Conformally invariant quantum field theory in static Einstein space-times

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The conformal transformation law for the renormalized one-loop effective action is used to give approximate renormalized stress tensors for conformally invariant scalar, spinor, and vector field theories on static Einstein space-times, paying particular attention to the ambiguous coefficient of the term $\Box R$ in the vector trace anomaly. In black-hole space-times these tensors for the Hartle-Hawking thermal state are shown to be regular on the horizon. Geometrical expressions are given for the approximate energy densities there; these can all be expressed in terms of the scalar invariant $C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau}$. A number of explicit examples are presented, including Schwarzschild space-time. In flat and conformally flat space-times these approximate stress tensors agree with the exact results wherever the latter are known. In more complicated space-times, where we do not always obtain the exact results, a brief discussion is given of how to obtain a more accurate approximation.

I. INTRODUCTION

A central problem in the study of quantum field theory on curved space-time is the calculation of the renormalized expectation value of the stress tensor operator. This quantity determines the back reaction on the metric of space-time through the semiclassical Einstein equations. The methods for calculating this tensor have been generally agreed upon for a number of years. Unfortunately, in practice, the actual calculations involved are very complicated for anything but extremely simple space-times. Therefore, it is important to construct schemes for approximating renormalized stress tensors in more complicated space-times, such as, for example, Schwarzschild space-time.

Such a scheme for obtaining expectation values in thermal states of a conformally invariant scalar field on a static Einstein space-time ($R_{\mu\nu}=\Lambda g_{\mu\nu}$) was introduced by one of us (D.N.P.) a few years ago.[1] This involved making a conformal transformation to the ultrastatic optical metric for which the coincidence limit of the second DeWitt coefficient,[2] $a_2(x,x)$, was noted to vanish. This suggests that the WKB approximation of the thermal propagator in this space-time should be especially good. The approximate propagator which ignores contributions from indirect geodesics can be obtained in closed form by following the work of Bekenstein and Parker.[3] The approximate thermal stress tensor in the physical space-time is then taken to be the one obtained by conformally transforming this approximate propagator back to the physical space-time.

This method yielded exact results for the renormalized stress tensors of the SO(4,1)-invariant state on de Sitter space and the SO(3)$\times$SO(2,1)-invariant state on the Lorentzian analytic continuation of $S^2\times S^2$ with the Nariai metric.[4] In addition, it gave a simple closed-form expression for the renormalized stress tensor for the Hartle-Hawking[5] state on Schwarzschild space-time which agreed remarkably well with the exact result obtained by extensive numerical computation by Howard.[6]

Later, Brown and Ottewill[7] showed how this approximation could be understood in terms of properties of the one-loop effective action for a conformally invariant field under conformal transformations. This approach also gave approximate renormalized stress tensors for thermal states of conformally invariant fields of higher spins. In addition, a discussion was given of the relationship between this approximation scheme and the argument that the vanishing of $a_2(x,x)$ in the optical space-time implied the existence of local states in that space-time for which the renormalized stress tensor vanished identically.[8]

The purpose of the present paper is twofold: first to unite the work of our two previous papers and second to address a problem that exists when the scheme as given in Ref. 7 is applied to thermal states of spin-1 fields. The problem is that using the expression for the thermal stress tensor in a static Einstein space given there one finds the absurd result that at high temperatures the approximate energy density becomes negative. This problem can be traced to the use in that paper of the value of the trace anomaly given by dimensional renormalization.[9] Although it was acknowledged that there was an ambiguity in the anomaly in the coefficient of $\Box R$, it was incorrectly assumed that this was unimportant. Below we shall show that if one uses point-separation[10] or ζ-function[11] renormalization and proceeds as before then one can obtain sensible answers for spin-1 fields.

In Sec. II we discuss the general theory associated with the conformal transformation law for the (one-loop) effective action allowing for different renormalization schemes. We then discuss the use of this law for the point-separated or ζ-function renormalized effective action to give an approximate renormalized stress tensor in static Einstein space-times. This is more complicated than with the dimensionally renormalized effective action since now the trace anomaly does not vanish for spin-1 fields in the optical space-time. Thus, the previous ansatz of equating to zero the zero-temperature renormalized stress tensor in this space-time is no longer possible.

In Sec. III we deal with static, but possibly spatially distorted, Einstein black-hole space-times. We shall show that the approximate stress tensors for the Hartle-Hawking vacuum on these space-times are always regular on the event horizon of the black hole. Furthermore, we obtain geometrical expressions for the energy of these stress tensors on the horizon, and so generalize the work of Frolov and Sanchez.[12]

In Sec. IV we give some specific examples, and in Sec. V we review the successes and failures of the scheme. In addition, we discuss possible ways of improving the approximation when it fails to give good agreement with known results.

Throughout we shall use natural units ($\hbar=c=G=k=1$) and follow the space-time conventions of Misner, Thorne, and Wheeler.[13]

II. GENERAL THEORY

The conformal transformation law for the dimensionally renormalized one-loop effective action for a free conformally invariant field theory on curved space-time was given in Ref. 7. We restate this law here in a more general form allowing for different renormalization schemes. We shall then discuss the use of this formula for approximating the renormalized stress tensor in static Einstein space-times with emphasis on the relationship to the work of Ref. 1.

Under a conformal transformation the renormalized one-loop effective action transforms according to the equation[7,14,15]

$$W_R[e^{-2\omega}g_{\mu\nu}]=W_R[g_{\mu\nu}]-aA[\omega;g]-bB[\omega;g]-cC[\omega;g]\ , \tag{2.1}$$

where

$$A[\omega;g]=\int d^4x\, g^{1/2}\{(R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}-2R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{3}R^2)\omega+\tfrac{2}{3}[R+3(\Box\omega-\omega_{;\mu}\omega^{;\mu})](\Box\omega-\omega_{;\nu}\omega^{;\nu})\}\ , \tag{2.2}$$

$$B[\omega;g]=\int d^4x\, g^{1/2}[(R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}-4R_{\mu\nu}R^{\mu\nu}+R^2)\omega+4R_{\mu\nu}\omega^{;\mu}\omega^{;\nu}-2R\omega_{;\mu}\omega^{;\mu}+2(\omega_{;\mu}\omega^{;\mu})^2-4\omega_{;\mu}\omega^{;\mu}\Box\omega]\ , \tag{2.3}$$

$$C[\omega;g]=\int d^4x\, g^{1/2}\{[R+3(\Box\omega-\omega_{;\mu}\omega^{;\mu})](\Box\omega-\omega_{;\nu}\omega^{;\nu})\}\ , \tag{2.4}$$

and the coefficients a, b, and c depend on the number of helicity states of each spin there are and on the renormalization scheme used. Point-separation, ζ-function, and dimensional renormalization all agree that

$$a=(2^9 45\pi^2)^{-1}[12h(0)+18h(\tfrac{1}{2})+72h(1)]\ , \tag{2.5}$$

$$b=(2^9 45\pi^2)^{-1}[-4h(0)-11h(\tfrac{1}{2})-124h(1)]\ , \tag{2.6}$$

where $h(s)$ denotes the number of helicity states for fields of spin s. On the other hand, there is disagreement on the value of c: dimensional renormalization gives

$$c=0\ , \tag{2.7}$$

while point-separation and ζ-function renormalization give

$$c=(2^9 45\pi^2)^{-1}[-120h(1)]\ . \tag{2.8}$$

As discussed in the Introduction, in this paper we shall take the latter value.

