Calabi-Yau manifolds from arbitrary weighted homogeneous spaces

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Three new Calabi-Yau spaces relevant for superstring compactifications with Euler characteristic -144, -156, and -120 are constructed from the weighted complex projective space WCP^5. No nontrivial discrete isometry group has been found for these spaces.

I. INTRODUCTION

Unification of strong interactions, electroweak interactions, and gravity is a main concern of theoretical physicists. Kaluza-Klein supergravities have been considered in these contexts, but have the following difficulties. (1) Although the maximal $d=11$ supergravity can be compactified with the $\mathrm{SU(3)\times SU(2)\times U(1)}$ isometry group,[1] the quantum numbers of fermions on harmonic expansion do not agree with quark-lepton representations. Furthermore, fermions in four dimensions compactified from $d=11$ supergravity are vectorlike. (2) A chiral fermion in four dimensions can be obtained from higher even-dimensional theory with nontrivial background.[2] However for $d=10$ theory, the scalar equation of motion is difficult to satisfy. (3) The space-time after compactification is anti–de Sitter space[3] of Planck size. These problems have led to speculation that the Kaluza-Klein supergravity theory is the theory of preon dynamics.[4]

Recently, Green and Schwarz[5] have shown that the gravitational and gauge anomalies of ten-dimensional theories have been miraculously canceled in SO(32) and $E_8\times E_8$ gauge groups. Furthermore Candelas, Horowitz, Strominger, and Witten[6] have shown the above three problems can be resolved in compactification of $d=10$ superstring theory into four-dimensional Minkowski and Calabi-Yau (CY) spaces.[7–9]

But there remains an important problem: to choose the "correct" one out of a huge number of conjectured Calabi-Yau spaces. (However only a very limited number of CY spaces have been constructed until now.) The following criteria may be useful in selecting the right compactification: (1) Compactification into Calabi-Yau space should be stable against string quantum fluctuations; (2) in addition, one requires a global anomaly[10] freedom. Presently our understanding of superstring theories is not deep enough to settle this problem immediately. At this stage, it is important to find as many Calabi-Yau spaces as possible. Then further developments in superstring theories will allow one to determine the right Calabi-Yau space.

In this paper, we report three new Calabi-Yau spaces with Euler characteristics of -144, -156, and -120 constructed from the weighted complex projective space WCP^n. No nontrivial discrete isometry group has been found for these spaces.

II. CONSTRUCTION OF CALABI-YAU SPACES FROM THE WEIGHTED COMPLEX PROJECTIVE SPACE WCP^n

The weighted complex projective space WCP^n is constructed from complex $(n+1)$-dimensional space excluding the origin, $C^{n+1}-\{0\}$, by identifying

$$(z_1,\dots,z_{n+1})\simeq(\lambda^{i_1}z_z,\dots,\lambda^{i_{n+1}}z_{n+1}), \tag{1}$$

where $i_1,\dots,i_{n+1}$ are arbitrary positive integers. Usual CP^n spaces correspond to the case $i_1=i_2=\cdots=i_{n+1}=1$. The total Chern class of WCP^n is

$$c=\prod_{k=1}^{n+1}(1+i_kJ), \tag{2}$$

where J can be interpreted as a normalized Ricci two-form.[11]

One can consider a submanifold of WCP^n with k constraints which are homogeneous polynomials of degree $d_1,\dots,d_k$ counted with weight given in Eq. (1). The metric and complex structure depend on the detailed form of constraints. One of the conditions for a smooth manifold is

$$dI_1\wedge\cdots\wedge dI_k\neq 0 \tag{3}$$

for every point. The total Chern class with constraints is[12]

$$c=\frac{\prod_{m=1}^{n+1}(1+i_mJ)}{\prod_{m=1}^{k}(1+d_mJ)} \tag{4}$$

and the Euler characteristic is $c_3\prod d_m/\prod i_m$, where c_3 is the coefficient of J^3 in the expansion of Eq. (4). The requirement of the SU(3) holonomy group is satisfied if

$$\sum_{m=1}^{k+4} i_m=\sum_{m=1}^{k} d_m\ . \tag{5}$$

This is more relaxed than the corresponding formula

$$k+4=\sum_{m=1}^{k} d_m\quad \text{for } \mathrm{CP}^n\ . \tag{6}$$

Three new Calabi-Yau spaces are constructed from weighted CP^5 as follows.

(1) CY space from WCP^5 with $\chi=-144$.

