_i^T(\mathbf{R}_\alpha,\omega)}{C(\omega)}\ , \tag{38}$$

where

$$M_{\alpha i;\beta j}(\omega)=\left[\delta_{\alpha\beta}\delta_{ij}+(1-\delta_{\alpha\beta})\frac{\eta_{ij}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)}{C(\omega)}\right]. \tag{39}$$

From Eqs. (32) and (38),

$$x_{\alpha i}(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\exp(-i\omega t)\times\left[\frac{e}{m}\sum_{\beta=1}^{N}\sum_{j=1}^{3}[M^{-1}(\omega)]_{\alpha i;\beta j}\times\frac{\tilde{E}_j^T(\mathbf{R}_\beta,\omega)}{C(\omega)}\right]. \tag{40}$$

Using this solution for the displacement of the αth oscillating particle, the expectation value of the Lorentz force on the αth oscillator can be obtained. Again approximating each oscillator as an electric dipole, the Lorentz force on the oscillator is given by

$$F_{\alpha i}(t)=[e\mathbf{x}_\alpha(t)\cdot\nabla]E_i(\mathbf{x},t)\,|_{\mathbf{x}=\mathbf{R}_\alpha}+\left[\frac{e\dot{\mathbf{x}}_\alpha}{c}\otimes\mathbf{B}(\mathbf{R}_\alpha,t)\right]_i\ , \tag{41}$$

where **E** and **B** represent the total electric and magnetic fields due to the radiation fields of Eqs. (1) and (2) and the dipole fields of all the other $(N-1)$ oscillators. When taking the expectation value of Eq. (41), the relationship of

$$\langle[\dot{\mathbf{x}}_\alpha\otimes\mathbf{B}(\mathbf{R}_\alpha,t)]\rangle=\left\langle\frac{d}{dt}[\mathbf{x}_\alpha(t)\otimes\mathbf{B}(\mathbf{R}_\alpha,t)]\right\rangle+c\langle\mathbf{x}_\alpha(t)\otimes[\nabla\otimes\mathbf{E}(\mathbf{x},t)]\,|_{\mathbf{x}=\mathbf{R}_\alpha}\rangle \tag{42}$$

is helpful, since the first term on the right-hand side of Eq. (42) equals zero, as will be proven shortly. Hence, from Eqs. (41) and (42),

$$\langle F_{\alpha i}^T(t)\rangle=\left\langle\sum_{j=1}^{3}ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t)\right\rangle+\left\langle\sum_{j=1}^{3}ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\left[\sum_{\beta\neq\alpha}E_j^{D\beta}(\mathbf{R}_\alpha,t)\right]\right\rangle\ , \tag{43}$$

where a superscript T has been added in $\langle F_{\alpha i}^T(t)\rangle$ in order to designate the thermal situation at temperature T.

In order to prove that the first term of Eq. (42) equals zero, the operation of taking the time derivative should be interchanged with taking the expectation value. From Eqs. (14), (20), (23), (33), and (40), one can then show that

$$\left\langle\mathbf{x}_\alpha(t)\otimes\left[\mathbf{B}^T(\mathbf{R}_\alpha,t)+\sum_{\beta\neq\alpha}\mathbf{B}^{D\beta}(\mathbf{R}_\alpha,t)\right]\right\rangle$$

is independent of time. Alternatively, one may present a more general argument by physically demanding that the behavior of the set of oscillators constitutes a process that is stationary in time. The expectation value of two quantities connected to the behavior of the oscillators must then depend only upon the difference in the time arguments of the two quantities.

In order to evaluate the first quantity on the right-hand side of Eq. (43), the consideration of the following quantity will prove helpful:

$$\left\langle\tilde{x}_{\alpha j}(\omega')\frac{\partial}{\partial R_{\alpha i}}\tilde{E}_j^T(\mathbf{R}_\alpha,\omega'')\right\rangle=\sum_{\beta=1}^{N}\sum_{l=1}^{3}[M^{-1}(\omega')]_{\alpha j;\beta l}\frac{e}{mC(\omega')}\left\langle\tilde{E}_l^T(\mathbf{R}_\beta,\omega')\frac{\partial}{\partial R_{\alpha i}}\tilde{E}_j^T(\mathbf{R}_\alpha,\omega'')\right\rangle. \tag{44}$$

From the inverse of Eq. (33),

$$\left\langle\tilde{E}_l^T(\mathbf{R}_\beta,\omega')\frac{\partial}{\partial R_{\alpha i}}\tilde{E}_j^T(\mathbf{R}_\alpha,\omega'')\right\rangle=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\exp(i\omega't')\int_{-\infty}^{\infty}dt''\exp(i\omega''t'')\left\langle E_l^T(\mathbf{R}_\beta,t')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t'')\right\rangle. \tag{45}$$

As will be proven shortly,

$$\left\langle E_l^T(\mathbf{R}_\beta,t')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t'')\right\rangle=\int_0^\infty d\omega\cos[\omega(t''-t')]\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})f_{lj}^T(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]\ , \tag{46}$$

where f_{lj}^T is given by Eq. (15), with $h_{\text{in}}(\omega)$ replaced by $h_T(\omega)$. From Eqs. (45) and (46),

$$\left\langle E_l^T(\mathbf{R}_\beta,\omega')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,\omega'')\right\rangle=\pi\int_0^\infty d\omega[\delta(\omega'-\omega)\delta(\omega''+\omega)+\delta(\omega'+\omega)\delta(\omega''-\omega)]\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})f_{lj}^T(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]\ . \tag{47}$$