By employing the identity

$$\frac{\delta}{\delta\omega}F[e^{-2\omega}g_{\rho\tau}]\bigg|_{\omega=0}=-2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}F[g_{\rho\tau}]\ , \tag{2.9}$$

we find

$$T_R{}^{\rho}{}_{\rho}[g]=a(H+\tfrac{2}{3}\Box R)+bG+c\Box R\ , \tag{2.10}$$

where

$$H=C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau}=R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}-2R_{\mu\nu}R^{\mu\nu}+\tfrac{1}{3}R^2 \tag{2.11}$$

and

$$G={}^*R_{\mu\nu\rho\tau}{}^*R^{\mu\nu\rho\tau}=R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau}-4R_{\mu\nu}R^{\mu\nu}+R^2\ . \tag{2.12}$$

Here, and elsewhere, the renormalized effective stress tensor $T_R{}^{\mu\nu}$ is defined by

$$T_R{}^{\mu\nu}=2g^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}W_R[g]\ . \tag{2.13}$$

Equation (2.10) is the usual trace anomaly.

We shall be interested in cases where ω itself is a scalar function on space-time which may depend on the metric. It is now convenient to introduce the symmetric tensors $T_A{}^{\mu\nu}[\omega]$, $T_B{}^{\mu\nu}[\omega]$, and $T_C{}^{\mu\nu}[\omega]$ defined by the equation

$$\begin{aligned} T_S{}^{\mu\nu}[\omega]&=2g^{-1/2}\frac{\Delta}{\Delta g_{\mu\nu}}S[\omega[g];g]\\ &=2g^{-1/2}\left[\frac{\delta}{\delta g_{\mu\nu}}+\frac{\delta\omega}{\delta g_{\mu\nu}}\frac{\delta}{\delta\omega}\right]S[\omega[g];g]\ , \end{aligned} \tag{2.14}$$

for $S=A,B,C$ where $\Delta/\Delta g_{\mu\nu}$ and $\delta/\delta g_{\mu\nu}$ denote, respectively, the total and partial derivatives with respect to the metric, $\delta/\delta\omega$ denotes the partial derivative with respect to ω, and an integration over space-time is to be understood in the last term.

From Eqs. (2.1) and (2.10) it follows that

$$2g^{-1/2}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}A[\omega;g]=(H+\tfrac{2}{3}\Box R)-e^{-4\omega}(\tilde{H}+\tfrac{2}{3}\tilde{\Box}\tilde{R})\ , \tag{2.15}$$

$$2g^{-1/2}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}B[\omega;g]=G-e^{-4\omega}\tilde{G}\ , \tag{2.16}$$

$$2g^{-1/2}g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}C[\omega;g]=\Box R-e^{-4\omega}\tilde{\Box}\tilde{R}\ , \tag{2.17}$$

and

$$2g^{-1/2}\frac{\delta}{\delta\omega}A[\omega;g]=e^{-4\omega}(\tilde{H}+\tfrac{2}{3}\tilde{\Box}\tilde{R})\ , \tag{2.18}$$

$$2g^{-1/2}\frac{\delta}{\delta\omega}B[\omega;g]=e^{-4\omega}\tilde{G}\ , \tag{2.19}$$

$$2g^{-1/2}\frac{\delta}{\delta\omega}C[\omega;g]=e^{-4\omega}\tilde{\Box}\tilde{R}\ , \tag{2.20}$$

where a tilde denotes that the quantity is evaluated in the space with metric $\tilde{g}_{\mu\nu}=e^{-2\omega}g_{\mu\nu}$. It follows that if either $a\delta A/\delta\omega$, $b\delta B/\delta\omega$, and $c\delta C/\delta\omega$ are all zero or $\omega[g]$ satisfies the scaling law

$$g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\omega[g]=1\ , \tag{2.21}$$

then $aT_A{}^{\mu\nu}$, $bT_B{}^{\mu\nu}$, and $cT_C{}^{\mu\nu}$ have traces $a(H+\frac{2}{3}\Box R)$, bG, and $c\Box R$, respectively. In addition, as total derivatives of actions with scalar Lagrangians they are conserved.

In Ref. 7 we always considered conformal factors $\omega[g]$ such that $\tilde{H}+\frac{2}{3}\tilde{\Box}\tilde{R}$ and $\tilde{G}$ vanished, and by choosing the dimensional renormalization value $c=0$ we never needed to consider the action C. In this way the problem of determining $\delta\omega/\delta g_{\mu\nu}$ did not arise. In this paper we shall choose $c\neq 0$ and consider conformal factors for which $\tilde{\Box}\tilde{R}$ does not vanish. Hence we must now face this problem.

We shall restrict our attention to space-times possessing a Killing vector field K^μ (at this stage we do not assume that K^μ is timelike or hypersurface orthogonal). In such space-times it is of special interest to consider the conformal transformation defined by

$$\omega=\tfrac{1}{2}\ln|K^2|\ , \tag{2.22}$$

where $K^2\equiv g_{\mu\nu}K^\mu K^\nu$. In this case one can prove the following theorem.

If $S[\omega;g]\equiv\int d^4x\, g^{1/2}L[\omega;g]$ where L is a scalar function on space-time then the symmetric tensor

$$2g^{-1/2}\frac{\Delta}{\Delta g_{\mu\nu}}S[\omega;g]=2g^{-1/2}\left[\frac{\delta}{\delta g_{\mu\nu}}+\frac{K^\mu K^\nu}{2K^2}\frac{\delta}{\delta\omega}\right]S[\omega;g]$$

is conserved.

In other words, it is consistent to define

$$\frac{\delta\omega}{\delta g_{\mu\nu}}=\frac{K^\mu K^\nu}{2K^2}\ ; \tag{2.23}$$

that is, in varying ω with respect to $g_{\mu\nu}$, K^μ is kept fixed. It is clear that Eq. (2.23) is consistent with Eq. (2.21). Hereafter we shall assume Eq. (2.23) to hold.

Let us now further restrict our attention to static Einstein spaces, so that K^μ is now assumed to be both timelike and hypersurface orthogonal (although the first requirement is inessential and will occasionally be dropped). In this case we have[1,7] $\tilde{H}+\frac{2}{3}\tilde{\Box}\tilde{R}=0$ and $\tilde{G}=0$ so that

$$T_A{}^{\mu\nu}=2g^{-1/2}\frac{\delta A}{\delta g_{\mu\nu}}\ , \tag{2.24}$$

$$T_B{}^{\mu\nu}=2g^{-1/2}\frac{\delta B}{\delta g_{\mu\nu}}\ , \tag{2.25}$$

and

$$T_C{}^{\mu\nu}=2g^{-1/2}\left[\frac{\delta C}{\delta g_{\mu\nu}}\right]-\frac{3}{2}\frac{K^\mu K^\nu}{K^2}H\ , \tag{2.26}$$

where in writing the last equation we have used the fact that $\tilde{\Box}\tilde{R}=-\frac{3}{2}\tilde{H}=-\frac{3}{2}e^{4\omega}H$.

