PHYSICAL REVIEW D VOLUME 33, NUMBER 8 15 MAY 1986

Superluminal reference frames and generalized Lorentz transformations

Roderick I. Sutherland
Department of Applied Science, Hawkesbury College of Advanced Education, Richmond, New South Wales, Australia
John R. Shepanski
Department of Theoretical Physics, The University of New South Wales, Sydney, Australia
(Received 2 January 1985; revised manuscript received 2 December 1985)

It has recently been shown by Marchildon, Antippa, and Everett that it is not possible to maintain the principle of relativity in constructing a superluminal generalization of the Lorentz transformations without obtaining obvious conflicts with experiment. Our aim here is to show that this, at first sight, disturbing result is, rather, what we should have expected from the known properties of space-time. To this end, we present a superluminal formulation which differs in some respects from the theories of previous authors.

I. INTRODUCTION

In a recent paper,[1] Marchildon, Antippa, and Everett have made an important contribution on the question of extending the Lorentz transformations to incorporate superluminal frames.[2] Their work concerns the so-called extended principle of relativity, viz., that all inertial frames, including those with relative speeds greater than that of light, should be equivalent with respect to the laws of physics. Restricting themselves to linear coordinate transformations, Marchildon *et al.* prove theorems which essentially exclude the possibility of implementing the extended principle of relativity by means of either real or complex transformations. For both cases they show that imposing this principle in conjunction with any extension of the proper orthochronous Lorentz group $L_+^\uparrow$ implies a large number of new symmetries, in conflict with what is observed in nature. In particular, for the case of real transformations, the smallest group consistent with the principle and containing $L_+^\uparrow$ is found to be the full linear group SL(4;R). Since many papers on faster-than-light generalizations have aimed to maintain the equivalence of all frames, the importance of these recent proofs is evident.

Our intention here is to demonstrate that this impossibility of constructing an experimentally tenable superluminal generalization which incorporates the extended principle is, in fact, to be expected on simple geometric grounds. We aim to show that, although the complete set of transformations (including those involving superluminal frames) must form a group, it should not have been expected that the principle of relativity would hold for this larger group.

The structure of the paper is as follows. In Sec. II we explain the basis of our formulation in terms of the usual Minkowski space-time picture and consider precisely what is meant by a superluminal frame. The way in which such frames are related to subluminal frames is discussed and we point out the natural existence of a geometrically distinguishable spatial axis in every superluminal frame. (This distinguishable direction should not lead one to confuse the present formulation with the "tachyon corridor" approach of Antippa and Everett.[3,4] Their model is quite different, involving as it does a preferred spatial direction in *subluminal* frames as well.) This geometrically different direction is the reason for the breakdown of the principle of relativity. In Secs. III and IV the equations for transforming between subluminal and superluminal frames are derived by way of our geometric approach. We then reexpress the invariant interval in terms of superluminal coordinates in Sec. V and thereby obtain the modified form of Pythagoras's theorem that holds for superluminal three-space. It is shown in Sec. VI that the case of transforming between two superluminal frames requires two different types of transformation, one of which has not been considered in the literature before. In Sec. VII we point out that it is this fact of different transformations applying in different circumstances which allows the overall group in our scheme to be smaller than what would otherwise be permitted by the arguments of Marchildon *et al.* Finally, in Sec. VIII the appropriate equations for describing a rotation of superluminal spatial axes are obtained.

Our basic aim in this paper is to demonstrate that the emphasis in the past on maintaining the principle of relativity has been misplaced and that the correct superluminal generalization of special relativity is indicated quite naturally by the usual space-time picture.

II. SUPERLUMINAL REFERENCE FRAMES

By superluminal frames we mean frames moving faster than light relative to the usual (subluminal) frames of our experience. Our terminology here differs from that of some other authors who ascribe a purely relative meaning to the words subluminal and superluminal. For them, a reference frame can be subluminal relative to some frames and, at the same time, superluminal relative to other frames. The reason we do not follow this mode of description is that we consider subluminal and superluminal frames to be geometrically different (and hence absolutely distinguishable), as will be discussed below. Another point of terminology is that coordinate systems with different spatial axes, but sharing the same time axis, will be referred to here as the same frame of reference rather than as different frames relatively at rest.