From Eqs. (16), (34), (37), and (39),

$$C(-\omega)=C^*(\omega)\ , \tag{48}$$

$$\eta_{ij}^D(\mathbf{R},-\omega)=\eta_{ij}^{D*}(\mathbf{R},\omega)\ , \tag{49a}$$

$$\eta_{ij}(\mathbf{R},-\omega)=\eta_{ij}^{*}(\mathbf{R},\omega)\ , \tag{49b}$$

$$M_{\alpha i;\beta j}(-\omega)=M_{\alpha i;\beta j}^{*}(\omega)\ . \tag{50}$$

From Eqs. (44), (47), (48), and (50), one can then show that the first term on the right-hand side of Eq. (43) is given by

$$\left\langle \sum_{j=1}^{3} ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t)\right\rangle=\frac{e^2}{m}\sum_{\beta=1}^{N}\sum_{j,l=1}^{3}\int_0^\infty d\omega\,\mathrm{Re}\left[\frac{[M^{-1}(\omega)]_{\alpha j;\beta l}}{C(\omega)}\right]\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})f_{lj}^T(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]\ . \tag{51}$$

This expression may be put into a more convenient form by the use of the following relationship:

$$\begin{aligned}\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})f_{lj}^T(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]&=\frac{-2\pi h_T{}^2(\omega)}{\frac{e^2}{m}\omega}\frac{\partial}{\partial R_{\alpha i}}\,\mathrm{Im}[\delta_{\alpha\beta}\delta_{lj}C(\omega)+(1-\delta_{\alpha\beta})\eta_{lj}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]\\&=\frac{-2\pi h_T{}^2(\omega)}{\frac{e^2}{m}\omega}\frac{\partial}{\partial R_{\alpha i}}\,\mathrm{Im}[C(\omega)M_{\alpha l;\beta j}(\omega)]\ ,\end{aligned} \tag{52}$$

which can be verified by using Eqs. (15), (34), and (39). Hence,

$$\left\langle \sum_{j=1}^{3} ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t)\right\rangle=-2\pi\sum_{\beta=1}^{N}\sum_{j,l=1}^{3}\int_0^\infty \frac{d\omega}{\omega}h_T{}^2(\omega)\,\mathrm{Re}\left[\frac{M^{-1}(\omega)_{\alpha j;\beta l}}{C(\omega)}\right]\mathrm{Im}\left[C(\omega)\frac{\partial}{\partial R_{\alpha i}}M_{\alpha l;\beta j}(\omega)\right]\ . \tag{53}$$

The missing proof of Eq. (46) will now be given. First, consider the case where $\alpha\neq\beta$. Combining the obvious relationship of

$$\left\langle E_l^T(\mathbf{R}_\beta,t')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t'')\right\rangle=\frac{\partial}{\partial R_{\alpha i}}\langle E_l^T(\mathbf{R}_\beta,t')E_j^T(\mathbf{R}_\alpha,t'')\rangle \tag{54}$$

with Eq. (14), one can easily verify Eq. (46) when $\alpha\neq\beta$. When $\alpha=\beta$, then the following identity must be used:

$$\langle\cos[A+\theta(\mathbf{k}',\lambda')]\sin[B+\theta(\mathbf{k}'',\lambda'')]\rangle=\tfrac{1}{2}\delta_{\lambda'\lambda''}\delta^3(\mathbf{k}''-\mathbf{k}')\sin(B-A)\ . \tag{55}$$

From Eqs. (1), (55), and (5), Eq. (46) becomes, for $\alpha=\beta$,

$$\left\langle E_l^T(\mathbf{R}_\alpha,t')\frac{\partial}{\partial R_{\alpha i}}E_j^T(\mathbf{R}_\alpha,t'')\right\rangle=\tfrac{1}{2}\int d^3k\,h^2(\omega)\left[\delta_{lj}-\frac{k_lk_j}{k^2}\right](-k_i)\sin[0-\omega(t_2-t_1)]\ . \tag{56}$$

By inspection, each of the two terms making up the integrand in Eq. (56) must be odd in either k_1, k_2, or k_3, thereby resulting in the integral being identically equal to zero when $\alpha=\beta$. Hence, Eq. (46) has been verified.

The second term of Eq. (43) will now be evaluated. From Eq. (34) and the Fourier transforms of Eqs. (17) and (40),

$$\begin{aligned}&\left\langle \tilde{x}_{\alpha j}(\omega')\frac{\partial}{\partial R_{\alpha i}}\tilde{E}_j^{D\beta}(\mathbf{R}_\alpha,\omega'')\right\rangle\\&\quad=-\frac{e}{m}\sum_{\gamma,\delta=1}^{N}\sum_{l,m,n=1}^{3}\frac{[M^{-1}(\omega')]_{\alpha j;\gamma l}}{C(\omega')}\left[\frac{\partial}{\partial R_{\alpha i}}\eta_{jm}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega'')\right]\frac{[M^{-1}(\omega'')]_{\beta m;\delta n}}{C(\omega'')}\langle\tilde{E}_l^T(\mathbf{R}_\gamma,\omega')\tilde{E}_n^T(\mathbf{R}_\delta,\omega'')\rangle\ .\end{aligned} \tag{57}$$

From the inverse of Eq. (33), along with Eq. (14),

$$\langle\tilde{E}_l^T(\mathbf{R}_\gamma,\omega')\tilde{E}_n^T(\mathbf{R}_\delta,\omega'')\rangle=\pi\int_0^\infty d\omega[\delta(\omega'-\omega)\delta(\omega''+\omega)+\delta(\omega'+\omega)\delta(\omega''-\omega)]f_{ln}^T(\mathbf{R}_\delta-\mathbf{R}_\gamma,\omega)\ . \tag{58}$$

With the use of Eqs. (57), (58), (48), (49b), and (50), the second term of Eq. (43) can be expressed as

$$\begin{aligned}&\left\langle \sum_{j=1}^{N} ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\left[\sum_{\beta\neq\alpha}E_j^{D\beta}(\mathbf{R}_\alpha,t)\right]\right\rangle\\&\quad=\frac{-e^2}{m}\sum_{\beta,\gamma,\delta=1}^{N}\sum_{j,l,m,n=1}^{3}\int_0^\infty d\omega\,f_{ln}^T(\mathbf{R}_\delta-\mathbf{R}_\gamma,\omega)\frac{\mathrm{Re}\left[[M^{-1}(\omega)]_{\alpha j;\gamma l}^{*}[M^{-1}(\omega)]_{\beta m;\delta n}\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})\eta_{jm}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]\right]}{|C(\omega)|^2}\ .\end{aligned} \tag{59}$$

Here, it should be noted that a factor of $(1-\delta_{\alpha\beta})$ has been included on the right-hand side of Eq. (59), thereby enabling the index β to be summed from 1 to N without restriction.