For completeness we now give $\delta A/\delta g_{\mu\nu}$, $\delta B/\delta g_{\mu\nu}$, and $\delta C/\delta g_{\mu\nu}$ for arbitrary space-times:[1,7]

$$\begin{aligned}2g^{-1/2}\frac{\delta A}{\delta g_{\mu\nu}}&=4(C^{\rho\mu\nu\tau}R_{\rho\tau}+2C^{\rho\mu\nu\tau}{}_{;\rho\tau})\omega+8R_\rho{}^{(\mu;\nu)}\omega^{;\tau}-8R^{\mu\nu}{}_{;\rho}\omega^{;\rho}+8\omega_{;\rho\tau}R^{\rho\mu\nu\tau}-8\omega^{;\rho(\mu}R^{\nu)}{}_\rho+4g^{\mu\nu}R_{\rho\tau}\omega^{;\rho\tau}+\tfrac{4}{3}R\omega^{;\mu\nu}\\&\quad+\tfrac{2}{3}g^{\mu\nu}R_{;\rho}\omega^{;\rho}+\tfrac{8}{3}R^{\mu\nu}\Box\omega-\tfrac{4}{3}g^{\mu\nu}R\Box\omega+\tfrac{4}{3}(\Box\omega)^{;\mu\nu}-\tfrac{4}{3}g^{\mu\nu}(\Box^2\omega)\\&\quad+8(\Box\omega)^{;(\mu}\omega^{;\nu)}-4g^{\mu\nu}\omega_{;\rho}(\Box\omega)^{;\rho}-2g^{\mu\nu}(\Box\omega)^2+\tfrac{4}{3}R\omega^{;\mu}\omega^{;\nu}\\&\quad-\tfrac{2}{3}g^{\mu\nu}R(\omega_{;\rho}\omega^{;\rho})-\tfrac{4}{3}(\omega_{;\rho}\omega^{;\rho})^{;\mu\nu}+\tfrac{4}{3}g^{\mu\nu}\Box(\omega_{;\rho}\omega^{;\rho})+\tfrac{4}{3}R^{\mu\nu}(\omega_{;\rho}\omega^{;\rho})\\&\quad-8\omega^{;(\mu}(\omega_{;\rho}\omega^{;\rho})^{;\nu)}+8(\Box\omega)\omega^{;\mu}\omega^{;\nu}+4g^{\mu\nu}\omega_{;\rho}(\omega_{;\tau}\omega^{;\tau})^{;\rho}-8(\omega_{;\rho}\omega^{;\rho})\omega^{;\mu}\omega^{;\nu}+2g^{\mu\nu}(\omega_{;\rho}\omega^{;\rho})^2 &(2.27)\\&=4e^{-6\omega}[2(\tilde{C}^{\rho\mu\nu\tau}\omega)_{:\rho\tau}+\tilde{R}_{\rho\tau}\tilde{C}^{\rho\mu\nu\tau}\omega]\ , &(2.28)\end{aligned}$$

where a colon denotes covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$;

$$\begin{aligned}2g^{-1/2}\frac{\delta B}{\delta g_{\mu\nu}}&=8g^{\mu\nu}R_{\rho\tau}\omega^{;\rho\tau}+4R\omega^{;\mu\nu}+8R^{\mu\nu}\Box\omega-4g^{\mu\nu}R\Box\omega-16R^{\rho(\mu}\omega^{;\nu)}{}_{;\rho}\\&\quad+8R^{\rho\mu\nu\tau}\omega_{;\rho\tau}+8R^{\rho\mu\nu\tau}\omega_{;\rho}\omega_{;\tau}-2g^{\mu\nu}R(\omega_{;\rho}\omega^{;\rho})+8g^{\mu\nu}R_{\rho\tau}\omega^{;\rho}\omega^{;\tau}\\&\quad-4g^{\mu\nu}(\Box\omega)^2+4g^{\mu\nu}(\omega_{;\rho\tau}\omega^{;\rho\tau})+4R^{\mu\nu}\omega_{;\rho}\omega^{;\rho}+4R\omega^{;\mu}\omega^{;\nu}+8\omega^{;\mu\nu}\Box\omega-16R_\rho{}^{(\mu}\omega^{;\nu)}\omega^{;\rho}-8\omega^{;\mu\rho}\omega^{;\nu}{}_\rho+8\omega^{;\mu}\omega^{;\nu}\Box\omega\\&\quad+8g^{\mu\nu}\omega_{;\rho\tau}\omega^{;\rho}\omega^{;\tau}-16\omega^{;\rho(\mu}\omega^{;\nu)}\omega_{;\rho}+2g^{\mu\nu}(\omega_{;\rho}\omega^{;\rho})^2-8\omega^{;\mu}\omega^{;\nu}(\omega_{;\rho}\omega^{;\rho}) &(2.29)\\&=e^{-6\omega}(4\tilde{C}^{\rho\mu\nu\tau}\tilde{R}_{\rho\tau}+2\tilde{H}^{\mu\nu})-(4C^{\rho\mu\nu\tau}R_{\rho\tau}+2H^{\mu\nu})\ , &(2.30)\end{aligned}$$

where

$$H^{\mu\nu}=-R^{\mu\rho}R_{\rho}{}^{\nu}+\tfrac{2}{3}RR^{\mu\nu}+(\tfrac{1}{2}R^{\rho\tau}R_{\rho\tau}-\tfrac{1}{4}R^{2})g^{\mu\nu}\,, \tag{2.31}$$

and

$$\begin{aligned}2g^{-1/2}\frac{\delta C}{\delta g_{\mu\nu}}&=2\omega^{;(\mu}R^{;\nu)}-g^{\mu\nu}R_{;\rho}\omega^{;\rho}-2R^{\mu\nu}\Box\omega+2(\Box\omega)^{;\mu\nu}-2g^{\mu\nu}(\Box^{2}\omega)\\&\quad+12(\Box\omega)^{;(\mu}\omega^{;\nu)}-6g^{\mu\nu}\omega_{;\rho}(\Box\omega)^{;\rho}-3g^{\mu\nu}(\Box\omega)^{2}+2R\,\omega^{;\mu}\omega^{;\nu}\\&\quad-g^{\mu\nu}R\,(\omega_{;\rho}\omega^{;\rho})-2(\omega_{;\rho}\omega^{;\rho})^{;\mu\nu}+2g^{\mu\nu}\Box(\omega_{;\rho}\omega^{;\rho})+2R^{\mu\nu}(\omega_{;\rho}\omega^{;\rho})\\&\quad-12\omega^{;(\mu}(\omega_{;\rho}\omega^{;\rho})^{;\nu)}+12(\Box\omega)\omega^{;\mu}\omega^{;\nu}+6g^{\mu\nu}\omega_{;\rho}(\omega_{;\tau}\omega^{;\tau})^{;\rho}-12(\omega_{;\rho}\omega^{;\rho})\omega^{;\mu}\omega^{;\nu}+3g^{\mu\nu}(\omega_{;\rho}\omega^{;\rho})^{2} \qquad (2.32)\\&=\tfrac{1}{6}(e^{-6\omega}\tilde{I}^{\mu\nu}-I^{\mu\nu})-\tfrac{8}{3}e^{-6\omega}[2(\tilde{C}^{\rho\mu\nu\tau}\omega)_{;\rho\tau}+\tilde{R}_{\rho\tau}\tilde{C}^{\rho\mu\nu\tau}\omega]\,, \qquad (2.33)\end{aligned}$$

where

$$\begin{aligned}I^{\mu\nu}&=g^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int d^{4}x\,g^{1/2}R^{2}\\&=2R^{;\mu\nu}-2RR^{\mu\nu}+(\tfrac{1}{2}R^{2}-2\Box R)g^{\mu\nu}\,.\end{aligned} \tag{2.34}$$

So far we have dealt entirely with exact properties of the theory under conformal transformations. We now turn to the question of using the above properties as the basis for an approximation scheme for calculating renormalized stress tensors in static Einstein space-times. We first discuss zero-temperature stress tensors and then later discuss the finite-temperature contributions.

