Renormalizing $(\phi^3)_6$ theory in curved space-time

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The renormalization of ϕ^3 theory in a six-dimensional conformally flat space-time is discussed at the two-loop level. The background-field method and the momentum-space representation of the Feynman propagator are used to calculate the ultraviolet divergences of the effective action. It is shown that $(\phi^3)_6$ theory is renormalizable in a conformally flat space-time at the two-loop level. The counterterms are given to the two-loop order. The next-to-leading-order term in the β function for the coupling constant of the $R\phi^2$ term is then obtained.

I. INTRODUCTION

Quantum field theory in curved space-time has been one of the interesting subjects of investigation.[1] In particular, the possibility of explaining the evolution of the early Universe based on particle physics must be carefully examined through a closer study of quantum field theories in curved space-time.[2] Furthermore, the examination of the ultraviolet structure of quantum field theory in curved space-time is thought to give some clue to the ultraviolet problem of quantum gravity itself.

Renormalizability is the first problem which arises when one considers an interacting field theory in curved space-time. This problem has been considered by many authors.[3] Although a general scheme to discuss the renormalizability of a given theory,[4] especially a gauge theory,[5] has not yet been given, the current folklore is that a theory may be renormalizable in curved space-time if it is renormalizable in flat space-time. The explicit calculations of scalar field theories and gauge theories in curved space-time with various topologies, in fact, support the renormalizability in curved space-time.

In this paper we consider ϕ^3 theory, $(\phi^3)_6$, in a six-dimensional conformally flat space-time. We perform an explicit calculation at the two-loop level. This is of interest because this theory is the simplest one which has asymptotic freedom. This theory was considered in Ref. 6 in flat space-time. The two-loop renormalization was performed in Ref. 7 in spherical space-time. In Ref. 8, the one-loop calculation was presented in a general curved space-time by using the heat-kernel technique. The authors in Ref. 9 investigated this theory at the two-loop level in a conformally flat space-time and found a peculiar structure of this theory.

The purpose of this paper is to prove the renormalizability of $(\phi^3)_6$ theory in a conformally flat space-time at the two-loop level. We will obtain as a byproduct the β function for the coupling constant of the $\xi R\phi^2$ term. We use the background-field method[10–12] to obtain the effective action (Green's function). The momentum-space representation of the Feynman propagator[13,14] and dimensional regularization are employed to analyze the ultraviolet divergences.

In Sec. II we present our notations and give a brief review of the momentum-space representation of the Feynman propagator which was developed in Ref. 14. In Sec. III we calculate the divergences at the two-loop level, and renormalizability is proved to this order. The counterterms and the β functions mentioned above are explicitly given. Section IV contains a brief summary.

II. $(\phi^3)_6$ THEORY IN A CONFORMALLY FLAT SPACE-TIME

In this section we explain our notations and some technical points which will be used in Sec. III. We restrict ourselves to a conformally flat space-time. Therefore, the metric tensor can be conformally transformed into a flat space-time:

$$ds^2=\Omega^2\eta_{\mu\nu}dx^\mu dx^\nu=\Omega^2(\eta)\left[d\eta^2-\sum_{i=1}^{n-1}dx^{i^2}\right], \quad (2.1)$$

where n is the dimension of the space-time.

The bare action for ϕ^3 theory is given by[15]

$$I[\phi_B]=\int dv_x\left[-\tfrac{1}{2}\phi_B\Box\phi_B-\tfrac{1}{2}m_B{}^2\phi_B{}^2-\tfrac{1}{2}\xi_B R\phi_B{}^2-\frac{1}{3!}g_B\phi_B{}^3\right], \quad (2.2)$$

where

$$dv_x\equiv\sqrt{-g(x)}d^nx,\quad \sqrt{-g(x)}=\Omega^n(\eta)\,.$$

According to the background-field method,[16] Eq. (2.2) will be expanded about the background field $\hat{\phi}_B$:

$$I[\hat{\phi}_B+\phi]=I[\hat{\phi}_B]+\phi\left[\frac{\delta I}{\delta\phi}\right]_{\phi=\hat{\phi}_B}+I[\hat{\phi}_B,\phi] \quad (2.3a)$$

and

$$I[\hat{\phi}_B,\phi]=-\tfrac{1}{2}\int dv_x\phi(\Box+m_B{}^2+\xi_B R)\phi-\tfrac{1}{2}g_B\int dv_x\phi^2\hat{\phi}_B-\frac{1}{3!}g_B\int dv_x\phi^3\,. \quad (2.3b)$$