Several substitutions can be made to simplify Eq. (59). First, from Eq. (39),

$$\frac{\partial}{\partial R_{\alpha i}}[(1-\delta_{\alpha\beta})\eta_{jm}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)]=\frac{\partial}{\partial R_{\alpha i}}[C(\omega)M_{\alpha j;\beta m}(\omega)] . \tag{60}$$

Second, from Eqs. (15) and (34),

$$f^T_{ln}(\mathbf{R}_\delta-\mathbf{R}_\gamma,\omega)=\frac{-2\pi h_T{}^2(\omega)}{\frac{e^2}{m}\omega}\,\mathrm{Im}\eta_{ln}(\mathbf{R}_\delta-\mathbf{R}_\gamma,\omega) . \tag{61}$$

By using $\sin(kR)\approx kR-\frac{1}{6}(kR)^3$ and $\cos(kR)\approx 1-\frac{1}{2}(kR)^2$ for $kR \ll 1$, one may verify from Eqs. (16), (34), and (37) that

$$\lim_{R\to 0}\mathrm{Im}\eta_{ln}(\hat{\mathbf{R}}R,\omega)=-\delta_{ln}\frac{2}{3}\frac{e^2}{mc^3}\omega^3=\delta_{ln}\mathrm{Im}C(\omega) . \tag{62}$$

Consequently, from Eqs. (62) and (39),

$$\begin{aligned}\mathrm{Im}\eta_{ln}(\mathbf{R}_\delta-\mathbf{R}_\gamma,\omega)&=\mathrm{Im}[\delta_{\delta\gamma}\delta_{ln}C(\omega)+(1-\delta_{\delta\gamma})\eta_{ln}(\mathbf{R}_\beta-\mathbf{R}_\gamma,\omega)]\\&=\mathrm{Im}[C(\omega)M_{\delta l;\gamma n}(\omega)] .\end{aligned} \tag{63}$$

Substituting Eqs. (60), (61), and (63) into Eq. (59) yields

$$\begin{aligned}&\left\langle \sum_{j=1}^{N} ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\left[\sum_{\beta\neq\alpha}E_j^{p\beta}(\mathbf{R}_\alpha,t)\right]\right\rangle\\&=2\pi\sum_{\beta,\gamma,\delta=1}^{N}\sum_{j,l,m,n=1}^{3}\int_0^\infty\frac{d\omega}{\omega}h_T{}^2(\omega)\frac{\mathrm{Im}[C(\omega)M_{\delta l;\gamma n}(\omega)]}{|C(\omega)|^2}\\&\qquad\times\mathrm{Re}\left[[M^{-1}(\omega)]^*_{\alpha j;\gamma l}[M^{-1}(\omega)]_{\beta m;\delta n}C(\omega)\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta m}(\omega)\right] .\end{aligned} \tag{64}$$

A fair amount of algebraic manipulations must now be employed in order to bring Eq. (64) into a form that is compatible with Eq. (53). First, the imaginary and real terms below may be expanded as shown:

$$\begin{aligned}&\sum_{\gamma,\delta=1}^{N}\sum_{l,n=1}^{3}\mathrm{Im}(CM_{\delta l;\gamma n})\mathrm{Re}\left[(M^{-1})^*_{\alpha j;\gamma l}(M^{-1})_{\beta m;\delta n}C\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta m}\right]\\&=\frac{1}{4i}\left[C\left[\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta m}\right]\left[\sum_{\gamma,l}C(M^{-1})^*_{\alpha j;\gamma l}\sum_{\delta,n}[(M^{-1})_{\beta m;\delta n}M_{\delta l;\gamma n}]-\sum_{\delta,n}C^*(M^{-1})_{\beta m;\delta n}\sum_{\gamma,l}[(M^{-1})^*_{\alpha j;\gamma l}M^*_{\delta l;\gamma n}]\right]\right.\\&\qquad\left.+C^*\left[\frac{\partial}{\partial R_{\alpha i}}M^*_{\alpha j;\beta m}\right]\left[\sum_{\delta,n}C(M^{-1})^*_{\beta m;\delta n}\sum_{\gamma,l}[(M^{-1})_{\alpha j;\gamma l}M_{\delta l;\gamma n}]-\sum_{\gamma,l}C^*(M^{-1})_{\alpha j;\gamma l}\sum_{\delta,n}[(M^{-1})^*_{\beta m;\delta n}M^*_{\delta l;\gamma n}]\right]\right] .\end{aligned} \tag{65}$$

From Eqs. (39), (34), and (16), the following symmetry rules for $M_{\alpha i;\beta j}$ may readily be verified:

$$M_{\alpha i;\beta j}=M_{\beta i;\alpha j} , \tag{66}$$

$$M_{\alpha i;\beta j}=M_{\alpha j;\beta i} . \tag{67}$$

By using Eqs. (66) and (67), a pair of Kronecker δ's may be obtained from each of the terms enclosed in square brackets in Eq. (65). For example,

$$\sum_{\delta,n}[(M^{-1})_{\beta m;\delta n}M_{\delta l;\gamma n}]=\sum_{\delta,n}[(M^{-1})_{\beta m;\delta n}M_{\delta n;\gamma l}]=\delta_{\beta\gamma}\delta_{ml} . \tag{68}$$