In the Minkowski geometric interpretation of special relativity, the four-dimensional distance (interval) between any two points in space-time is given by

$$ds = |\,dx^2+dy^2+dz^2-c^2dt^2\,|^{1/2}\ . \tag{1}$$

(The absolute value has been chosen here so that ds is always real.) The minus sign in this expression indicates that one of the dimensions is of a different type from the other three; i.e., space-time is non-Euclidean and nonisotropic. This difference is the basis of our viewpoint. Another point that will be relevant is that the notion of orthogonality remains meaningful in the geometry defined by (1). Two vectors are defined to be orthogonal if their Minkowskian scalar product is zero. For any subluminal reference frame, the three-dimensional spatial hyperplanes corresponding to single instants of time are all orthogonal to the frame's time axis. (This follows once the assumption is made that the speed of light is the same in all directions.)

We define superluminal reference frames as the rest frames of particles moving faster than light relative to us (if such particles were to exist). With regard to notation, we will distinguish between the coordinates of subluminal and superluminal frames by placing a bar above the latter: $\bar{x},\bar{y},\bar{z},\bar{t}$. Let us consider a faster-than-light particle, or tachyon, which is moving at constant velocity relative to the subluminal frame S and which passes through the space-time origin 0 of S. The tachyon's world line is spacelike and lies outside the light cone associated with 0. We are interested in the superluminal rest frame $\bar{S}$ of this tachyon, i.e., the $\bar{x}$, $\bar{y}$, $\bar{z}$, and $\bar{t}$ axes in space-time. If we take the tachyon to be at the spatial origin of its rest frame then, in terms of this frame's coordinates, the world line of the tachyon must comprise the set of events $(0,0,0,\bar{t})$; i.e., it will lie along the time axis of the frame.[5] Having located the $\bar{t}$ axis, it remains to find the spatial part of the tachyon's rest frame, i.e., the three-dimensional hyperplanes of constant $\bar{t}$. By analogy with subluminal frames, the hyperplanes of simultaneity are taken as orthogonal to the $\bar{t}$ axis. This ensures that light will travel at speed c in every direction with respect to this superluminal frame (see Sec. V). Thus, if we choose the three spatial axes to form an orthogonal set, the four axes of the superluminal frame will all be mutually orthogonal. The nature of a superluminal frame has now been completely determined. Figure 1 shows a subluminal frame and a superluminal frame moving at constant velocity relative to each other along their common x direction.

Before continuing, it is worthwhile listing the assumptions we are aware of having made, since the preceding considerations are already at variance with the ideas of some other authors. In addition to assuming the validity, for subluminal frames, of the Minkowski geometric picture, we have assumed (i) that a superluminal frame should be defined to have three spatial axes and one time axis and (ii) that the spatial hyperplanes of constant time for such a frame should be orthogonal to the time axis, in analogy with subluminal frames. It has also been implicitly assumed that the four superluminal axes in space-time have appropriate scales, i.e., that the scales $\bar{x},\bar{y},\bar{z},c\bar{t}$

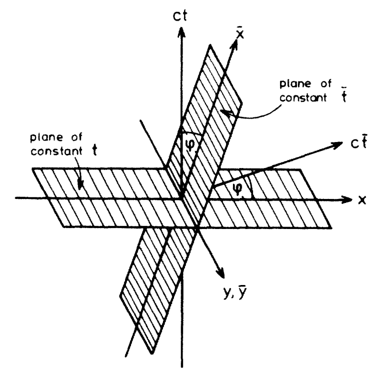

FIG. 1. Minkowski diagram for a subluminal and a superluminal frame.

give a true measure of intervals along the axes. We are, of course, requiring that the spatial axes lie within our present four-dimensional space-time continuum. Otherwise, further dimensions would be introduced for which there is neither experimental evidence nor theoretical need and, in addition, the present four dimensions of space-time would not be completely spanned unless the superluminal frame had more than four axes.

Let us now examine the above notions further. It should be realized that for every superluminal frame there is a corresponding subluminal frame with the same four axes, the only difference being that the labels on the t axis and one of the spatial axes are interchanged. The Lorentz transformations describe rotations of four orthogonal axes in space-time and, since orthogonality requires that there must always be one axis inside the light cone, there is a subluminal frame corresponding to every possible orientation. There are, however, $4!=24$ different ways of labeling four orthogonal axes with the letters x, y, z, and t, and for only six of these will the axis inside the light cone be labeled t. These are subluminal frames, and the other 18 are superluminal frames. (The six subluminal permutations actually correspond to only one subluminal frame with various interchanges of the spatial axes. Likewise, the 18 superluminal cases actually correspond to only three superluminal frames with various permutations of the spatial axes.)