Now define the renormalized quantities and counterterms as follows by taking into account that quantum fields need not be renormalized:

$$\mu^{3-n/2}\hat{\phi}_B = Z^{1/2}\hat{\phi}, \quad Z = 1+\delta Z \;, \tag{2.4a}$$

$$m_B{}^2 = m^2 + \delta m^2 \;, \tag{2.4b}$$

$$\xi_B = \xi + \delta\xi \;, \tag{2.4c}$$

$$g_B = \mu^{3-n/2}(g+\delta g) \;, \tag{2.4d}$$

where an arbitrary parameter μ of mass unit is introduced so that g and $\hat{\phi}$ have the same dimension for all n as they do for $n=6$. By substituting Eqs. (2.4) into Eq. (2.3), we obtain the action in terms of the renormalized quantities and the counterterms. $I[\hat{\phi}_B]$ becomes

$$I[\hat{\phi}_B] = I^R[\hat{\phi}] + I^{\rm ct}[\hat{\phi}] \;, \tag{2.5a}$$

where

$$I^R[\hat{\phi}] = \mu^{n-6}\int dv_x \left[-\tfrac{1}{2}\hat{\phi}\Box\hat{\phi} - \tfrac{1}{2}m^2\hat{\phi}^2 - \tfrac{1}{2}\xi R\hat{\phi}^2 - \frac{1}{3!}g\hat{\phi}^3 \right] \tag{2.5b}$$

and

$$I^{\rm ct}[\hat{\phi}] = \mu^{n-6}\int dv_x \{ -\tfrac{1}{2}\delta Z\,\hat{\phi}\Box\hat{\phi} - \tfrac{1}{2}(m^2\delta Z + \delta m^2 + \delta m^2\delta Z)\hat{\phi}^2 - \tfrac{1}{2}(\xi\,\delta Z + \delta\xi + \delta\xi\,\delta Z)R\hat{\phi}^2 - (1/3!)[(g+\delta g)(1+\delta Z)^{3/2} - g]\hat{\phi}^3 \} \;. \tag{2.5c}$$

The terms which are quadratic and cubic in ϕ take the following form:[17]

$$I[\hat{\phi},\phi] = -\tfrac{1}{2}\int dv_x \phi(\Box + m^2 + \xi R)\phi - \tfrac{1}{2}g\int dv_x \phi^2\hat{\phi} - \frac{1}{3!}\mu^{3-n/2}g\int dv_x\phi^3 - \tfrac{1}{2}\int dv_x\phi(\delta m^2 + \delta\xi\, R)\phi - \tfrac{1}{2}[(g+\delta g)(1+\delta Z)^{1/2} - g]\int dv_x\phi^2\hat{\phi} - \frac{1}{3!}\mu^{3-n/2}\delta g\int dv_x\phi^3 \;. \tag{2.6}$$

Since we are interested in only the divergences which arise in the matter action, it is enough to treat the background field $\hat{\phi}$ perturbatively.[18] Therefore, the first term of Eq. (2.6) is regarded as a free part of the theory and the remaining terms as interactions:

$$I = I_0 + I_{\rm int} \;,$$

with

$$I_0 = -\tfrac{1}{2}\int dv_x \phi(\Box + m^2 + \xi R)\phi \;. \tag{2.7}$$

The form of $I_{\rm int}$ is obvious in Eq. (2.6). Equation (2.7) defines the free propagator

$$G(x,x') = -i\langle 0 \,|\, T[\phi(x)\phi(x')] \,|\, 0\rangle \;,$$

which satisfies

$$(\Box + m^2 + \xi R)G(x,x') = -(-g)^{-1/2}\delta^n(x-x') \;. \tag{2.8}$$

The effective action $\Gamma(\hat{\phi})$ can be obtained by calculating the one-particle-irreducible diagrams with the help of the Wick reduction formula. The propagator G is, of course, used on the internal lines.