Both terms in the large parentheses of Eq. (65) may then be shown to be equal to

$$-2i\,\mathrm{Im}[C^*(M^{-1})_{\alpha j;\beta m}] .$$

Combining terms enables Eq. (65) to be expressed as

$$\sum_{\gamma,\delta=1}^{N}\sum_{l,n=1}^{3}\mathrm{Im}(CM_{\delta l;\gamma n})\mathrm{Re}\left[(M^{-1})^{*}_{\alpha j;\gamma l}(M^{-1})_{\beta m;\delta n}C\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta m}\right]=-\mathrm{Im}[C^{*}(M^{-1})_{\alpha j;\beta m}]\mathrm{Re}\left[C\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta m}\right] . \quad (69)$$

By now substituting Eq. (69) into Eq. (64), using the simple relationship of

$$\frac{\mathrm{Im}[C^{*}(M^{-1})_{\alpha j;\beta m}]}{|C|^{2}}=\mathrm{Im}\left[\frac{(M^{-1})_{\alpha j;\beta m}}{C}\right] , \quad (70)$$

and relabeling the m dummy index as l, finally yields

$$\left\langle\sum_{j=1}^{N}ex_{\alpha j}(t)\frac{\partial}{\partial R_{\alpha i}}\left[\sum_{\beta\neq\alpha}E_{j}^{D\beta}(\mathbf{R}_{\alpha},t)\right]\right\rangle=-2\pi\sum_{\beta=1}^{N}\sum_{j,l=1}^{3}\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\mathrm{Im}\left[\frac{(M^{-1})_{\alpha j;\beta l}}{C}\right]\mathrm{Re}\left[C\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta l}\right] . \quad (71)$$

If Eq. (67) is used to switch the l,j indices of $M_{\alpha l;\beta j}$ in Eq. (53), then it can immediately be seen that Eqs. (53) and (71) are of the same form. Combining Eqs. (43), (53), and (71), then yields

$$\langle F_{\alpha i}^{T}(t)\rangle=-2\pi\sum_{\beta=1}^{N}\sum_{j,l=1}^{3}\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\mathrm{Im}\left[(M^{-1})_{\alpha j;\beta l}\frac{\partial}{\partial R_{\alpha i}}M_{\alpha j;\beta l}\right] . \quad (72)$$

Let the determinant of $[M]$ be denoted by $|M|$; let $\Delta_{\alpha j;\beta l}$ be the cofactor of the matrix element $M_{\alpha j;\beta l}$. Since $|M|$ may be expressed as

$$|M|=\sum_{\beta=1}^{N}\sum_{l=1}^{3}M_{\beta l;\alpha j}\Delta_{\beta l;\alpha j} , \quad (73)$$

then the inverse matrix elements may be written as

$$(M^{-1})_{\alpha j;\beta l}=\frac{\Delta_{\beta l;\alpha j}}{|M|}=\frac{1}{|M|}\frac{\partial}{\partial M_{\beta l;\alpha j}}|M|=\frac{\partial}{\partial M_{\beta l;\alpha j}}\ln|M| . \quad (74)$$

Consequently, with the use of Eqs. (66) and (67),

$$\langle F_{\alpha i}^{T}(t)\rangle=-2\pi\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\mathrm{Im}\left[\sum_{\beta=1}^{N}\sum_{j,l=1}^{3}\left[\frac{\partial\ln|M|}{\partial M_{\alpha j;\beta l}}\right]\left[\frac{\partial M_{\alpha j;\beta l}}{\partial R_{\alpha i}}\right]\right] . \quad (75)$$

Using Eq. (66) to write Eq. (75) more symmetrically,

$$\langle F_{\alpha i}^{T}(t)\rangle=-\pi\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\mathrm{Im}\left[\sum_{\beta}\sum_{j,l}\left[\frac{\partial\ln|M|}{\partial M_{\alpha j;\beta l}}\right]\left[\frac{\partial M_{\alpha j;\beta l}}{\partial R_{\alpha i}}\right]+\sum_{\beta}\sum_{j,l}\left[\frac{\partial\ln|M|}{\partial M_{\beta j;\alpha l}}\right]\left[\frac{\partial M_{\beta j;\alpha l}}{\partial R_{\alpha i}}\right]\right] . \quad (76)$$

This expression for $\langle F_{\alpha i}^{T}(t)\rangle$ may be simplified significantly by considering the quantity within the square brackets. From Eq. (39), the quantity $(\partial/\partial R_{\alpha i})M_{\beta j;\gamma l}$ is nonzero only when $\beta=\alpha$ and $\gamma\neq\alpha$, or when $\beta\neq\alpha$ and $\gamma=\alpha$. By inspection of Eq. (76), all such nonzero contributions of this quantity have already been included within the square brackets. Hence, one may simply include the remaining terms within the indicated summation, since the remaining terms equal zero. More specifically,

$$\langle F_{\alpha i}^{T}(t)\rangle=-\pi\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\,\mathrm{Im}\left[\sum_{\gamma,\beta=1}^{N}\sum_{j,l=1}^{3}\left[\frac{\partial\ln|M|}{\partial M_{\beta j;\gamma l}}\right]\left[\frac{\partial M_{\beta j;\gamma l}}{\partial R_{\alpha i}}\right]\right] . \quad (77)$$

Hence, as may be seen from Eq. (77), $\langle F_{\alpha i}^{T}(t)\rangle$ may be written as

$$\langle F_{\alpha i}^{T}(t)\rangle=-\frac{\partial}{\partial R_{\alpha i}}U^{T} , \quad (78)$$

where

$$U^{T}=\pi\int_{0}^{\infty}\frac{d\omega}{\omega}h_{T}^{2}(\omega)\mathrm{Im}[\ln|M(\omega)|] . \quad (79)$$

Equations (78) and (79) generalize the van der Waals expressions of Eqs. (8) and (9) in Ref. 7. The latter result dealt with the expectation value of the component of the Lorentz force along the axis separating two electric dipole oscillators situated in thermal plus zero-point electromagnetic radiation. From Eqs. (16), (34), and (39), Eq. (79) may be readily shown to reduce to Eq. (9) of Ref. 7 when $N=2$ and $\mathbf{R}=\hat{\mathbf{z}}R$. [The configuration of $\mathbf{R}=\hat{\mathbf{z}}R$ was chosen in Refs. 5 and 7. From Eqs. (16) and (34), $\eta_{ij}(\hat{\mathbf{z}}R,\omega)=\delta_{ij}\eta_{i}$, where η_{i} is given in Eqs. (19) and (20) of Ref. 7.]