Because one of the spatial axes of a superluminal frame must lie within the light cone (i.e., must have the properties of the subluminal time dimension), it will be different from the other two spatial axes and hence the geometry of the spatial part of a superluminal frame cannot be isotropic. In general, we can determine the special axis of a particular superluminal frame by considering the corresponding subluminal frame with the same four axes. The geometrically different axis is always that spatial axis of the superluminal frame along which the subluminal frame is moving. The existence of a distinguishable superluminal spatial direction[6] is nothing more than a restatement of the fact that four-dimensional space-time is not isotropic, having a distinguishable dimension (the subluminal time dimension).

We can now see the extent to which the principle of relativity is valid in the context of superluminal frames. It is one of the cornerstones of special relativity that subluminal frames satisfy this principle and, from our preceding discussion, it should be apparent that any two superluminal frames will also be in accordance with the principle. The existence, however, of a geometrically different spatial direction in superluminal frames makes them distinguishable from subluminal frames, which have no such dissimilar spatial direction. This breakdown of the principle of relativity due to the lack of equivalence between subluminal and superluminal frames clearly should not be viewed as surprising. It is simply a consequence of the fact [indicated by Eq. (1)] that the time direction of any subluminal frame differs geometrically from the spatial directions.

The geometric approach we have followed thus provides a physical explanation for the impossibility of implementing the extended principle of relativity.

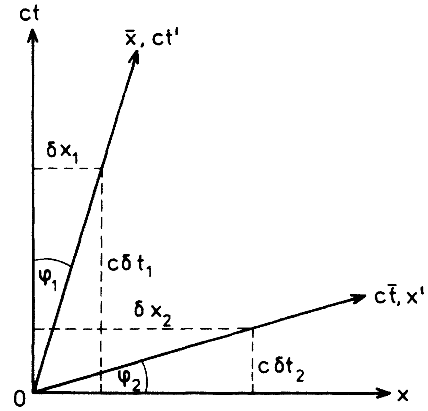

FIG. 2. Relationship between the coordinates of the frames S, S', and $\bar{S}$.

III. TRANSFORMATION BETWEEN A SUBLUMINAL AND A SUPERLUMINAL FRAME

The transformation equations relating superluminal to subluminal frames will now be derived. The usual Lorentz transformations connecting two subluminal frames, whose relative velocity v is along their common x direction and whose spatial origins coincide at $t=t'=0$, are

$$x'=\gamma(x-vt),\quad y'=y,\quad z'=z,\quad t'=\gamma\left[t-\frac{vx}{c^2}\right], \tag{2}$$

where we have

$$\gamma=\left[1-\frac{v^2}{c^2}\right]^{-1/2} \tag{3}$$

and $|v|$ is less than c. Let us now consider a subluminal frame S and a superluminal frame $\bar{S}$ moving with relative velocity $\bar{v}>c$ along their common x direction (the spatial origins of the frames coinciding at $t=\bar{t}=0$). Corresponding to $\bar{S}$ there will be a subluminal frame S' with the same four axes in space-time but with the x and t labels interchanged (see Fig. 2). It will be moving with a velocity $v<c$ relative to S. We wish to find the relationship between v and $\bar{v}$. From Fig. 2 we have

$$\tan\phi_1=\frac{\delta x_1}{c\delta t_1}=\frac{v}{c},$$

$$\tan\phi_2=\frac{c\delta t_2}{\delta x_2}=\frac{c}{\bar{v}}.$$

Hence, since ϕ_1 is always equal to ϕ_2 on a Minkowski diagram, we may combine these equations to get

$$v=\frac{c^2}{\bar{v}}. \tag{4}$$

The transformation from S to $\bar{S}$ is simply a combination of two steps: a Lorentz transformation from S to S' [Eqs. (2) and (3)] followed by the changes

$$x'\to c\bar{t},\quad ct'\to\bar{x},\quad y'\to\bar{y},\quad z'\to\bar{z}. \tag{5}$$

Inserting (4) in (2), we obtain the equations

$$x'=\gamma\left[x-\frac{c^2t}{\bar{v}}\right],\quad y'=y,\quad z'=z,\quad t'=\gamma\left[t-\frac{x}{\bar{v}}\right]$$

and combining these with (5) yields

$$\bar{t}=\frac{\gamma}{c}\left[x-\frac{c^2t}{\bar{v}}\right],\quad \bar{y}=y,\quad \bar{z}=z,\quad \bar{x}=c\gamma\left[t-\frac{x}{\bar{v}}\right].$$