The conformal transformation defined by Eq. (2.22) takes the static physical metric to the ultrastatic optical metric (in which K^{μ} is a covariantly constant vector field). In this space, as discussed in Refs. 1 and 7, one expects the Gaussian approximation to be especially good so that one can make the approximation $\tilde{T}_{R}{}^{\mu\nu}=0$, with Ref. 8 arguing that there exists a class of states for which this is exact. This equation is consistent for all spins for dimensional renormalization; however, it is not so for ζ-function or point-separation renormalization for spin 1 since for these schemes the trace anomaly does not vanish for the optical metric. To motivate a consistent ansatz, take the total derivative of Eq. (2.1) with respect to $g_{\mu\nu}$: on using the chain rule and Eq. (2.23) this yields the equation

$$\begin{aligned}\frac{\Delta}{\Delta g_{\mu\nu}}W_{R}[e^{-2\omega}g]&=e^{-6\omega}(\tilde{T}_{R}{}^{\mu\nu}-K^{\mu}K^{\nu}\tilde{T}_{R}{}^{\rho}{}_{\rho})\\&=T_{R}{}^{\mu\nu}-aT_{A}{}^{\mu\nu}-bT_{B}{}^{\mu\nu}-cT_{C}{}^{\mu\nu}\,,\end{aligned} \tag{2.35}$$

where $T_{A}{}^{\mu\nu}$, $T_{B}{}^{\mu\nu}$, and $T_{C}{}^{\mu\nu}$ are defined by Eqs. (2.24)–(2.26).

We now make the revised approximation ansatz (cf. Ref. 16) that the left-hand side of this equation vanishes so that

$$\tilde{T}_{R}{}^{\mu\nu}=K^{\mu}K^{\nu}\tilde{T}_{R}{}^{\rho}{}_{\rho}\,, \tag{2.36}$$

or, equivalently,

$$T_{R}{}^{\mu\nu}=aT_{A}{}^{\mu\nu}+bT_{B}{}^{\mu\nu}+cT_{C}{}^{\mu\nu}\,. \tag{2.37}$$

This approximation ansatz is consistent since the left-hand side of Eq. (2.35) is the variation of an action which by Eq. (2.21) is conformally invariant; as such it is certainly a conserved, trace-free tensor. Of course, the transformation law (2.1) can only ever determine $W_{R}[g]$ up to conformally invariant additions. It appears[1,7] that in the optical metric of a static Einstein space-time any additions make little contribution to the renormalized stress tensor for many states of interest.

Although our discussion has followed the lines of Ref. 7, Eq. (2.36) is a natural ansatz in the spirit of Refs. 1 and 16. One can postulate it directly as a property of the optical metric and then transform it back to the physical metric to obtain Eq. (2.37) thereby avoiding worries of defining $\delta\omega/\delta g_{\mu\nu}$. In this approach the additional term in $T_{C}{}^{\mu\nu}$, as defined by Eq. (2.26), arises as the conformal transform of Eq. (2.36).

We should note that in abandoning the requirement that the trace anomaly in the conformal space vanish we have lost the original motivation for expecting the approximation to be reasonable.[17] It may yet be better to solve the equation $T_{R}{}^{\rho}{}_{\rho}[e^{-2\omega}g]=0$ for $\omega[g]$ and use that as the basis for our approximation. Our reason for not doing so is simply that it is too difficult, and the solution may not be unique. Instead we shall work with Eq. (2.37) and let its value as an approximation scheme be judged by the results it yields.

Now we shall discuss thermal states in static Einstein space-times. In Ref. 1 an approximate thermal stress tensor for scalar fields was obtained by explicitly calculating the Gaussian approximation to the thermal propagator in the ultrastatic optical space-time. In Ref. 7, working in the static physical space-time, it was shown that the conformal factor

$$\bar{\omega}=\alpha t+\tfrac{1}{2}\ln(-K^{2}) \tag{2.38}$$

yielded, under the assumptions $c=0$ and $T_{R}{}^{\mu\nu}[e^{-2\bar{\omega}}g]=0$, an approximation to the renormalized stress tensor for a thermal state with temperature $\alpha/2\pi$. To see the connection between these two approaches we first observe that the actions A, B, and C have the property that[14]

$$S[\omega_{1}+\omega_{2};g]=S[\omega_{1};e^{-2\omega_{2}}g]+S[\omega_{2};g] \tag{2.39}$$

for arbitrary conformal factors ω_{1} and ω_{2}. Choosing $\omega_{1}=\alpha t$ and $\omega_{2}=\frac{1}{2}\ln(-K^{2})$ it is clear that taking conformal factor $\bar{\omega}$ in the physical space-time is equivalent to taking conformal factor ω_{1} in the optical space-time and then transforming the answer back to the physical space-time.

The prescription of choosing ϖ, and assuming that the stress tensor vanishes in the metric $e^{-2\varpi}g_{\mu\nu}$, generically yields a tensor consisting of three terms, one independent of α which coincides with that arising from ω_2, one quadratic in α and proportional (with a spin-dependent coefficient which vanishes for spin 0) to a tensor which we shall denote $T_2{}^{\mu\nu}[\alpha]$, and one quartic in α and proportional to a tensor $T_4{}^{\mu\nu}[\alpha]$. Written in the optical space-time these tensors may be normalized to take the form

$$\tilde{T}_2{}^{\mu\nu}[\alpha]=4\alpha^2(-\tilde{G}^{\mu\nu}+\tilde{R}K^\mu K^\nu)\ , \tag{2.40}$$

$$\tilde{T}_4{}^{\mu\nu}[\alpha]=\alpha^4(\tilde{g}^{\mu\nu}+4K^\mu K^\nu)\ , \tag{2.41}$$

where $\tilde{G}^{\mu\nu}$ denotes the Einstein tensor, and written in the physical metric they take the form

$$T_2{}^{\mu\nu}[\alpha]=\frac{\alpha^2}{K^4}\left[\left(\frac{K^2{}_{;\rho}K^{2;\rho}}{K^2}+\frac{4}{3}\Lambda K^2\right)\left(g^{\mu\nu}-4\frac{K^\mu K^\nu}{K^2}\right)+8K_\rho C^{\rho\mu\nu\tau}K_\tau\right]\ , \tag{2.42}$$

$$T_4{}^{\mu\nu}[\alpha]=\frac{\alpha^4}{K^4}\left(g^{\mu\nu}-4\frac{K^\mu K^\nu}{K^2}\right)\ . \tag{2.43}$$

It is of some interest to observe that these tensors can be obtained as the metric variation of the conformally invariant actions

$$S_2[\alpha]=\alpha^2\int d^4x\, g^{1/2}(-K^2)^{-1/2}\times(\Box-\tfrac{1}{6}R)(-K^2)^{-1/2} \tag{2.44}$$

and

$$S_4[\alpha]=\alpha^4\int d^4x\, g^{1/2}\frac{1}{K^4}\ , \tag{2.45}$$

where the variation is understood in the sense of Eq. (2.23).