When the temperature T equals zero, then Eq. (79) may be compared to the result of Eq. (18) in Ref. 13 that was obtained via the means of quantum electrodynamics. When $T=0$, then Eq. (8) must be used in Eq. (79). From the line following Eq. (13) of Ref. 13 and the comment $\overleftrightarrow{G}_{\alpha\alpha}(z)=0$ at the top of p. 202 in Ref. 13, one can deduce that $\overleftrightarrow{G}_{\alpha\beta}$ may be written as

$$G_{\alpha i;\beta j}(\omega)=(1-\delta_{\alpha\beta})\left[\nabla_{\alpha i}\nabla_{\beta j}-\frac{\omega^2}{c^2}\delta_{ij}\right]\times\left[\frac{\exp\left[\frac{i\omega}{c}|\mathbf{R}_\alpha-\mathbf{R}_\beta|\right]}{|\mathbf{R}_\alpha-\mathbf{R}_\beta|}\right]=\frac{m}{e^2}(1-\delta_{\alpha\beta})\eta_{ij}(\mathbf{R}_\alpha-\mathbf{R}_\beta,\omega)\ . \tag{80}$$

From Eq. (16) of Ref. 13, $\tilde{\alpha}(\omega)=(e^2/m)[1/C(\omega)]$. By now following the steps in the first part of Sec. IV B of Ref. 5, from Eq. (82) to Eq. (87), the above result of Eq. (79) can be shown to be equivalent to Eq. (18) of Ref. 13.

The calculations in this section may be extended to the situation of N oscillators located in a plane undergoing uniform acceleration through classical electromagnetic zero-point radiation. The equations that allow this extension to be made consist of the relationships between the two-point radiation field correlation functions and the fields of a fluctuating electric dipole found in Eqs. (14), (15), (17), (20), (21), and (23) of the present article, and in Eqs. (A35), (A36), and (C1)–(C4) of Ref. 11. Although this extension will not be carried out here, a careful reading of the present section and of Ref. 11 will indicate how this extension may be accomplished.

IV. EXPECTATION VALUE OF POYNTING VECTOR IN PRESENCE OF AN ELECTRIC DIPOLE OSCILLATOR

If a classical charged harmonic oscillator is bathed in classical electromagnetic radiation, then the oscillator will be forced into a steady-state motion by the radiation. Consequently, the charged oscillator will emit electromagnetic radiation of its own. Consider the case where the oscillator is taken in the electric dipole limit. Let the statistical properties of the electromagnetic radiation causing the oscillator's forced motion be isotropic and homogeneous in space. Under these conditions, one can show that the expectation value of the Poynting vector, due to the total electromagnetic radiation, is exactly equal to zero. This is precisely the situation that occurs when the dipole oscillator is not present. Hence, the presence of the oscillator does not alter the basic flow pattern of electromagnetic radiation.

The proof of the above statement has been given previously in Appendix B of Ref. 1. This proof will be reconstructed in the present section of this article in such a way as to explicitly use the relationships found in Sec. II. As will be seen, what enables the proof to be carried out are precisely these relationships between the two-point correlation functions of classical electromagnetic radiation fields and the electromagnetic fields due to a fluctuating electric dipole. At the end of this section, the means for extending this proof to the case of a uniformly accelerating oscillator will be demonstrated.

Let a single oscillator be situated at the origin of a Cartesian coordinate system in an inertial reference frame. The model assumed for the oscillator will be the same as that of Sec. III. Let the background electromagnetic radiation be described by Eqs. (1) and (2). As may readily be deduced from Eq. (31), with $\mathbf{E}^{D\beta}$ omitted on the right-hand side, along with Eqs. (32), (33), and (37), the motion of a single oscillator is given by

$$x_i(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}d\omega\,\exp(-i\omega t)\left[\frac{e}{m}\frac{\tilde{E}_i^{\rm in}(0,\omega)}{C(\omega)}\right]\ . \tag{81}$$

The expectation value of the Poynting vector due to the total electromagnetic radiation at a location $\mathbf{R}$ and time t is given by

$$\langle S_k(\mathbf{R},t)\rangle=\left\langle\frac{c}{4\pi}[\mathbf{E}_{\rm total}(\mathbf{R},t)\otimes\mathbf{B}_{\rm total}(\mathbf{R},t)]_k\right\rangle$$
$$=\frac{c}{4\pi}\sum_{i,j=1}^{3}\epsilon_{ijk}[\langle E_i^{\rm in}(\mathbf{R},t)B_j^{\rm in}(\mathbf{R},t)\rangle+\langle E_i^{\rm in}(\mathbf{R},t)B_j^{D}(\mathbf{R},t)\rangle+\langle E_i^{D}(\mathbf{R},t)B_j^{\rm in}(\mathbf{R},t)\rangle+\langle E_i^{D}(\mathbf{R},t)B_j^{D}(\mathbf{R},t)\rangle]\ . \tag{82}$$

The four terms in Eq. (82) can readily be evaluated by the use of Eq. (81) and the relationships in Sec. II. Following the same method used in deducing Eq. (62), one can show from Eqs. (22) and (35) that

$$\lim_{R\to 0}\mathrm{Re}[\rho_{ij}(R\hat{\mathbf{R}},\omega)]=0\ . \tag{83}$$

From Eq. (83), the result follows that the correlation function of Eq. (20) equals zero when $\mathbf{x}_2=\mathbf{x}_1$; consequently, the first term on the right-hand side of Eq. (82) equals zero. More specifically, from Eqs. (20), (21), (35), and (83),

$$\langle E_i^{\rm in}(\mathbf{R},t)B_j^{\rm in}(\mathbf{R},t)\rangle=0\ . \tag{84}$$