Now, in view of (4), γ may be written in the form

$$\gamma=\frac{|\bar{v}|}{c}\left[\frac{\bar{v}^2}{c^2}-1\right]^{-1/2}.$$

Therefore, since $\bar{v}$ is positive in the case of Fig. 2, we can now write

$$\bar{t}=-\left[\frac{\bar{v}^2}{c^2}-1\right]^{-1/2}\left[t-\frac{\bar{v}x}{c^2}\right]$$

and

$$\bar{x}=-\left[\frac{\bar{v}^2}{c^2}-1\right]^{-1/2}(x-\bar{v}t).$$

Hence, dropping the bars from the v's, the desired transformations for $v>c$ can finally be expressed as

$$x=-\gamma(x-vt),\quad \bar{y}=y,\quad \bar{z}=z,\quad \bar{t}=-\gamma\left[t-\frac{vx}{c^2}\right],$$

where we have generalized (3) to

$$\gamma=\left|1-\frac{v^2}{c^2}\right|^{-1/2}$$

(this expression for γ now being applicable for both $|v|<c$ and $|v|>c$).

IV. FURTHER ROTATIONS IN THE XT PLANE

The transformation equations derived in the previous section hold for any superluminal frame whose $\bar{t}$ axis points forwards in time relative to the initial subluminal frame (although the numerical value of v will not always be positive). In this section we will consider other rotations in the XT plane, including those for which the time direction is reversed.

Starting from an initial set of subluminal axes, one can perform any of the rotations tabulated in Fig. 3. The appropriate transformation equations for the various cases can be obtained via arguments analogous to that in Sec. III. One finds that the equations only differ by the sign in front of γ, there being four different forms.

A: Subluminal to subluminal.

(i) t' axis forward in time relative to frame S:

$$x' = +\gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = +\gamma\left[t - \frac{vx}{c^2}\right] .$$

(ii) t' axis backward in time relative to frame S:

$$x' = -\gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = -\gamma\left[t - \frac{vx}{c^2}\right] .$$

B: Subluminal to superluminal.

(i) $\bar{t}$ axis forward in time relative to frame S:

$$\bar{x} = -\gamma(x - vt), \quad \bar{y} = y, \quad \bar{z} = z, \quad \bar{t} = -\gamma\left[t - \frac{vx}{c^2}\right] .$$

(ii) $\bar{t}$ axis backward in time relative to frame S:

$$\bar{x} = +\gamma(x - vt), \quad \bar{y} = y, \quad \bar{z} = z, \quad \bar{t} = +\gamma\left[t - \frac{vx}{c^2}\right] .$$

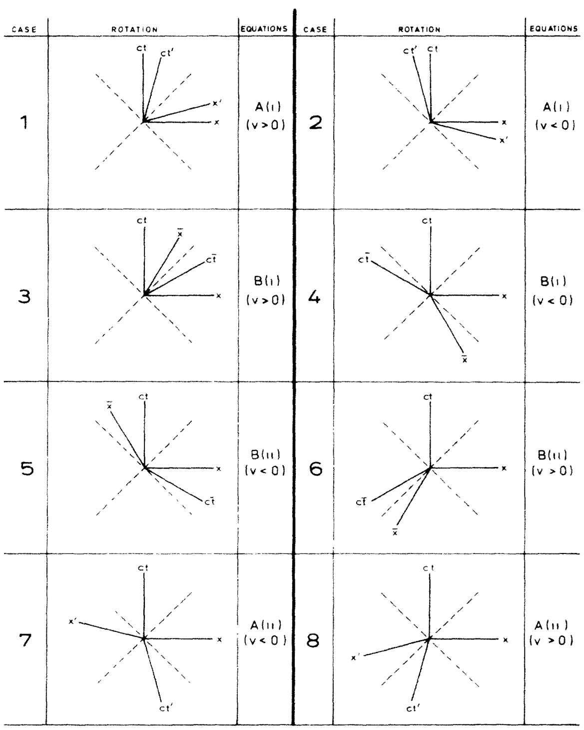

FIG. 3. Rotations in the xt plane.