The Calabi-Yau space is constructed with identification

$$(z_1,z_2,w_1,\ldots,w_4)\simeq(\lambda^2 z_1,\lambda^2 z_2,\lambda w_1,\ldots,\lambda w_4) \tag{7}$$

and with polynomial constraints

$$I_1=z_1{}^2+z_2{}^2+\sum_{i=1}^{4} w_i{}^4=0\,, \tag{8}$$

$$I_2=z_1{}^2+az_2{}^2+\sum_{i=1}^{4} b_i w_i{}^4=0\,, \tag{9}$$

where $(a,b_1,\ldots,b_4)$ are complex parameters and any two of $(1,a,b_1,\ldots,b_4)$ should not coincide. The power of z_j $(j=1,2)$ with weight 2 is identical to the power of w_i $(i=1,2,3,4)$ with weight 1 in Eqs. (8) and (9) for consistency with the identification of Eq. (7). The Ricci-flat condition Eq. (5) is satisfied with $\sum i_m=8$ and $\sum d_m=8$ obviously.

We want to show that spaces constructed with the equivalence relation Eq. (7) and constraint equations (8) and (9) are smooth. Fixed points in the equivalence relation Eq. (7), if they exist, give rise to singularities. For $\lambda=-1$, points with $w_i=0$ $(i=1,2,3,4)$ and arbitrary z_j $(j=1,2)$ can be fixed points, if they satisfy constraint equations (8) and (9). However constraint equations (8) and (9) with $w_i=0$ $(i=1,2,3,4)$ require $z_i=0$, since a is not equal to unity in Eq. (9). But the origin is excluded from WCP^5. There is no fixed point in the equivalence relation Eq. (7).

Singularities can also originate from constraint equations (8) and (9). Each constraint eliminates one complex coordinate; thus, the normal directions of two constraints should be independent everywhere to eliminate two complex coordinates smoothly from WCP^5. If the two-form

$$\Omega=dI_1\wedge dI_2 \tag{10}$$

vanishes at any point, we cannot eliminate two complex coordinates smoothly, and this point becomes singular. Equation (10) for constraints (8) and (9) is

$$\Omega=\left[z_1dz_1+z_2dz_2+2\sum_{i=1}^{4} w_i{}^3dw_i\right]\wedge\left[z_1dz_1+az_2dz_2+2\sum_{i=1}^{4} b_iw_i{}^3dw_i\right]. \tag{11}$$

Equation (11) vanishes if dI_1 and dI_2 are parallel as

$$z_1=\mu z_1,\quad z_2=\mu a z_2,\quad w_i=\mu b_i w_i\quad (i=1,2,3,4)\,. \tag{12}$$

For $\mu=1$, Eq. (12) can be satisfied by arbitrary z_1, $z_2=0$ and $w_i=0$ $(i=1,2,3,4)$. Constraint equations (8) and (9) with $z_2=0$ and $w_i=0$ $(i=1,2,3,4)$ require $z_1=0$, but the origin is already eliminated. Similar arguments hold for the $\mu a=1$ and $\mu b_i=1$ $(i=1,2,3,4)$ cases, respectively. Therefore this complex three-dimensional manifold is smooth and Ricci flat with Euler characteristic -144.

(2) CY space from WCP^5 with $\chi=-156$.

The equivalence relation

$$(z_0,z_1,z_2,w_1,w_2,w_3)\simeq(\lambda^3 z_0,\lambda^2 z_1,\lambda^2 z_2,\lambda w_1,\lambda w_2,\lambda w_3) \tag{13}$$

and constraints

$$I_1=z_0{}^2+z_1{}^3+z_2{}^3+\sum_{i=1}^{3} w_i{}^6=0\,, \tag{14}$$

$$I_2=z_1{}^2+az_2{}^2+\sum_{i=1}^{3} b_iw_i{}^4=0\,, \tag{15}$$

give a Calabi-Yau space. Here complex parameters $(a,b_1,\ldots,b_3)$ should avoid $i+ja^3+kb_1{}^3+lb_2{}^3+mb_3{}^3=0$ with $(i,j,k,l,m)=0$ or 1. This space is Ricci flat and has Euler characteristic -156.

Fixed points can arise for $\lambda=-1$, $e^{2\pi i/3}$, or $e^{4\pi i/3}$, but they are not consistent with constraints Eqs. (14) and (15). Thus there is no singularity in equivalence relation Eq. (13).

The transversality condition is satisfied if the two-form

$$\Omega=\left[2z_0dz_0+3z_1{}^2dz_1+3z_2{}^2dz_2+6a\sum_{i=1}^{3} w_i{}^5dw_i\right]\wedge\left[z_1dz_1+az_2dz_2+2\sum_{i=1}^{3} b_iw_i{}^3dw_i\right] \tag{16}$$

does not vanish everywhere. Singularities can arise for the cases

$$z_0=0,\quad 3z_1{}^2=\mu z_1,\quad 3z_2{}^2=\mu a z_2\,,\qquad 3w_i{}^5=\mu b_i w_i{}^3\quad (i=1,2,3)\,, \tag{17}$$

where dI_1 and dI_2 are parallel. Equation (17) can be satisfied by

$$z_1=0 \text{ or } z_1=\mu/3\,, \tag{18}$$

$$z_2=0 \text{ or } z_2=\mu a/3\,, \tag{19}$$

$$w_i=0 \text{ or } w_i{}^2=\mu b_i/3\quad (i=1,2,3)\,. \tag{20}$$

To eliminate these possible singular points, the complex parameters $(a,b_1,\ldots,b_3)$ in Eqs. (14) and (15) should avoid the following:

$$i+ja^3+kb_1{}^3+lb_2{}^3+mb_3{}^3=0\,, \tag{21}$$

where i, j, k, l, and m are either 0 or 1 to cover in Eqs. (18)–(20).