The second term on the right-hand side of Eq. (82) may be evaluated by reexpressing B_j^D by Eqs. (23), (81), and (33) and then using Eqs. (14), (15), (34), and (48). From Eqs. (22) and (35),

$$\rho_{ij}^D(\mathbf{R},-\omega)=\rho_{ij}^{D*}(\mathbf{R},\omega)\ , \tag{85a}$$

$$\rho_{ij}(\mathbf{R},-\omega)=\rho_{ij}^{*}(\mathbf{R},\omega)\ . \tag{85b}$$

Combining the equations mentioned above,

$$\langle E_i^{\text{in}}(\mathbf{R},t)B_j^{p}(\mathbf{R},t)\rangle=\frac{\pi}{2i(e^2/m)}\sum_{l=1}^{3}\int_0^\infty\frac{d\omega}{\omega}\frac{h_{\text{in}}{}^2(\omega)}{|C(\omega)|^2}[C^*(\omega)\eta_{li}(\mathbf{R},\omega)\rho_{jl}(\mathbf{R},\omega)+C(\omega)\eta_{li}(\mathbf{R},\omega)\rho_{jl}^*(\mathbf{R},\omega)-C^*(\omega)\eta_{li}^*(\mathbf{R},\omega)\rho_{jl}(\mathbf{R},\omega)-C(\omega)\eta_{li}^*(\mathbf{R},\omega)\rho_{jl}^*(\mathbf{R},\omega)]\ . \tag{86}$$

Using virtually identical treatments, one may obtain the following expressions for the third and fourth terms of Eq. (82):

$$\langle E_i^{p}(\mathbf{R},t)B_j^{\text{in}}(\mathbf{R},t)\rangle=\frac{\pi}{2i(e^2/m)}\sum_{l=1}^{3}\int_0^\infty\frac{d\omega}{\omega}\frac{h_{\text{in}}{}^2(\omega)}{|C(\omega)|^2}[C(\omega)\eta_{il}^*(\mathbf{R},\omega)\rho_{lj}(-\mathbf{R},\omega)-C^*(\omega)\eta_{il}(\mathbf{R},\omega)\rho_{lj}(-\mathbf{R},\omega)+C(\omega)\eta_{il}^*(\mathbf{R},\omega)\rho_{lj}^*(-\mathbf{R},\omega)-C^*(\omega)\eta_{il}(\mathbf{R},\omega)\rho_{lj}^*(-\mathbf{R},\omega)]\ , \tag{87}$$

$$\langle E_i^{p}(\mathbf{R},t)B_j^{p}(\mathbf{R},t)\rangle=\frac{\pi}{2i(e^2/m)}\sum_{l=1}^{3}\int_0^\infty\frac{d\omega}{\omega}\frac{h_{\text{in}}{}^2(\omega)}{|C(\omega)|^2}[-C(\omega)\eta_{il}(\mathbf{R},\omega)\rho_{jl}^*(\mathbf{R},\omega)+C^*(\omega)\eta_{il}(\mathbf{R},\omega)\rho_{jl}^*(\mathbf{R},\omega)-C(\omega)\eta_{il}^*(\mathbf{R},\omega)\rho_{jl}(\mathbf{R},\omega)+C^*(\omega)\eta_{il}^*(\mathbf{R},\omega)\rho_{jl}(\mathbf{R},\omega)]\ . \tag{88}$$

Hence, all four terms of Eq. (82) have been expressed in terms of the functions η_{ij} and ρ_{ij} that appear in the expressions for the electric and magnetic fields of a fluctuating electric dipole. At this point, the symmetries of

$$\rho_{ij}(-\mathbf{R},\omega)=-\rho_{ij}(\mathbf{R},\omega)\ , \tag{89}$$

$$\eta_{ij}(\mathbf{R},\omega)=\eta_{ji}(\mathbf{R},\omega)\ , \tag{90}$$

$$\rho_{ij}(\mathbf{R},\omega)=-\rho_{ji}(\mathbf{R},\omega)\ , \tag{91}$$

which are readily deduced from Eqs. (16), (22), (34), and (35), may be introduced to show that all terms immediately cancel upon combining Eqs. (82), (84), and (86)–(88). Moreover, as may be noted by examining the integrands in Eqs. (86)–(88), this cancellation occurs exactly at each frequency. The relationships derived in Sec. II are what enabled this precise cancellation to arise. Hence, $\langle S_k(\mathbf{R},t)\rangle=0$.

Thus ends the proof of this section of the fact that the presence of an electric dipole oscillator does not alter $\langle S_k(\mathbf{R},t)\rangle$ from its zero value. The important assumptions used in this proof were that the linear dipole oscillator was stationary in an inertial frame and bathed with homogeneous, isotropic electromagnetic radiation described by Eqs. (1) and (2).

Without too much difficulty, however, this proof may be extended to the case of a dipole oscillator uniformly accelerating through classical electromagnetic zero-point radiation. This analogous situation requires that $\langle S_k(\mathbf{R},t)\rangle$ be evaluated in the instantaneous inertial rest frame of the accelerating oscillator. Instead of using the relationships in Sec. II of the present article, the analogous relationships of Appendixes A and C of Ref. 11 must be employed. The steps of the proof given in this section for an unaccelerated oscillator may then be followed up through Eq. (88). The symmetries of Eqs. (89)–(91) may not be employed, however, as these do not carry over to the acceleration case.