In each case we are using the notation

$$\gamma = \left|1 - \frac{v^2}{c^2}\right|^{-1/2} . \tag{6}$$

Referring to Fig. 3, cases 1 and 2 correspond to the usual subluminal transformations and cases 3 and 4 correspond to the transformations derived in the previous section. In each of cases 5, 6, 7 and 8, the time axis of the frame resulting from the rotation points backward in time relative to the initial frame. The final frames in these cases are actually not distinct from those in the first four cases: they are the same as those in cases 4, 3, 2, and 1, respectively, except that the directions of the axes are reversed.

The sign of the final frame's velocity v relative to the initial frame in each case can be determined from the definition $v \equiv \delta x / \delta t$, where the changes δx and δt in the initial frame's coordinates correspond to a displacement $\delta t'$ or $\delta \bar{t}$ along the final frame's time axis. Thus, the numerical value of v in cases 2, 4, 5, and 7 will be negative.

The equations describing a Lorentz rotation in the YT or ZT plane will, of course, be analogous to those above.

V. INVARIANT INTERVAL IN SUPERLUMINAL COORDINATES

Let us determine the superluminal form of Eq. (1). For, say, the subluminal frame S' in Sec. III, Eq. (1) takes the form

$$ds = |dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2|^{1/2} .$$

We may rewrite this expression in terms of the coordinates of the frame $\bar{S}$ by using (5), to obtain

$$ds = |c^2 d\bar{t}^2 + d\bar{y}^2 + d\bar{z}^2 - d\bar{x}^2|^{1/2} .$$

This equation reexpresses the invariant interval in terms of superluminal coordinates. By setting $d\bar{t} = 0$ we then obtain the distance $d\bar{r}$ between any two points in the three-dimensional superluminal space:

$$d\bar{r} = |d\bar{y}^2 + d\bar{z}^2 - d\bar{x}^2|^{1/2} .$$

This expression describes quantitatively the geometrically distinguishable nature of one of the superluminal spatial axes. Finally, since light propagates along space-time paths satisfying $ds = 0$, the equation for a light pulse emanating from the spatial origin of $\bar{S}$ at $\bar{t} = 0$ is

$$d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 = c^2 d\bar{t}^2 ,$$

which corresponds to a hyperboloid expanding in superluminal three-space at speed $d\bar{r}/d\bar{t} = c$ (in contrast with the usual sphere of Euclidean space).

VI. TRANSFORMATIONS BETWEEN SUPERLUMINAL FRAMES

We have considered transformations between two subluminal frames and between a subluminal and a superluminal frame. The transformations relating two superluminal frames will now be discussed.

In Sec. III we were concerned with a subluminal frame S and a superluminal frame $\bar{S}$ whose relative motion was along their common x direction. We will now introduce a second superluminal frame $\bar{S}'$ moving along the common x direction of the three frames, its speed relative to $\bar{S}$ being v, where $|v|<c$. By considerations analogous to those in Secs. III and IV one finds that the transformation between the two frames $\bar{S}$ and $\bar{S}'$ takes one of the following two forms [with γ defined by Eq. (6) as usual].

C: Superluminal to superluminal.

(i) $\bar{t}'$ axis forward in time relative to frame $\bar{S}$:

$$\bar{x}'=+\gamma(\bar{x}-v\bar{t}),\ \ \bar{y}'=\bar{y},\ \ \bar{z}'=\bar{z},\ \ \bar{t}'=+\gamma\left[\bar{t}-\frac{v\bar{x}}{c^2}\right].$$

(ii) $\bar{t}'$ axis backward in time relative to frame $\bar{S}$:

$$\bar{x}'=-\gamma(\bar{x}-v\bar{t}),\ \ \bar{y}'=\bar{y},\ \ \bar{z}'=\bar{z},\ \ \bar{t}'=-\gamma\left[\bar{t}-\frac{v\bar{x}}{c^2}\right].$$

These transformations are identical in form with those relating two subluminal frames.

There is, however, another type of transformation which holds for superluminal frames in certain circumstances. Let us consider again a subluminal frame S and two superluminal frames $\bar{S}$ and $\bar{S}'$. This time the relative velocity of S and $\bar{S}$ is chosen to be along their common y direction, while the relative velocity v of $\bar{S}$ and $\bar{S}'$ is still along their common x direction (see Fig. 4). We wish to determine the transformation equations relating $\bar{S}$ and $\bar{S}'$ now. One is tempted to assume that these two superluminal frames should still be related via Eqs. (C). Surprisingly, this is not the case. [Applying Eqs. (C) to $\bar{S}$ does not just produce a rotation of axes in space-time but actually results in a set of four axes which are no longer mutually orthogonal and which have dilated scales.] To understand why this is so and to find the correct transformation connecting $\bar{S}$ and $\bar{S}'$, we will construct a Minkowski diagram (Fig. 5) to illustrate how the axes of these two frames are related.