In the case of the conformally flat space-time, Eq. (2.8) can be solved by going into the momentum space as discussed in Ref. 14. If a new function $g(x,x')$ is defined as

$$g(x,x') \equiv \Omega(\eta)^{n/2-1}G(x,x')\Omega(\eta')^{n/2-1} \;, \tag{2.9}$$

the equation for $g(x,x')$ reads

$$\eta^{\mu\nu}\partial_\mu\partial_\nu g(x,x') + \{m^2\Omega^2 + [\xi - \xi(n)]R\,\Omega^2\}g(x,x') = -\delta^n(x-x') \;, \tag{2.10}$$

where $\xi(n) = (n-2)/4(n-1)$. By Fourier transforming Eq. (2.10) into momentum space,

$$g(x,x') = \frac{1}{(2\pi)^n}\int \exp[ik_0\eta - ik_0'\eta' - i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')] \times g(k_0,k_0',\mathbf{k})dk_0 dk_0' d^{n-1}\mathbf{k} \;, \tag{2.11}$$

it is straightforward to drive the following integral equation for $g(k_0,k_0',\mathbf{k})$:[19]

$$g(k_0,k_0',\mathbf{k}) = (k_0{}^2 - \mathbf{k}^2)^{-1}\delta(k_0 - k_0') - (k_0{}^2 - \mathbf{k}^2)^{-1}\int dp_0\hat{V}(k_0,p_0)g(p_0,k_0',\mathbf{k}) \;, \tag{2.12}$$

where $\hat{V}(k_0,p_0)$ is the Fourier transform of $V(\eta)$,

$$V(\eta) \equiv -m^2\Omega(\eta)^2 - [\xi - \xi(n)]R\,\Omega(\eta)^2 \tag{2.13a}$$

then

$$\hat{V}(k_0,p_0) = \frac{1}{2\pi}\int e^{-i(k_0-p_0)\eta}V(\eta)d\eta \;. \tag{2.13b}$$

Since Eq. (2.12) can be solved iteratively, we will get the solution for G by putting back Eq. (2.12) into Eq. (2.11) and using Eq. (2.9). For our purpose of finding the pole terms, only one iteration of Eq. (2.12) will be needed as easily seen by power counting. Explicitly the form which will be used in the next section is

$$g(k_0,k_0',\mathbf{k}) = g_0(k_0,k_0',\mathbf{k}) + g_V(k_0,k_0',\mathbf{k}) + g_f(k_0,k_0',\mathbf{k}) \;, \tag{2.14}$$

with

$$g_0(k_0,k_0',\mathbf{k})=\delta(k_0-k_0')(k_0{}^2-\mathbf{k}^2)^{-1}\ , \tag{2.15a}$$

$$g_V(k_0,k_0',\mathbf{k})=-(k_0{}^2-\mathbf{k})^{-1}\hat{V}(k_0,k_0')(k_0'^2-\mathbf{k}^2)^{-1}\ . \tag{2.15b}$$

It turns out that the explicit form of g_f is not required to discuss the renormalizability to the order we are working.

Since all the technical devices which will be needed have been prepared, we can proceed to the explicit calculations.

III. DIVERGENCES IN $(\phi^3)_6$ THEORY

In this section we calculate the pole terms to the two-loop order. For completeness we start our computations at the one-loop level.

A. One-loop renormalization

At the one-loop level, the divergences of the effective action (for the matter parts) are contained in the terms

$$\Gamma^{(1)}(\hat{\phi})=-\frac{i}{4}g^2\int dv_x dv_{x'}\hat{\phi}(x)G^2(x,x')\hat{\phi}(x') -\frac{i}{6}g^3\int dv_x dv_{x'} dv_{x''}\hat{\phi}(x)G(x,x')\hat{\phi}(x')\times G(x',x'')\hat{\phi}(x'')G(x'',x)\ . \tag{3.1}$$

The pole terms of $\Gamma^{(1)}$ are given by those of $G^2(x,x')$ and $G(x,x')G(x',x'')G(x'',x)$. From Eqs. (2.9) and (2.11) we can immediately write down the expressions for G^2 and GGG. Power counting only the first and the second (the first) terms in Eq. (2.14) leads to the divergence of G^2 (GGG) at $n=6$. We will write these contributions schematically in Fig. 1. In Fig. 1 the bold line represents the full propagator $g(k_0,k_0',\mathbf{k})$. The solid line corresponds to g_0 [Eq. (2.15a)], the line with the triangle to g_V [Eq. (2.15b)].