A sketch of this calculation is given in the Appendix. Let $\mathbf{a}$ denote the proper acceleration of the oscillator; let R denote the distance from the oscillator to the point at which the Poynting vector is evaluated, taken along a perpendicular to the acceleration. Provided that a small laboratory condition is imposed such that terms $O(aR/c^2)$ are ignored, then a null value is obtained for the expectation value of the Poynting vector in the instantaneous rest frame of the accelerating oscillator. This result agrees with the exact value of zero that is obtained for the expectation value of the Poynting vector when the oscillator is not present, as given in a nonrotating coordinate system uniformly accelerating through classical electromagnetic zero-point radiation. Hence, for terms up to $O(aR/c^2)$, the presence of an oscillator within a plane uniformly accelerating through classical electromagnetic zero-point radiation does not alter the expectation value of the electromagnetic momentum density calculated at some point in the inertial rest frame of this accelerating plane.

V. CLOSING REMARKS

Relationships were derived in Sec. II between the two-point field correlation functions for homogeneous and isotropic random (Gaussian) classical electromagnetic radiation and the electromagnetic fields of a classical fluctuating electric dipole. Section III explicitly used these relationships in order to obtain the van der Waals force on an oscillator surrounded by $(N-1)$ other dipole oscillators, all of which were bathed in classical electromagnetic zero-point plus thermal zero-point radiation. Section IV used the relationships of Sec. II to show that the expectation value of the Poynting vector in the presence of an oscillator, bathed with homogeneous and isotropic classical electromagnetic radiation, is unaltered from its null value that occurs when the oscillator is not present. Brief discussions were also given in Secs. III and IV and the Appendix as to how the calculations of Secs. III and IV could be extended to the case of a small laboratory uniformly accelerating through classical electromagnetic zero-point radiation. What enables this extension to be made is the result found in Ref. 11: namely, when the two-point field correlation functions for classical electromagnetic zero-point radiation are evaluated along a trajectory described by uniform acceleration, they are found to be related to the electromagnetic fields of a uniformly accelerating electric dipole in the same way that occurs for the relationships of Sec. II in the unaccelerated-thermal case.

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APPENDIX

Section IV presented a proof that the expectation value of the Poynting vector, evaluated in the presence of isotropic and homogeneous random (Gaussian) classical electromagnetic radiation, is unchanged from its null value when a harmonic dipole oscillator is also present. Generalizing the relationships of Sec. II to the relationships found in Ref. 11, this proof may be extended to the case of an oscillator uniformly accelerating through classical electromagnetic zero-point radiation. In order to make this extension, a fair degree of familiarity is required of Ref. 11. Consequently, the calculation sketched below assumes that Ref. 11 is readily accessible to the reader.

In keeping with the work of Ref. 11, let $\hat{\mathbf{x}}$ be along the direction of acceleration. The corresponding relationships to Eqs. (16) and (22) of the present article were evaluated in Ref. 11 for the case where the vector position $\mathbf{R}$ [here, $\mathbf{R}$ corresponds to the argument of η_{ij} and ρ_{ij} in Eqs. (16) and (22)] was contained in the plane that was accelerating along with the oscillator and oriented such as to be perpendicular to the $\hat{\mathbf{x}}$ direction [see Eqs. (A37)—(A44) of Ref. 11]. The Fermi-Walker transported coordinate system used in Refs. 10 and 11 was constructed so as to have one coordinate axis along the $\hat{\mathbf{x}}$ direction and two orthogonal coordinate axes lying in this accelerating plane and parallel to the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions. Let ξ indicate a vector in this coordinate system; let $\xi=\hat{\mathbf{y}}R$ be a point in the accelerating plane at a distance R along the y axis from the accelerating oscillator. It is at this point that the Poynting vector will be evaluated.

Equations (82)—(88) still hold for this accelerating system when $\mathbf{R}$ is replaced by $\xi=\hat{\mathbf{y}}R$, t is replaced by the proper time τ_e of the accelerating oscillator's equilibrium point, all fields are evaluated in the instantaneous inertial rest frame of the oscillator's equilibrium point, $h_{\text{in}}{}^2(\omega)$ is replaced by Eq. (7) for $T=\hbar a/2\pi ck$, and the functions C, η_{ij}, and ρ_{ij} are replaced by their appropriate generalizations of C_l^a, η_{ij}^a, and ρ_{ij}^a that occur for the accelerated situation.[18] The latter functions are given in Eqs. (16) and (A37)—(A44) of Ref. 11. Combining these functions with Eqs. (82)—(88) then yields

$$\langle S^{\text{zp}}_{\tau_e k}(\hat{\mathbf{y}}R,\tau_e)\rangle=\frac{c}{(e^2/m)}\int_0^\infty \frac{d\omega}{\omega} h_T{}^2(\omega)\,|_{\,T=\hbar a/2\pi ck}\,\text{Im}(\eta_{12}^a)\left[\text{Re}\left[\frac{\rho_{32}^a}{C_2^a}\right]\delta_{k2}+\text{Re}\left[\frac{\rho_{31}^a}{C_1^a}\right]\delta_{k1}\right]. \tag{A1}$$

The subscript τ_e on $S^{\text{zp}}_{\tau_e k}$ indicates that the Poynting vector is to be evaluated in the inertial reference frame instantaneously at rest with respect to the accelerating oscillator at proper time τ_e. The superscript zp on $S^{\text{zp}}_{\tau_e k}$ indicates that the background radiation, through which the trajectory of uniform acceleration takes place, consists of classical electromagnetic zero-point radiation.

As will be noticed in deducing Eq. (A1) from Eqs. (82)—(88), a large number of the terms cancel and drop out; nevertheless, Eq. (A1) does not equal zero exactly, as was the situation in the unaccelerated case.[19] As shown in Ref. 11, however, when $i\neq j$, then the magnitude of η_{ij}^a is approximately (aR/c^2) times the magnitude of η_{ii}^a. Thus, when the small laboratory condition of

$$\frac{aR}{c^2}\ll 1 \tag{A2}$$

is considered, then the result of Eq. (A1) shows that the only terms that remain after combining Eqs. (82)—(88) are terms that are first order in (aR/c^2), or higher. All terms of zeroth order in (aR/c^2) cancel precisely.