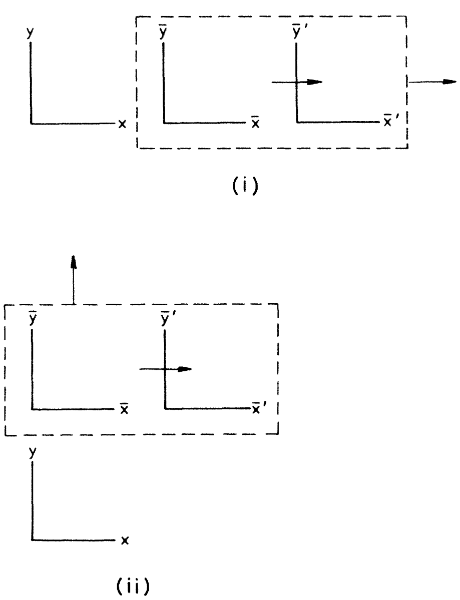

FIG. 4. A subluminal frame S and two superluminal frames $\bar{S}$ and $\bar{S}'$. In (i), the three frames are moving relative to each other along their common x direction. In (ii), the relative motion of S and $\bar{S}$ is now along their common y direction, while the relative motion of $\bar{S}$ and $\bar{S}'$ is still along their common x direction.

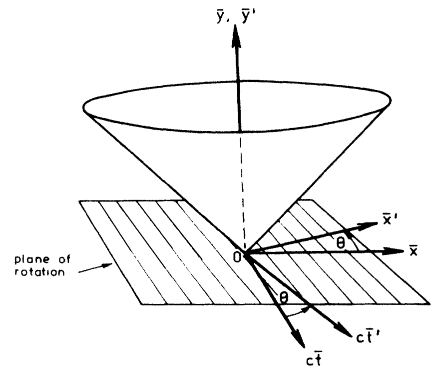

FIG. 5. Two superluminal frames $\bar{S}$ and $\bar{S}'$ whose spatial axes of relative motion ($\bar{x}$ and $\bar{x}'$) lie outside the light cone in space-time.

All superluminal frames have one of their spatial axes inside the light cone. Since the relative motion of S and $\bar{S}$ is along their common y direction, it must be the $\bar{y}$ axis of the frame $\bar{S}$ that lies inside the light cone. (Having established this fact, the subluminal frame S is not needed further and its axes will not be included on the diagram.) Also, the $\bar{y}$ and $\bar{z}$ axes of $\bar{S}$ in space-time will coincide with the $\bar{y}'$ and $\bar{z}'$ axes of $\bar{S}'$, respectively. (It is, of course, not possible to represent all the four axes of a frame on a space-time diagram and we have chosen to leave off the $\bar{z},\bar{z}'$ axis.) $\bar{S}$ and $\bar{S}'$ will have different x axes and different t axes in space-time due to their relative motion along the x direction. In particular, the transformation from $\bar{S}$ to $\bar{S}'$ will correspond to some sort of rotation in the $\bar{x},\bar{t}$ space-time plane. From the diagram, however, it is clear that the geometry within this plane is Euclidean, since its orientation in space-time is that of a subluminal spatial plane. This fact, together with the requirement that the rotation must maintain the orthogonality of the axes, establishes the correct transformation equations uniquely. The rotation must be of the Euclidean form

$$\bar{x}'=\bar{x}\cos\theta-c\bar{t}\sin\theta,\ \ c\bar{t}'=c\bar{t}\cos\theta+\bar{x}\sin\theta,$$

where the angle θ is given by

$$\tan\theta=\frac{\delta\bar{x}}{c\delta\bar{t}}=\frac{v}{c}.$$

For the case of Fig. 5 (i.e., $0<\theta<90^\circ$) we have

$$\cos\theta\equiv+(1+\tan^2\theta)^{-1/2}=+\left[1+\frac{v^2}{c^2}\right]^{-1/2}$$

and

$$\sin\theta \equiv +\tan\theta(1+\tan^2\theta)^{-1/2} = +\frac{v}{c}\left[1+\frac{v^2}{c^2}\right]^{-1/2} .$$

Hence, the required transformations for this case are

$$\bar{x}\,' = +\left[1+\frac{v^2}{c^2}\right]^{-1/2}(\bar{x}-v\bar{t}) ,$$

$$\bar{t}\,' = +\left[1+\frac{v^2}{c^2}\right]^{-1/2}\left[\bar{t}+\frac{v\bar{x}}{c^2}\right] .$$

More generally, for any rotation in the plane shown in Fig. 5, the transformation takes one of the following two forms.