The pole part of G^2 [Fig. 1(a)] is easily calculated to be

$$P_P[\mu^{6-n}G^2(x,x')]=-i\frac{1}{(4\pi)^3}\frac{1}{n-6}\{-\tfrac{1}{3}[\Omega(\eta)\Omega(\eta')]^{-1-n/2}\eta^{\mu\nu}\partial_\mu\partial_\nu\delta^n(x-x')+2[\Omega(\eta)]^{-8}V(\eta)\delta^n(x-x')\}$$
$$=-i\frac{1}{(4\pi)^3}\frac{1}{n-6}(-\tfrac{1}{3}[\Omega(\eta')]^{-n}[\Box+\xi(n)R]\delta^n(x-x')-2[\Omega(\eta)]^{-6}\{m^2+[\xi-\xi(n)]R\}\delta^n(x-x'))\ . \tag{3.2}$$

In Eq. (3.2) use has been made of the following identity:

$$[\Omega(\eta)\Omega(\eta')]^{-1-n/2}[\eta^{\mu\nu}\partial_\mu\partial_\nu\delta^n(x-x')]=[\Box_x+\xi(n)R]\delta^n(x-x')[\Omega(\eta')]^{-n}\ . \tag{3.3}$$

Similarly the pole part of GGG [Fig. 1(b)] is easily found which is the same as in flat space-time:

$$P_P[\mu^{6-n}G(x,x')G(x',x'')G(x'',x)]=\frac{i}{(4\pi)^3}\frac{1}{n-6}\Omega(\eta)^{-12}\delta^n(x-x')\delta^n(x-x'')\ . \tag{3.4}$$

Substituting Eqs. (3.2) and (3.3) into Eq. (3.1), we get the pole term of $\Gamma^{(1)}(\hat{\phi})$:

$$P_P\Gamma^{(1)}(\hat{\phi})=\frac{g^2}{(4\pi)^3}\frac{1}{n-6}\mu^{n-6}\int dv_x\{\tfrac{1}{12}\hat{\phi}\Box\hat{\phi}+\tfrac{1}{2}m^2\hat{\phi}^2+\tfrac{1}{2}[\xi-\tfrac{5}{6}\xi(6)]R\hat{\phi}^2+\tfrac{1}{6}g\hat{\phi}^3\}\ . \tag{3.5}$$

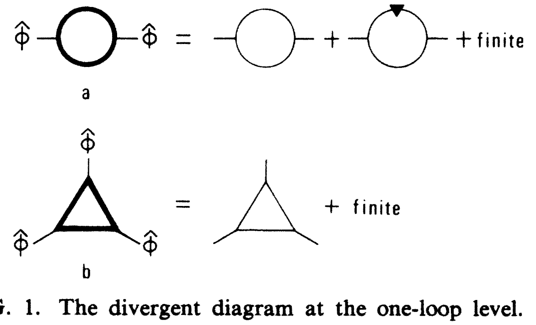

FIG. 1. The divergent diagram at the one-loop level. The bold line represents the full propagator g. The solid-line (line with triangle) corresponds to g_0 (g_V).

Adding the counterterms at the one-loop level of Eq. (2.5c), it will be seen that the pole terms are removable if the counterterms are fixed as follows:

$$\delta Z^{(1)}=\frac{1}{6}\frac{g^2}{(4\pi)^3}\frac{1}{n-6}\ ,$$
$$\delta m^{2(1)}=\frac{5}{6}\frac{g^2}{(4\pi)^3}\frac{1}{n-6}m^2\ ,$$
$$\delta\xi^{(1)}=\frac{5}{6}\frac{g^2}{(4\pi)^3}\frac{1}{n-6}(\xi-\tfrac{1}{5})\ ,$$
$$\delta g^{(1)}=\frac{3}{4}\frac{g^3}{(4\pi)^3}\frac{1}{n-6}\ . \tag{3.6}$$

These results are in agreement with those in Ref. 8.

B. Two-loop renormalization

We next turn to the two-loop calculations. As in flat space-time, a power-counting argument tells us that we have only to consider the two-point (self-energy diagram) and the three-point (vertex diagram) Green's functions.

The divergent diagrams we should compute are listed in Figs. 2 and 3. The contributions from each propagator of Eq. (2.14) are also shown as in Fig. 1. The dashed line means that the corresponding propagator contains any combination of g_0, g_V, and g_f other than those explicitly shown in the figure. The vertex with a closed circle represents the counterterm vertex[20] which is given by the fifth term of Eq. (2.6). The closed circle on the line corresponds to the counterterm contribution from the fourth term of Eq. (2.6). It should be noted that the integral over the dashed lines does not produce any new divergence as easily seen by power counting. Therefore, the diagrams with dashed lines have only the single pole which is just that of the one-loop subdiagram. This means that such poles are canceled out trivially among the two-loop diagrams and the counterterm diagrams. The nontrivial diagrams which should be calculated are those which do not contain dashed lines.