(3) CY space from WCP^5 with $\chi=-120$.

This space is constructed from the identification

$$(z_1,z_2,w_1,w_2,v_1,v_2)\simeq(\lambda^3 z_1,\lambda^3 z_2,\lambda^2 w_1,\lambda^2 w_2,\lambda v_1,\lambda v_2) \tag{22}$$

and constraints

$$I_1=(z_1{}^2+z_2{}^2)+(w_1{}^3+w_2{}^3)+(v_1{}^6+v_2{}^6)=0\,, \tag{23}$$

$$I_2=(z_1{}^2+\alpha z_2{}^2)+\beta(w_1{}^3+\gamma w_2{}^3)+\delta(v_1{}^6+\epsilon v_2{}^6)=0\,, \tag{24}$$

where any two out of $(1,\alpha,\beta,\delta,\beta\gamma,\delta\epsilon)$ should be different.

Fixed points in the equivalence relation Eq. (22) can arise for the case of $\lambda=-1$, $e^{2\pi i/3}$, or $e^{4\pi i/3}$. Constraint equations (23) and (24) require these fixed points to lie at the origin, which has been already eliminated in WCP^5. Thus there is no singularity in the equivalence relation Eq. (22).

The transversality condition is satisfied if

$$\Omega=(2z_1dz_1+2z_2dz_2+3w_1{}^2dw_1+3w_2{}^2dw_2+6v_1{}^5dv_1+6v_2{}^5dv_2) \wedge(2z_1dz_1+2\alpha z_2dz_2+3\beta w_1{}^2dw_1+3\beta\gamma w_2{}^2dw_2+6\delta v_1{}^5dv_1+6\delta\epsilon v_2{}^5dv_2)\neq 0 \ . \tag{25}$$

Singularity can arise for

$$z_1=\mu z_1,\ \ z_2=\mu\alpha z_2,\ \ w_1{}^2=\mu\beta w_1{}^2, \quad w_2{}^2=\mu\beta\gamma w_2{}^2,\ \ v_1{}^5=\mu\delta v_1{}^5,\ \ v_2{}^5=\mu\delta\epsilon v_2{}^5, \tag{26}$$

where dI_1 is parallel to dI_2. To avoid the singularity given by Eq. (26), any two of $(1,\alpha,\beta,\delta,\beta\gamma,\delta\epsilon)$ should not equal each other. Thus the constructed space is smooth and Ricci flat with Euler characteristic -120.

We now turn to the familiar Calabi-Yau spaces. Strominger and Witten have found CY spaces with branched coverings of CP^n where only one of the weights differs from one and the remaining weight equals one. They are

(1) $(z_1,z_2,\ldots,z_5)\simeq(\lambda^2z_1,\lambda z_2,\ldots,\lambda z_5)$ with $\chi=-204$,
(2) $(z_1,z_2,\ldots,z_5)\simeq(\lambda^4z_1,\lambda z_2,\ldots,\lambda z_5)$ with $\chi=-296$,
(3) $(z_1,z_2,\ldots,z_6)\simeq(\lambda^3z_1,\lambda z_2,\ldots,\lambda z_6)$ with $\chi=-256$,
(4) $(z_1,z_2,\ldots,z_6)\simeq(\lambda^2z_1,\lambda z_2,\ldots,\lambda z_6)$ with $\chi=-156$.

Strominger and Witten[9] have found one space from arbitrary WCP^n:

$$(z_1,z_2,\ldots,z_5)\simeq(\lambda^5z_1,\lambda^2z_2,\lambda z_3,\lambda z_4,\lambda z_5)$$

with $\chi=-288$. The new spaces reported in this paper belong to this category of homogeneous space with an arbitrary weight.

There are no other Calabi-Yau spaces, from differently weighted (WCP^4, WCP^5, WCP^6, WCP^7, WCP^8), up to highest weight (100, 40, 30, 20, 10) in the equivalence relation Eq. (1).

It is not clear how these new Calabi-Yau spaces can be applied to the phenomenology of superstring theories. The number of generations for these spaces are 72, 78, and 60, respectively, and are quite large compared to three generations of the standard model. The Hosotani[13] mechanism of gauge-symmetry breaking by a twisted boundary condition cannot be applied to the Calabi-Yau spaces in this paper due to the absence of a nontrivial discrete isometry group.

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