D: Superluminal to superluminal.
(i) $\bar{t}\,'$ axis forward in time relative to frame $\bar{S}$:

$$\bar{x}\,' = +\left[1+\frac{v^2}{c^2}\right]^{-1/2}(\bar{x}-v\bar{t}),\quad \bar{y}\,'=\bar{y},\quad \bar{z}\,'=\bar{z} ,$$

$$\bar{t}\,' = +\left[1+\frac{v^2}{c^2}\right]^{-1/2}\left[\bar{t}+\frac{v\bar{x}}{c^2}\right] .$$

(ii) $\bar{t}\,'$ axis backward in time relative to frame $\bar{S}$:

$$\bar{x}\,' = -\left[1+\frac{v^2}{c^2}\right]^{-1/2}(\bar{x}-v\bar{t}),\quad \bar{y}\,'=\bar{y},\quad \bar{z}\,'=\bar{z} ,$$

$$\bar{t}\,' = -\left[1+\frac{v^2}{c^2}\right]^{-1/2}\left[\bar{t}+\frac{v\bar{x}}{c^2}\right] .$$

The positive sign under the square roots in Eqs. (D) is not easily foreseen in an approach confined to the three-dimensional picture (events evolving in space), but becomes quite understandable once one examines the situation in space-time and sees that the transformation in question just corresponds to a rotation of the spatial axes of a subluminal frame. The above discussion emphasizes the fact that the four-dimensional picture is the more advantageous when formulating the superluminal generalization of special relativity. However, even visualizing the situation only in three dimensions, it is possible to get some idea of why apparently similar circumstances require different types of transformation by identifying the physical feature which makes them dissimilar. The two cases in question are illustrated in Figs. 4(i) and 4(ii). In both cases the relative motion of $\bar{S}$ and $\bar{S}\,'$ is the same. However, the situations differ with regard to the inherent distinguishable axes of these two superluminal frames. The geometrically distinguishable spatial axis of $\bar{S}$ is its $\bar{x}$ axis in case (i) but its $\bar{y}$ axis in case (ii). Hence, the motion of $\bar{S}\,'$ is *along* the distinguishable axis of $\bar{S}$ in (i), whereas it is *orthogonal* to the distinguishable axis of $\bar{S}$ in (ii). Thus, the difference between the two cases from a three-dimensional viewpoint is apparent.

Whether the transformation between two superluminal frames is correctly described by Eqs. (D) or by the more familiar Eqs. (C) depends on whether the corresponding rotation of axes in space-time is in a spacelike or timelike plane, respectively.[7] It should be noted that Eqs. (D) apply both for $|v| < c$ and $|v| > c$, there being no critical velocity since there is no special direction inherent in the geometry of a Euclidean plane. Also note that a frame moving at speed greater than c relative to a superluminal frame need not necessarily be subluminal.

VII. THE FULL GROUP OF TRANSFORMATIONS AND THE PROOFS OF MARCHILDON, ANTIPPA, AND EVERETT

The transformations presented in the preceding sections characterize the extension of the subluminal Lorentz group required by the generalization to superluminal frames. *The full group consists of all possible rotations of the time axis in space-time, including rotations where this axis is allowed to pass through the light-cone.*[8] (The transformations we have presented are merely intended to illustrate the various special cases encountered within this overall group.) This group is much smaller than the full linear group implied by the proofs of Marchildon, Antippa, and Everett. The essential reason why those proofs do not apply to the present formulation is, of course, that they are contingent on the validity of the extended principle of relativity. However, it is perhaps worth looking explicitly at just how this principle is involved in the proofs. If a particular set of transformation equations is known to describe correctly the relationship between two frames, the assumption that all frames are equivalent requires that this transformation can also be applied to any third frame and that the result will be the coordinates of yet another equivalent frame. It is this requirement which necessitates that all sorts of different transformations must exist between different pairs of frames and which implies that the full linear group is needed to encompass all of them. Since, however, our approach does not entail the equivalence of all frames, it need not satisfy this requirement that any transformation can be applied to any frame. Indeed, our scheme obviously conflicts with this requirement. For example, although Eqs. (C) can be applied to some superluminal frames, they cannot correctly be applied to a superluminal frame whose x axis does not lie within the light cone. Also, if one wishes to apply a faster-than-light boost along a particular spatial axis of a superluminal frame, one must use equations of the form B if the space-time representation of that axis lies inside the light cone, whereas one must use equations of the form (D) otherwise.