The calculations of each diagram are tedious but straightforward. The diagrams which contain only the g_0 propagator, Eq. (2.15a) (the first diagram of the two-point and three-point functions), are the same as in flat spacetime except for an additional $\ln\Omega(\eta)$-type contribution. In order to explain from where this kind of contribution comes out, let us consider the diagram of Fig. 2(a). The expression for Fig. 2(a) is

$$\Gamma^{(2)}_{2a}=\tfrac{1}{4}g^4\mu^{6-n}\int dv_x dv_{x'} dv_{x''} dv_{x'''}\hat{\phi}(x)G(x,x')G(x,x'')\times G^2(x'',x''')G(x''',x')\hat{\phi}(x')\ .$$

Substituting Eqs. (2.9) and (2.11), we have

$$\begin{aligned}\Gamma^{(2)}_{2a}=\tfrac{1}{4}g^4\mu^{6-n}\int dv_x dv_{x'}\hat{\phi}(x)\hat{\phi}(x')(2\pi)^{-5n}[\Omega(\eta)\Omega(\eta')]^{2-n}d^nx''d^nx'''[\Omega(\eta'')\Omega(\eta''')]^{3-n/2}\\ \times\int \exp[ik_0\eta-ik_0'\eta'-i\mathbf{k}(\mathbf{x}-\mathbf{x}')]g(k_0,k_0',\mathbf{k})dk_0dk_0'd^{n-1}\mathbf{k}\\ \times\int \exp[ip_0\eta-ip_0'\eta''-i\mathbf{p}(\mathbf{x}-\mathbf{x}'')]g(p_0,p_0',\mathbf{p})dp_0dp_0'd^{n-1}\mathbf{p}\\ \times\left[\int \exp[iq_0\eta''-iq_0'\eta'''-i\mathbf{q}(\mathbf{x}''-\mathbf{x}''')]g(q_0,q_0',\mathbf{q})dq_0dq_0'd^{n-1}\mathbf{q}\right]^2\\ \times\int \exp[ir_0\eta'''-ir_0'\eta'-i\mathbf{r}(\mathbf{x}'''-\mathbf{x}')]g(r_0,r_0',\mathbf{r})dr_0dr_0'd^{n-1}\mathbf{r}\ .\end{aligned} \tag{3.7}$$

From this expression, it is easy to see the origin of the $\ln\Omega$-type contribution. This is due to the incomplete compensation of Ω factor at the vertices x'' and x'''. In fact,

$$[\Omega(\eta'')\Omega(\eta''')]^{3-n/2}=1-\tfrac{1}{2}(n-6)\ln[\Omega(\eta'')\Omega(\eta''')]+O((n-6)^2)\ ,$$

and the second term multiplied by the double pole produces a single pole. The factor of $[\Omega(\eta)\Omega(\eta')]^{2-n}$ also gives the similar contribution to pole terms as seen in the subsequent explicit calculations.

Now let us list our results of calculations. Writing $\Gamma^{(2)}$ as

$$P_P\Gamma^{(2)}=\mu^{n-6}\int dv_x dv_{x'}\hat{\phi}(x)\hat{\phi}(x')[\Omega(\eta)\Omega(\eta')]^{2-n}\times\frac{1}{(2\pi)^n}\int e^{iq(x-x')}\times d^nq\left[\frac{-q^2}{4\pi\mu^2}\right]^{n-6}\frac{g^4}{(4\pi)^6}I\ , \tag{3.8}$$

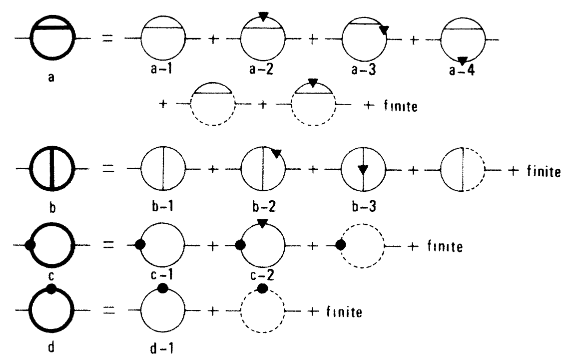

FIG. 2. The divergent two-point diagram at the two-loop level. The meaning of each line is the same as that in Fig. 1. The dashed line means any combination of g_0, g_V, and g_f. The closed circle corresponds to the counterterm.