As an aside, we note that in the spirit of Ref. 1 it is straightforward to guess the forms (2.40) and (2.41) on general grounds: Both must be trace-free and conserved, and furthermore they should be constructed from the covariantly constant vector field K^μ and the geometry with, on dimensional grounds, $\alpha^{-2}T_2{}^{\mu\nu}$ linear in the curvature and $\alpha^{-4}T_4{}^{\mu\nu}$ of order zero in the curvature.

This procedure of using the conformal factor ϖ defined by Eq. (2.38) works whenever $c=0$ and hence gives the thermal forms $\overline{T}_A{}^{\mu\nu}$ and $\overline{T}_B{}^{\mu\nu}$ of the zero-temperature tensors $T_A{}^{\mu\nu}$ and $T_B{}^{\mu\nu}$. Unfortunately, to obtain $\overline{T}_C{}^{\mu\nu}$ in this way it is necessary in general to give a meaning to $(\delta/\delta g_{\mu\nu})(\alpha t)$ and it is not clear how to do this. Alternatively, one would need a consistent approximation for $T_R{}^{\mu\nu}[e^{-2\varpi}g]$ when $c\neq 0$. Instead we proceed by demanding agreement with the values in the open Einstein universe, where one knows that the Gaussian approximation is exact.[18] The form of the thermal stress tensor on this space for massless fields of spin 0, $\frac{1}{2}$, and 1 are given by Candelas and Dowker,[19] with the following corrected evaluation of the integral:

$$\int_0^\infty d\nu\frac{\nu(\nu^2+s^2)}{e^{\nu/T}-(-1)^{2s}}=\tfrac{1}{240}\{[15+(-1)^{2s}]\pi^4T^4+[30+10(-1)^{2s}]s^2\pi^2T^2\}\ . \tag{2.46}$$

In this way we find that the thermal tensors are given by

$$\overline{T}_A{}^{\mu\nu}=T_A{}^{\mu\nu}+\tfrac{1}{3}T_2{}^{\mu\nu}+2T_4{}^{\mu\nu}\ , \tag{2.47}$$

$$\overline{T}_B{}^{\mu\nu}=T_B{}^{\mu\nu}+T_2{}^{\mu\nu}+2T_4{}^{\mu\nu}\ , \tag{2.48}$$

$$\overline{T}_C{}^{\mu\nu}=T_C{}^{\mu\nu}-\tfrac{1}{2}T_2{}^{\mu\nu}-T_4{}^{\mu\nu}\ . \tag{2.49}$$

In Sec. III we shall consider static Einstein black-hole space-times and shall show that these combinations are precisely those which make the stress tensors for the thermal state with $\alpha=\kappa_0$, the surface gravity of the black hole, regular on the horizon.

In passing we should note that in certain cases, such as flat space or the open Einstein universe, $\tilde{\Box}\tilde{R}$ vanishes for conformal transformations with conformal factors of the form ϖ. In such cases the problem of dealing with $(\delta/\delta g_{\mu\nu})(\alpha t)$ does not arise; however, even here $T_C{}^{\mu\nu}[\varpi]$ does not agree with $\overline{T}_C{}^{\mu\nu}$ as defined by Eq. (2.48) which is puzzling. On the one hand, these thermal terms correspond to conformally invariant additions to the effective action and so, bearing in mind the remarks after Eq. (2.37), one might not expect to be able to obtain any information about them directly from Eq. (2.1). On the other hand, it is strange that one can do so for the actions A and B but not for the action C.

To conclude this section we shall give explicit formulas for the tensors introduced above for static Einstein space-times. It is convenient to introduce the symmetric tensor $\Pi^{\mu\nu}$ defined by the equation

$$\Pi^{\mu\nu}=(K^2)^{-1}K_\rho C^{\rho\mu\nu\tau}K_\tau\ , \tag{2.50}$$

and the scalar χ defined by the equation

$$\chi=-\frac{K^2{}_{;\mu}K^{2;\mu}}{4K^2}-\frac{\Lambda}{3}K^2\ . \tag{2.51}$$

In terms of these quantities, the zero-temperature stress tensors, $T_A{}^{\mu\nu}$, $T_B{}^{\mu\nu}$, and $T_C{}^{\mu\nu}$, defined by $\omega=\frac{1}{2}\ln(-K^2)$, can be written as

$$T_A{}^{\mu\nu}=16\Pi^{\mu\rho}\Pi_\rho{}^\nu-\frac{8}{3K^2}(7\chi+\Lambda K^2)\Pi^{\mu\nu}+\frac{4}{3K^2}\chi^{;\mu\nu}+\frac{16}{3K^4}\chi^{;(\mu}K^{2;\nu)}+\frac{2}{K^4}(C^2K^4+2\chi_{;\rho}K^{2;\rho}+\tfrac{4}{3}\chi^2)\frac{K^\mu K^\nu}{K^2}-\frac{2}{3K^4}(C^2K^4+4\chi_{;\rho}K^{2;\rho}+\chi^2)g^{\mu\nu}\ , \tag{2.52}$$

$$T_B{}^{\mu\nu}=8\Pi^{\mu\rho}\Pi_\rho{}^\nu-\frac{8}{K^2}\chi\Pi^{\mu\nu}+\frac{1}{2K^2}(4\chi^2-C^2K^2)\left[g^{\mu\nu}-4\frac{K^\mu K^\nu}{K^2}\right]+\tfrac{2}{3}\Lambda^2 g^{\mu\nu}\,, \tag{2.53}$$

$$T_C{}^{\mu\nu}=-\frac{4}{K^2}\chi\Pi^{\mu\nu}+\frac{2}{K^2}\chi^{;\mu\nu}+\frac{2}{K^4}\chi^{;(\mu}K^{2;\nu)}+\frac{1}{K^4}(4\chi^2-\tfrac{3}{2}C^2K^4)\frac{K^\mu K^\nu}{K^2}+\frac{1}{2K^4}(C^2K^4-2\chi_{;\rho}K^{2;\rho}-2\chi^2)g^{\mu\nu}\,, \tag{2.54}$$

and the two thermal tensors $T_2{}^{\mu\nu}[\alpha]$ and $T_4{}^{\mu\nu}[\alpha]$ as

$$T_2{}^{\mu\nu}[\alpha]=\alpha^2\left[\frac{8}{K^2}\Pi^{\mu\nu}-\frac{4}{K^4}\chi\left[g^{\mu\nu}-4\frac{K^\mu K^\nu}{K^2}\right]\right]\,, \tag{2.55}$$

$$T_4{}^{\mu\nu}[\alpha]=\alpha^4\frac{1}{K^4}\left[g^{\mu\nu}-4\frac{K^\mu K^\nu}{K^2}\right]\,, \tag{2.56}$$

where $C^2\equiv C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau}$. In deriving these tensors and checking their trace and divergence it is necessary to use various properties of $\Pi^{\mu\nu}$ and χ in static Einstein spaces; we have listed these properties in the Appendix.