VIII. ROTATIONS OF SUPERLUMINAL SPATIAL AXES

As a final task, we will obtain the equations for describing rotations of spatial axes within a superluminal frame. Such rotations are relevant to the first proof of Marchildon, Antippa, and Everett and to a related proof by Gorini.[9] Both proofs essentially show the consequences of assuming all spatial rotations to be identical with subluminal spatial rotations (as would be required by the extended principle of relativity). We will assume, for the purposes of illustration, that it is the $\bar{x}$ axis of the superluminal frame in question that lies inside the light cone in space-time. This means that this axis will have the geometric properties of a subluminal time axis. Hence, a rotation in the superluminal $\bar{x}\,\bar{y}$ plane (for example) will be the same as a rotation in a subluminal ty plane, i.e., a subluminal Lorentz transformation in the y direction. Now, such a Lorentz transformation can be written in the form (see, e.g., Ref. 10):

$$ct' = ct\cosh\theta - y\sinh\theta\,,$$

$$y' = y\cosh\theta - ct\sinh\theta\,,$$

where

$$\tanh\theta = \frac{v}{c}\,.$$

By analogy with the arguments in Sec. III, we may reexpress these equations in terms of the corresponding superluminal coordinates by making the changes

$$ct \to \bar{x}\,,\quad y \to \bar{y}\,,$$

so that we obtain

$$\bar{x}\,' = \bar{x}\cosh\theta - \bar{y}\sinh\theta\,,$$

$$\bar{y}\,' = \bar{y}\cosh\theta - \bar{x}\sinh\theta\,.$$

These are the required equations for a rotation in the superluminal $\bar{x}\,\bar{y}$ plane.[11] A rotation in the $\bar{x}\,\bar{z}$ plane is described by analogous equations:

$$\bar{x}\,' = \bar{x}\cosh\theta - \bar{z}\sinh\theta\,,$$

$$\bar{z}\,' = \bar{z}\cosh\theta - \bar{x}\sinh\theta\,.$$

However, a rotation of axes in the superluminal $\bar{y}\,\bar{z}$ plane will be described by the usual Euclidean equations:

$$\bar{y}\,' = \bar{y}\cos\theta + \bar{z}\sin\theta\,,$$

$$\bar{z}\,' = \bar{z}\cos\theta - \bar{y}\sin\theta\,,$$

because the $\bar{y}$ and $\bar{z}$ axes are both outside the light cone and so the plane of rotation is spacelike.

ACKNOWLEDGMENT

The authors wish most sincerely to thank Dr. Louis Marchildon for helpful comments on several of the points discussed in this paper.

[1] L. Marchildon, A. F. Antippa, and A. E. Everett, Phys. Rev. D **27**, 1740 (1983).

[2] For a bibliography on this topic, the reader is referred to Ref. 1.

[3] A. F. Antippa and A. E. Everett, Phys. Rev. D **8**, 2352 (1973).

[4] A. F. Antippa, Phys. Rev. D **11**, 724 (1975).

[5] All lines in space-time which are parallel to the tachyon's world line must denote constant position in this superluminal frame.

[6] More precisely, in each superluminal frame we have a *range* of directions geometrically distinguishable from the rest, namely, that range of directions lying inside the light cone.

[7] "Spacelike" and "timelike" are employed here in their usual (subluminal) sense.

[8] If rotations of spatial axes within frames are to be included in the overall group as well, the full group then consists of all possible rotations of four orthogonal axes in space-time, including rotations where axes are allowed to pass through the light cone.

[9] V. Gorini, Commun. Math. Phys. **21**, 150 (1971).

[10] J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1958), Chap. IV, Sec. 2.

[11] To be precise, these equations only apply to rotations for which the $\bar{x}$ axis does not pass through the light cone. Equations describing the opposite case can also be constructed without much difficulty.