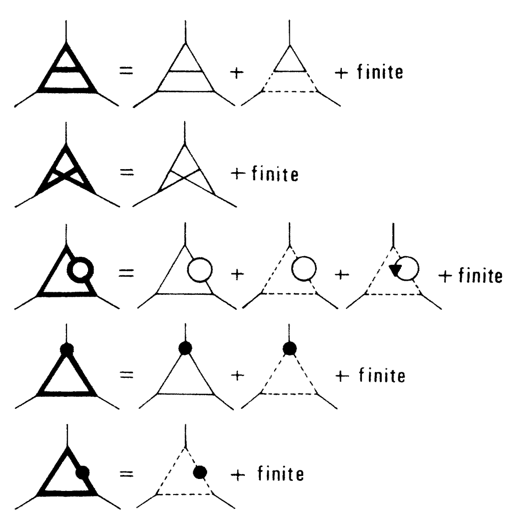

FIG. 3. The divergent three-point diagram at the two-loop level.

we show the results for I. Figure 2(a) gives

$$I_{2a-1}=(-q^2)\left[\frac{1}{72(n-6)^2}-\frac{1}{72(n-6)}\left[\frac{43}{12}-\gamma\right]-\frac{1}{144(n-6)}\ln\Omega(\eta)\Omega(\eta')\right],$$

$$I_{2a-2}=\frac{V(\eta)}{4(n-6)^2}-\frac{V(\eta)}{4(n-6)}\left[\frac{3}{4}-\gamma\right]+\frac{1}{2(n-6)}F(\eta,q)-\frac{V(\eta)}{4(n-6)}\ln\Omega(\eta),$$

$$I_{2a-3}=-\frac{V(\eta)}{12(n-6)^2}+\frac{V(\eta)}{12(n-6)}\left[\frac{13}{12}-\gamma\right]-\frac{1}{6(n-6)}F(\eta,q)+\frac{V(\eta)}{12(n-6)}\ln\Omega(\eta),\qquad(3.9)$$

$$I_{2a-4}=\tfrac{1}{2}I_{2a-3}$$

For Fig. 2(b),

$$I_{2b-1}=(-q^2)\left[-\frac{1}{12(n-6)^2}+\frac{1}{12(n-6)}(3-\gamma)+\frac{1}{24(n-6)}\ln\Omega(\eta)\Omega(\eta')\right],$$

$$I_{2b-2}=2I_{2a-2},\qquad(3.10)$$

$$I_{2b-3}=-\frac{V(\eta)}{8(n-6)}.$$

For Figs. 2(c) and 2(d), we use the one-loop counterterms of Eq. (3.5) to fix the value of the effective vertex:

$$I_{2c-1}=\left[\frac{-q^2}{4\pi\mu^2}\right]^{3-n/2}\times(-q^2)\left[\frac{5}{36(n-6)^2}-\frac{5}{36(n-6)}\left[\frac{4}{3}-\frac{\gamma}{2}\right]\right],$$

$$I_{2c-2}=\left[\frac{-q^2}{4\pi\mu^2}\right]^{3-n/2}\left[-\frac{5}{6(n-6)^2}V(\eta)+\frac{5}{12(n-6)}\left[\frac{1}{2}-\gamma\right]V(\eta)-\frac{5}{6(n-6)}F(\eta,q)\right],\qquad(3.11)$$

$$I_{2d-1}=\left[\frac{-q^2}{4\pi\mu^2}\right]^{3-n/2}\left[-\frac{5}{12(n-6)^2}V(\eta)+\frac{5}{24(n-6)}\left[\frac{1}{2}-\gamma\right]V(\eta)+\frac{1}{240(n-6)}R\Omega^2-\frac{5}{12(n-6)}F(\eta,q)\right].$$

In the above expressions $V(\eta)$ is given by Eq. (2.13a) and γ is the Euler constant. The function $F(\eta,q)$ is given by

$$F(\eta,q)\equiv\int dk_0e^{ik_0\eta}\hat{V}(k_0)\int d\xi\,d\zeta\,\xi\ln\left[1-\xi-2\zeta(1-\xi)\frac{k_0q_0}{q^2}+\zeta(1-\xi\zeta)\frac{k_0^{\,2}}{q^2}\right].\qquad(3.12)$$