The energy density ρ associated with each of these tensors is defined by the equation

$$\rho=-(K^2)^{-1}T^{\mu\nu}K_\mu K_\nu\,. \tag{2.57}$$

These energy densities are given by

$$\rho_A=-\frac{1}{K^4}(\tfrac{4}{3}C^2K^4+2\chi^2+2\chi_{;\mu}K^{2;\mu})\,, \tag{2.58}$$

$$\rho_B=-\frac{1}{K^4}(\tfrac{3}{2}C^2K^4-4\chi^2+\tfrac{2}{3}\Lambda^2K^4)\,, \tag{2.59}$$

$$\rho_C=+\frac{1}{K^4}(C^2K^4-3\chi^2)\,, \tag{2.60}$$

and

$$\rho_2[\alpha]=-\frac{12}{K^4}\alpha^2\chi\,, \tag{2.61}$$

$$\rho_4[\alpha]=+\frac{3}{K^4}\alpha^4\,. \tag{2.62}$$

III. BLACK-HOLE SPACE-TIMES

Here we shall study the properties of the tensors introduced in Sec. II for the special case of black-hole space-times. These space-times are taken to be static Einstein spaces but we do not assume any spatial symmetry. We shall be particularly interested in the energy of each tensor on the event horizon of the black hole and shall obtain a geometrical expression for each of these quantities. In so doing we shall extend the work of Frolov and Sanchez,[12] whose results we recover as a special case.

It is evident that the tensors $T_A{}^{\mu\nu}$, $T_B{}^{\mu\nu}$, $T_C{}^{\mu\nu}$, $T_2{}^{\mu\nu}$, and $T_4{}^{\mu\nu}$ are all singular on the event horizon, which coincides with the Killing horizon as the space-time is static. To study their behavior near the horizon we follow Frolov and Sanchez[12] and make use of the work of Israel.[20] We first write the metric in the form defined by the line element

$$ds^2=-\Omega^2dt^2+\kappa^{-2}d\Omega^2+h_{ab}d\theta^a d\theta^b\,, \tag{3.1}$$

where we have $\Omega^2=e^{-2\omega}=-K^2$, $h_{ab}=h_{ab}(\Omega,\theta^c)$ with $a,b\in\{2,3\}$, and $\kappa^2(\Omega,\theta^a)=\Omega_{;\mu}\Omega^{;\mu}$. κ is defined in such a way that $\kappa(0,\theta^a)=\kappa_0$, the surface gravity of the black hole. It is convenient to express Einstein's equations (with cosmological constant) in terms of geometrical quantities defined by the embedding of the two-surfaces $\Omega=$const in a three-surface $t=$const. The extrinsic curvature of such a two-surface is defined by the equation

$$k_{ab}=\tfrac{1}{2}\kappa\partial_\Omega h_{ab}\,. \tag{3.2}$$

We further define $k=h^{ab}k_{ab}$ and denote the intrinsic curvature of the two-surface by ${}^2R$. In terms of these quantities the Einstein equations take the form

$$\partial_\Omega\kappa=-k-\frac{\Lambda\Omega}{\kappa}\,, \tag{3.3}$$

$$\kappa(\partial_\Omega+\Omega^{-1})k_a{}^b=\tfrac{1}{2}\,{}^2R\,\delta_a{}^b-\Lambda\delta_a{}^b-\kappa(\kappa^{-1})_{:a}{}^{:b}-kk_a{}^b\,, \tag{3.4}$$

$${}^2R=\frac{2\kappa k}{\Omega}+2\Lambda-k_{ab}k^{ab}+k^2\,, \tag{3.5}$$

$$\partial_b\kappa=-\Omega(\partial_b k-k_b{}^c{}_{:c})\,, \tag{3.6}$$

where a colon denotes covariant differentiation with respect to the two-metric h_{ab} and all index manipulations are performed with this metric and its inverse. Given values of κ, h_{ab}, and k_{ab} on a two-surface $\Omega=$const satisfying Eqs. (3.5) and (3.6) as constraints, Eqs. (3.2)–(3.4) determine their values elsewhere, the constraints (3.5) and (3.6) being automatically preserved.[12]

An important consequence of Eq. (3.6) is that $\kappa_0=\kappa(0,\theta^a)$, the value of κ on the horizon, satisfies the equation

$$\kappa_{0:a}=0\,; \tag{3.7}$$

in other words, κ_0 is constant over the horizon. Approximate stress tensors for a thermal state with the Hawking temperature $\kappa_0(2\pi)^{-1}$ can now be defined using the function

$$\bar{\omega}=\kappa_0 t+\tfrac{1}{2}\ln(-K^2)=\kappa_0 t+\ln\Omega\,. \tag{3.8}$$

Since this function is smooth across the future event horizon of the black hole, it follows immediately that the thermal tensors $\bar{T}_A{}^{\mu\nu}\equiv T_A{}^{\mu\nu}[\bar{\omega}]$ and $\bar{T}_B{}^{\mu\nu}\equiv T_B{}^{\mu\nu}[\bar{\omega}]$ are regular on the future event horizon. In addition, these tensors are the same as those defined by the function $\bar{\omega}'\equiv-\kappa_0 t+\ln\Omega$, which is regular across the past horizon, so that $\bar{T}_A{}^{\mu\nu}$ and $\bar{T}_B{}^{\mu\nu}$ must also be regular there. Unfortunately, as discussed in the previous section, it is not possible to obtain the thermal tensor $\bar{T}_C{}^{\mu\nu}$ from $\bar{\omega}$ so that this argument for regularity breaks down. Instead we shall simply take the formula (2.49) and show that the corresponding tensor is regular.

We shall now evaluate $\bar{\rho}_A$, $\bar{\rho}_B$, and $\bar{\rho}_C$ on the horizon. To do this we use Eqs. (3.3)—(3.6) to obtain expansions for κ, and hence χ, about the horizon. Writing

$$\kappa=\kappa_0+\kappa_1(\theta^a)\Omega^2+\kappa_2(\theta^a)\Omega^4+\cdots \tag{3.9}$$

and

$$k=k_0(\theta^a)\Omega+k_1(\theta^a)\Omega^3+\cdots , \tag{3.10}$$

we find

$$\kappa_1=-\frac{1}{2\kappa_0}(\kappa_0 k_0+\Lambda) , \tag{3.11}$$

$$\kappa_2=\frac{1}{16\kappa_0{}^3}(\kappa_0{}^2k_0{}^2-2\Lambda\kappa_0k_0-2\Lambda^2+\kappa_0k_{0:a}{}^{:a}) . \tag{3.12}$$

In terms of the above variables the scalar χ is given by the equation

$$\chi=\kappa^2+\tfrac{1}{3}\Lambda\Omega^2 . \tag{3.13}$$

Hence, defining

$$\chi=\chi_0+\chi_1\Omega^2+\chi_2\Omega^4+\cdots , \tag{3.14}$$

we obtain

$$\chi_0=\kappa_0{}^2 , \tag{3.15}$$

$$\chi_1=-\kappa_0k_0-\tfrac{2}{3}\Lambda , \tag{3.16}$$

$$\chi_2=\frac{1}{8\kappa_0{}^2}(3\kappa_0{}^2k_0{}^2+2\Lambda\kappa_0k_0+\kappa_0k_{0:a}{}^{:a}) . \tag{3.17}$$

Furthermore, we can write[20]

$$C^2=\frac{8}{\Omega^2}[\kappa^2k_{ab}k^{ab}+2\kappa_{:a}\kappa^{:a}+(\kappa k+\Lambda\Omega)^2]-\tfrac{8}{3}\Lambda^2 , \tag{3.18}$$

which gives

$$C_0{}^2\equiv C^2|_{\mathscr{H}}=12(\kappa_0k_0+\tfrac{2}{3}\Lambda)^2 . \tag{3.19}$$

Comparison with Eq. (3.16) yields the interesting equation

$$\chi_1=-(C_0{}^2/12)^{1/2} . \tag{3.20}$$

It is worth noting that Eq. (3.18) can be written as

$$C^2=8\left[G_A{}^B+\frac{\Lambda}{3}\delta_A{}^B\right]\left[G_B{}^A+\frac{\Lambda}{3}\delta_B{}^A\right] , \tag{3.21}$$

where $A,B,\in\{1,2,3\}$ and G_{AB} is the Einstein tensor in a three-surface $t=$const, which is given by the equation $G_{AB}=\Omega^{-1}\Omega_{||AB}$ where parallel bars denotes the covariant derivative in the three-surface. In particular, it follows that C^2 is everywhere non-negative, which may readily be shown to be true for any static space-time, Einstein or not, by going to the ultrastatic metric.