It should be noted that restricting attention to terms of zeroth order in (aR/c^2) does not mean that $\langle S^{\text{zp}}_{\tau_e k}(\hat{\mathbf{y}}R,\tau_e)\rangle$ has simply been expanded in a power series in the acceleration and only the zeroth-order term in the acceleration examined. Such a case would be quite trivial, indeed, since then the null result for $\langle S^{\text{zp}}_{\tau_e k}(\hat{\mathbf{y}}R,\tau_e)\rangle$, obtained to zeroth order in the acceleration, would simply be a restatement of the unaccelerated result in Sec. IV. On the contrary, the zeroth-order terms in (aR/c^2) for the four expressions corresponding to Eqs. (84) and (86)—(88) in the acceleration case, do depend upon the acceleration; in particular, they depend upon the spectral function

$$h_T{}^2(\omega)\,|_{\,T=\hbar a/2\pi ck}=\frac{\hbar\omega}{2\pi^2}\coth\left[\frac{\pi c\omega}{a}\right]. \tag{A3}$$

In addition, these four expressions depend upon the functions C^a, η_{ij}^a, and ρ_{ij}^a, which contribute additional dependency upon acceleration even after dropping terms of $O(aR/c^2)$. Adding these four expressions together in order to form $\langle S^{\text{zp}}_{\tau_e k}(\hat{\mathbf{y}}R,\tau_e)\rangle$ then yields a null value for $\langle S^{\text{zp}}_{\tau_e k}(\hat{\mathbf{y}}R,\tau_e)\rangle$, when terms of $O(aR/c^2)$ are ignored.

References 8, 10, and 11 analyzed the equivalency that exists in certain physical properties between a system of classical dipole oscillators in a thermal radiation bath and a similar set of oscillators uniformly accelerating through classical electromagnetic zero-point radiation. As first noted in Ref. 8, the stochastic behavior of a single accelerating oscillator agrees with the behavior of an unaccelerated oscillator bathed in classical electromagnetic thermal radiation characterized by the spectral function of Eq. (A3). The calculation outlined above for the expectation value of the total Poynting vector, when a single oscillator is present, shows another property that has a correspondence between the accelerated and unaccelerated-thermal single-oscillator situations. In this case, the narrow linewidth approximation used in Refs. 8, 10, and 11 was not required; only the small laboratory condition was needed. Here, the expectation value of the Poynting vector was evaluated in the instantaneous rest frame of the oscillator and in the plane that included the oscillator and that was perpendicular to the direction of acceleration. This quantity was shown to equal zero, provided that terms of $O(aR/c^2)$ were ignored, thereby agreeing with the null effect upon the expectation value of the electromagnetic momentum density when a harmonic dipole oscillator is included within a thermal radiation bath.

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[12]Since the functional form for the electric (magnetic) field of an electric dipole agrees with the functional form for the magnetic (electric) field of a magnetic dipole, then the two-point field correlation functions of this article could just as easily have been related to the fields of a magnetic dipole.

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[14]See Refs. 1 and 2 for the motivation for using these expressions. Also see T. H. Boyer, Phys. Rev. **186**, 1304 (1969).

[15]Explicitly writing out the transformation of Eq. (11), as done in Eq. (12), is actually not required here in order to obtain Eqs. (14)–(16). As may be noted, the only properties of $[O]$ that were used here were $O_{i3}=R_i/R$, which may be obtained from Eq. (11), and $\sum_{m=1}^{3} O_{im}O_{jm}=\delta_{ij}$.

[16]See, for example, J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 395, Eq. (9.18).

[17]This equation of motion is the one most commonly used in stochastic electrodynamics for describing the behavior of a nonrelativistic charged particle acted upon by fluctuating electromagnetic fields and a harmonic potential. The reason usually given for using an equation of the form of Eq. (31), rather than the full Lorentz-Dirac equation, is that terms such as $e\gamma(\dot{\mathbf{x}}/c)\otimes(\mathbf{B}^{\text{in}}+\sum_{\beta\neq\alpha}\mathbf{B}^{D\beta})$ or $m\Gamma\dot{\mathbf{x}}\ddot{x}^{\nu}\ddot{x}_{\nu}/c^2$, which occur in the Lorentz-Dirac equation but not in Eq. (31), are taken to be small in the nonrelativistic limit. As described on p. 1091 of Ref. 8 and in more detail on pp. 1974 and 1978 of Ref. 10, the reasoning involved with ignoring certain terms in the Lorentz-Dirac equation is somewhat different under what has been termed the small oscillator approximation: namely, the terms retained in the equation of motion are those that are linear only either in the driving fields causing the particle's fluctuating motion or in the spatial coordinates describing the particle's fluctuations about some average position. Although the distinction between the nonrelativistic approximation and the small oscillator approximation is not significant for an unaccelerated oscillator, this distinction is important when analyzing the behavior of an accelerated oscillator, as was done in Refs. 8 and 10. [This distinction is also important in the unaccelerated oscillator situation when a constant magnetic field is added to the problem, as in Eq. (117) of Ref. 2. In that case, the use of the small oscillator approximation is the appropriate reason for retaining the $(\dot{\mathbf{x}}/c)\otimes\mathbf{B}_{\text{const}}$ term in the equation of motion.]

[18]As was done in Ref. 11, the notation of η_{ij}^{a} and ρ_{ij}^{a} stands for the functions $\eta_{ij}(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\omega)$ and $\rho_{ij}(\hat{\mathbf{x}}a,+\hat{\mathbf{y}}R,\omega)$, where R is assumed to be positive here.

[19]The expectation value of the Poynting vector in the Fermi-Walker-transported coordinate system may be readily shown to be exactly equal to zero when the oscillator is not present. This may be deduced from the fact that $\langle E_{\tau_e i}^{\text{zp}}(\xi,\tau_e)B_{\tau_e j}^{\text{zp}}(\xi,\tau_e)\rangle=0$, where ξ is a point in the Fermi-Walker-transported coordinate system.