Summing up Eqs. (3.9)—(3.11), we get, through Eq. (3.8),

$$P_P\Gamma^{(2)}_{\phi^2}=\mu^{n-6}\int dv_xdv_{x'}\hat{\phi}(x)\hat{\phi}(x')[\Omega(\eta)\Omega(\eta')]^{2-n}\frac{g^4}{(4\pi)^6}\frac{1}{(2\pi)^n}$$
$$\times\int e^{iq(x-x')}d^nq\left[(-q^2)\left[\frac{5}{72(n-6)^2}+\frac{13}{864(n-6)}\right]-V(\eta)\left[\frac{5}{8(n-6)^2}+\frac{23}{96(n-6)}\right]\right.$$
$$\left.+\frac{1}{240(n-6)}R\Omega^2+(-q^2)\frac{5}{144(n-6)}\ln\Omega(\eta)\Omega(\eta')-V(\eta)\frac{5}{8(n-6)}\ln\Omega(\eta)\right]$$
$$=\mu^{n-6}\int dv_xdv_{x'}\hat{\phi}(x)\hat{\phi}(x')\frac{g^4}{(4\pi)^6}$$
$$\times\left[\left[\frac{5}{72(n-6)^2}+\frac{13}{864(n-6)}\right][\Omega(\eta)\Omega(\eta')]^{-1-n/2}[\eta^{\mu\nu}\partial_\mu\partial_\nu\delta^n(x-x')]\right.$$
$$+\left[\frac{5}{8(n-6)^2}+\frac{23}{96(n-6)}\right]\{m^2+[\xi-\xi(n)]R\}\Omega^{-n}\delta^n(x-x')$$
$$\left.+\frac{1}{240(n-6)}R\Omega^{-n}\delta^n(x-x')\right].$$

Again using the formula Eq. (3.3), we reach the final expression for $P_P\Gamma^{(2)}_{\hat\phi^2}$:

$$P_P\Gamma^{(2)}_{\hat\phi^2}=\mu^{n-6}\int dv_x \frac{g^4}{(4\pi)^6}\left\{\left[\frac{5}{72(n-6)^2}+\frac{13}{864(n-6)}\right]\hat\phi\Box\hat\phi+\left[\frac{5}{8(n-6)^2}+\frac{23}{96(n-6)}\right]m^2\hat\phi^2 + \left[\left[\frac{5}{8(n-6)^2}+\frac{23}{96(n-6)}\right]\xi-\frac{1}{9(n-6)^2}-\frac{5}{108(n-6)}\right]R\hat\phi^2\right\}. \quad (3.13)$$

The three-point function, Fig. 3, can be calculated in the same way to be

$$P_P\Gamma^{(2)}_{\hat\phi^3}=\mu^{n-6}\int dv_x \frac{g^5}{(4\pi)^6}\left[\frac{5}{24(n-6)^2}+\frac{23}{288(n-6)}\right]\hat\phi^3 . \quad (3.14)$$

This result is exactly the same as that in flat space-time.[6]

The counterterm Eq. (2.5c) to this order reads

$$I^{\mathrm{ct}(2)}=\mu^{n-6}\int dv_x[-\tfrac{1}{2}\delta Z^{(2)}\hat\phi\Box\hat\phi-\tfrac{1}{2}(m^2\delta Z^{(2)}+\delta m^{2(1)}\delta Z^{(1)}+\delta m^{2(2)})\hat\phi^2 -\tfrac{1}{2}(\xi\delta Z^{(2)}+\delta\xi^{(1)}\delta Z^{(1)}+\delta\xi^{(2)})R\hat\phi^2-\tfrac{1}{6}(\delta g^{(2)}+\tfrac{1}{2}g\delta Z^{(2)}+\tfrac{3}{2}\delta g^{(1)}\delta Z^{(1)}+\tfrac{3}{8}g\delta Z^{(1)2})\hat\phi^3] . \quad (3.15)$$