In this way it is a straightforward to show that $\bar{T}_C{}^{\mu\nu}$ is regular on the horizon. In addition, we obtain the following formulas for the energies on the horizon:

$$\begin{aligned}\bar{\rho}_A&=-\tfrac{1}{3}(7\kappa_0k_0+4\Lambda)(3\kappa_0k_0+2\Lambda)+\kappa_0\Delta_2k_0\\&=-\chi_1(7\chi_1+\tfrac{2}{3}\Lambda)-\Delta_2\chi_1 ,\end{aligned} \tag{3.22}$$

$$\bar{\rho}_B=-\tfrac{4}{3}(3\kappa_0k_0+2\Lambda)^2-\tfrac{2}{3}\Lambda^2=-12\chi_1{}^2-\tfrac{2}{3}\Lambda^2 , \tag{3.23}$$

$$\bar{\rho}_C=(3\kappa_0k_0+2\Lambda)^2=9\chi_1{}^2 , \tag{3.24}$$

where Δ_2 denotes the Laplacian on $\mathscr{H}$. These results reduce to those of Frolov and Sanchez[12] for $\bar{\rho}_A$ and $\bar{\rho}_B$ when $\Lambda=0$.

IV. EXAMPLES

In this section we shall give the approximate renormalized stress tensor in a number of special cases, and compare the results with the exact tensors where these are known.

We start with a couple of examples in flat space-time for which we may immediately set $\Pi^{\mu\nu}=0$ and $\Lambda=0$ in Eqs. (2.52)—(2.56).

A. Rindler space

In Rindler coordinates (τ,ξ,y,z) the line element for Rindler space takes the form

$$ds^2=-\xi^2d\tau^2+d\xi^2+dy^2+dz^2 , \tag{4.1}$$

where $\xi\in(0,\infty)$ and $\tau,y,z\in(-\infty,\infty)$. We choose the Killing vector given by $K^\mu=\delta^\mu_\tau$, for which $K^2=-\xi^2$ and $\chi=1$. It can then readily be shown that

$$\begin{aligned}\bar{T}_R{}^\mu{}_\nu[\alpha]=[(-\tfrac{2}{3}-\tfrac{4}{3}\alpha^2+2\alpha^4)a+(2-4\alpha^2+2\alpha^4)b\\+(-1+2\alpha^2-\alpha^4)c]\text{diag}(-3,1,1,1)^\mu{}_\nu .\end{aligned} \tag{4.2}$$

In particular, choosing $\alpha=0$, corresponding to the Rindler vacuum, we obtain

$$\rho_R=-2a+6b-3c , \tag{4.3}$$

while choosing $\alpha=1$, corresponding to the Minkowski vacuum, we reassuringly obtain

$$\bar{\rho}_R[1]=0 . \tag{4.4}$$

Equation (4.3) gives the exact result for the energy density of the Rindler vacuum for all spins provided (and only provided) we choose the value of c given by ζ-function and point-separation renormalization.

B. The Casimir wedge

We can use our formalism to obtain an expression for the renormalized stress tensor for a field between two plane plates making a wedge with opening angle Φ, satisfying periodic boundary conditions, or equivalently on a cone with semiangle $\arcsin(\Phi/2\pi)$. We write the line element for flat space in the form

$$ds^2=-dt^2+d\rho^2+\rho^2d\phi^2+dz^2 , \tag{4.5}$$

and take the plates to be defined by the equation $\phi=0$ and $\phi=\Phi$. We now choose the Killing vector $K^\mu=\delta^\mu_\phi$. This, of course, is not timelike but, as we noted in Sec. II, this requirement was not essential. For this Killing vector field we have $K^2=\rho^2$ and $\chi=-1$.

The role of t is now taken by ϕ, with the "thermal" state with "inverse temperature" $i\Phi$ corresponding to a state periodic in ϕ with period Φ. Using Eqs. (2.52)–(2.56) we obtain the approximate renormalized stress tensor for this state as

$$T_R{}^{\mu}{}_{\nu}=\rho^{-4}\left\{\left[-\frac{2}{3}-\frac{4}{3}\left[\frac{2\pi}{\Phi}\right]^2+2\left[\frac{2\pi}{\Phi}\right]^4\right]a+\left[2-4\left[\frac{2\pi}{\Phi}\right]^2+2\left[\frac{2\pi}{\Phi}\right]^4\right]b+\left[-1+2\left[\frac{2\pi}{\Phi}\right]^2-\left[\frac{2\pi}{\Phi}\right]^4\right]c\right\}\times\mathrm{diag}(1,1,-3,1)^{\mu}{}_{\nu}\,. \tag{4.6}$$

In particular, we find

$$\rho_R(s=0)=\frac{1}{360\Phi^2\rho^4}\left[\frac{4\pi^2}{\Phi^2}-\frac{\Phi^2}{4\pi^2}\right] \tag{4.7}$$

and

$$\rho_R(s=1)=\frac{1}{720\pi^2\rho^4}\left[\frac{4\pi^2}{\Phi^2}+11\right]\left[\frac{4\pi^2}{\Phi^2}-1\right], \tag{4.8}$$

where we have chosen $h(0)=1$ and $h(1)=2$. These results agree with the exact results. To obtain the results for Dirichlet or Neumann boundary conditions one needs merely replace Φ by 2Φ in the above equations, as can be seen by an examination of the work of Deutsch and Candelas.[21] The standard result for the Casimir energy between two parallel plates with separation a can be obtained by letting $\rho\to\infty$ and $\Phi\to 0$ in Eq. (4.6) in such a way that $\rho\Phi=a$ remains constant.

We now turn to symmetric Einstein space-times, that is, space-times which satisfy the equations $R_{\mu\nu}=\Lambda g_{\mu\nu}$ and

$$R_{\mu\nu\rho\tau;\sigma}=0\,. \tag{4.9}$$

The only examples are flat space, de Sitter space, anti–de Sitter space, the Nariai metric,[4] and nonsimply connected spaces obtained from these by factoring out by a discrete group.