Although each diagram has the nonlocal divergences like $(n-6)^{-1}F(\eta,q)$ and/or $(n-6)^{-1}\ln\Omega(\eta)$, they cancel out among each other. Therefore, the divergences in Eqs. (3.13) and (3.14) can be removed by the counterterm of Eq. (3.15) with the following choice:

$$\delta Z^{(2)}=\frac{5}{36}\frac{g^4}{(4\pi)^6}\frac{1}{(n-6)^2}+\frac{13}{432}\frac{g^4}{(4\pi)^6}\frac{1}{n-6},$$
$$\delta m^{2(2)}=\frac{35}{36}\frac{g^4}{(4\pi)^6}\frac{1}{(n-6)^2}m^2+\frac{97}{216}\frac{g^4}{(4\pi)^6}\frac{1}{n-6}m^2, \quad (3.16)$$
$$\delta\xi^{(2)}=\frac{35}{36}\frac{g^4}{(4\pi)^6}\frac{1}{(n-6)^2}\left[\xi-\frac{1}{5}\right]+\frac{97}{216}\frac{g^4}{(4\pi)^6}\frac{1}{n-6}\left[\xi-\frac{20}{97}\right],$$
$$\delta g^{(2)}=\frac{27}{32}\frac{g^5}{(4\pi)^6}\frac{1}{(n-6)^2}+\frac{125}{288}\frac{g^5}{(4\pi)^6}\frac{1}{n-6} .$$

As we have seen, the theory is renormalizable at the two-loop level. This conclusion is in disagreement[21] with that of Ref. 9. The counterterms to this order are exactly equal to those in flat space-time[6] except $\delta\xi$ which is not present in flat space-time. Therefore, the 't Hooft pole identities[23] and the Symanzik identity are naturally satisfied. The beta function for the coupling constant g is also the same as in flat space-time:

$$\beta_g(g)=\mu\frac{\partial g}{\partial\mu}=-\frac{3}{4}\frac{g^3}{(4\pi)^3}-\frac{125}{144}\frac{g^5}{(4\pi)^6} .$$

The β function for ξ becomes

$$\beta_\xi(g,\xi)=\mu\frac{\partial\xi}{\partial\mu} =-\frac{5}{6}\left[\xi-\frac{1}{5}\right]\frac{g^2}{(4\pi)^3}-\frac{97}{108}\left[\xi-\frac{20}{97}\right]\frac{g^4}{(4\pi)^6} .$$

Since the theory is asymptotically free, the value of $\xi(\mu)$ approaches the conformal value $\frac{1}{5}$ when $\mu\to\infty$.[8] This fact is, of course, not affected by the higher- (two-loop) order effects.[2]

IV. CONCLUDING REMARKS

It has been shown that $(\phi^3)_6$ theory in a conformally flat space-time is renormalizable at the two-loop level. The background-field formulation was used to write down the effective action (Green's function). The momentum-space representation for the Feynman propagator was employed with dimensional regularization to handle the divergences in the effective action. In Sec. III the explicit calculation to the two-loop order was performed and we have seen that the counterterms Eqs. (3.6) and (3.16) can remove all the divergences present. As in flat space-time, the cancellation of the state-dependent infinities $F(\eta,q)$ of Eq. (3.12) and $\ln\Omega(\eta)$ which are present in the individual diagram can make the theory renormalizable. In this respect our results are not in agreement with a recent result in Ref. 9.

In this paper we have considered only the divergences which are present in the matter action. Strictly speaking, in order to claim that the theory is renormalizable at the two-loop level, one must also investigate the pure gravitational action and also the "tadpole" parts in the action.[8] However, it is beyond the scope of this paper. Although we investigated $(\phi^3)_6$ theory in a conformally flat space-time, it may be a simple exercise to do a similar calculation in an arbitrary space-time by using the momentum-space method of Ref. 24 as far as the topology of the space-time admits such a method. Such calculation may prove the correctness of the results of the present paper and is currently in progress.

As mentioned in Sec. I, it is usually supposed that the theory may be renormalizable in curved space-time if it is so in flat space-time. Although one cannot *a priori* claim this, there has been much evidence to support it. The results we have obtained in this paper may add a further indication that this is indeed the case.

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[16]A nice review is contained in Ref. 8.

[17]Note that the dimension of the quantum field ϕ is $[\phi]=n/2-1$.

[18]See, for example, Refs. 2 and 12.

[19]The expansion of g in terms of the massless propagator does not cause any problem in the actual calculations in Sec. III.

[20]In the background-field formalism, both the vertex and the wave-function counterterms in the usual sense are represented by a single "vertex" counterterm.

[21]The answer Eq. (19) for Fig. 1(c) of Ref. 9 should be replaced by $S_{43}(p)=1/2S_{41}(p)$. This calculational error seems to come from their misassignment of momenta in Eq. (IV,154) of Ref. 22.

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