C. de Sitter space and anti–de Sitter space

We write the line element in the static form

$$ds^2=-\left[1-\frac{\Lambda}{3}r^2\right]dt^2+\left[1-\frac{\Lambda}{3}r^2\right]^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\,, \tag{4.10}$$

where $t\in(-\infty,\infty)$, $r\in[0,\infty)$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi)$ and Λ is positive for de Sitter space but negative for anti–de Sitter space. Choosing $K^{\mu}=\delta^{\mu}_{t}$ we have $K^2=-[1-(\Lambda/3)r^2]$, $\chi=\Lambda/3$ and, of course, $\Pi^{\mu\nu}=0$. This gives

$$\overline{T}_R{}^{\mu\nu}=\left\{\left[-\frac{2}{3}\left[\frac{\Lambda}{3}\right]^2-\frac{4}{3}\left[\frac{\Lambda}{3}\right]\alpha^2+2\alpha^4\right]a+\left[2\left[\frac{\Lambda}{3}\right]^2-4\left[\frac{\Lambda}{3}\right]\alpha^2+2\alpha^4\right]b+\left[-\left[\frac{\Lambda}{3}\right]^2+2\left[\frac{\Lambda}{3}\right]\alpha^2-\alpha^4\right]c\right\}\times\frac{1}{K^4}\left[g^{\mu\nu}-4\frac{K^{\mu}K^{\nu}}{K^2}\right]+\tfrac{2}{3}\Lambda^2 b g^{\mu\nu}\,. \tag{4.11}$$

In particular, for the thermal state on de Sitter space with the Gibbons-Hawking[22] temperature, given by $\alpha=(\Lambda/3)^{1/2}$, we have simply

$$\overline{T}_R{}^{\mu\nu}\left[\left[\frac{\Lambda}{3}\right]^{1/2}\right]=\tfrac{2}{3}\Lambda^2 b g^{\mu\nu}\,. \tag{4.12}$$

D. Nariai space-time

The Nariai metric[4] is simply the standard Riemannian metric on $S^2\times S^2$ analytically continued to the Lorentzian section. We write the line element in the form

$$ds^2=-(1-\Lambda x^2)dt^2+(1-\Lambda x^2)^{-1}dx^2+\Lambda^2(d\theta^2+\sin^2\theta\,d\phi^2)\,, \tag{4.13}$$

where $t,x\in(-\infty,\infty)$, $\theta\in[0,\pi]$, $\phi\in[0,2\pi)$. We take $K^{\mu}=\delta^{\mu}_{t}$ which gives $K^2=-(1-\Lambda x^2)$ and $\chi_0=\Lambda$. We limit ourselves to considering the SO(2,1)$\times$SO(3)-invariant state which has a temperature defined by $\alpha=\chi_0{}^{1/2}=\Lambda^{1/2}$. For this state we find

$$\overline{T}_R{}^{\mu}{}_{\nu}[\Lambda^{1/2}]=\tfrac{4}{3}\Lambda^2[(a+\tfrac{3}{2}b)\mathrm{diag}(1,1,1,1)^{\mu}{}_{\nu}+(a+3b-3c)\times\mathrm{diag}(1,1,-1,-1)^{\mu}{}_{\nu}]\,. \tag{4.14}$$

The exact stress tensor for this state is given by

$$T_R{}^{\mu}{}_{\nu}[\mathrm{exact}]=\Lambda^2(\tfrac{4}{3}a+2b)\mathrm{diag}(1,1,1,1)^{\mu}{}_{\nu}\,. \tag{4.15}$$

As noted in Ref. 1, Eq. (4.13) agrees with the exact stress tensor for scalar fields, however, it clearly disagrees for higher spins. We shall discuss this in the Conclusion.

E. Schwarzschild space-time

To conclude this section we give the explicit form for the thermal tensors $\overline{T}_A{}^{\mu\nu}$, $\overline{T}_B{}^{\mu\nu}$, and $\overline{T}_C{}^{\mu\nu}$ in Schwarzschild coordinates for a field in the Hartle-Hawking thermal state, together with the tensors $T_2{}^{\mu\nu}$ and $T_4{}^{\mu\nu}$. We shall follow the notation of Ref. 7 and introduce tensors $U^{\mu\nu}$, $V^{\mu\nu}$, and $W^{\mu\nu}$ which are given in Schwarzschild coordinates (t,r,θ,ϕ) by

$$U^{\mu}{}_{\nu}=\mathrm{diag}(-3,1,1,1)^{\mu}{}_{\nu}\,, \tag{4.16}$$

$$V^{\mu}{}_{\nu}=\mathrm{diag}(0,2,-1,-1)^{\mu}{}_{\nu}\,, \tag{4.17}$$

and

$$W^{\mu}{}_{\nu}=\mathrm{diag}(3,1,0,0)^{\mu}{}_{\nu}\,. \tag{4.18}$$

We further define $R\equiv 2M$ and $w\equiv R/r\equiv 2M/r$, then

$$\overline{T}_A{}^{\mu\nu}[\kappa_0]=\frac{1}{24R^4}[(3+6w+9w^2+12w^3+13w^4+14w^5-41w^6)U^{\mu\nu}+8w^3(1+w+w^2+2w^3)V^{\mu\nu}+72w^6W^{\mu\nu}]\ , \tag{4.19}$$

$$\overline{T}_B{}^{\mu\nu}[\kappa_0]=\frac{1}{8R^4}[(1+2w+3w^2+4w^3+3w^4+2w^5-23w^6)U^{\mu\nu}+8w^3(1+w+w^2+2w^3)V^{\mu\nu}+24w^6W^{\mu\nu}]\ , \tag{4.20}$$

$$\overline{T}_C{}^{\mu\nu}[\kappa_0]=\frac{1}{16R^4}[-(1+2w+3w^2+4w^3+3w^4+2w^5-63w^6)U^{\mu\nu}-8w^3(1+w+w^2+9w^3)V^{\mu\nu}]\ , \tag{4.21}$$

while

$$T_2{}^{\mu\nu}[\alpha]=-\frac{\alpha^2w^4}{R^2(1-w)^2}U^{\mu\nu}+\frac{4\alpha^2w^3}{R^2(1-w)}V^{\mu\nu} \tag{4.22}$$

and

$$T_4{}^{\mu\nu}[\alpha]=\frac{\alpha^4}{(1-w)^2}U^{\mu\nu}\ . \tag{4.23}$$

In particular, on the horizon

$$\overline{T}_R{}^{\mu}{}_{\nu}[\kappa_0]=\frac{1}{R^4}(7a+12b-9c)\mathrm{diag}(1,1,0,0)^{\mu}{}_{\nu}+\frac{1}{R^4}(-a-6b+9c)\mathrm{diag}(0,0,1,1)^{\mu}{}_{\nu}\ . \tag{4.24}$$

This agrees with the general formulas (3.22)–(3.24), on calculating that $X_1=R^{-2}$ in this case. Indeed, these formulas could be used to derive Eq. (4.24) since the tensorial form on the horizon is completely determined by the trace and the requirement of regularity.

The numerical value of the approximate renormalized energy density for scalar fields

$$\overline{\rho}_R(\text{spin } 0)=-\frac{1}{640\pi^2R^4}\ , \tag{4.25}$$

is in good agreement with the value obtained numerically by Candelas[23] for the exact stress tensor. For the Maxwell field we obtain

$$\overline{\rho}_R(\text{spin } 1)=-\frac{1}{120\pi^2R^4}\ , \tag{4.26}$$

while Elster[24] gives the exact value as

$$\rho_R(\text{spin } 1,\text{exact})=+\frac{41}{480\pi^2R^4}\ . \tag{4.27}$$

Remarkably this is precisely the value given by Eq. (4.24) on choosing the dimensional renormalization value $c=0$. We do not understand this curious result. We cannot simply choose $c=0$ everywhere as it leads to absurd negative energy densities at infinity.

Clearly, as it stands Eq. (4.26) does not give a good approximation to the exact energy density. We shall discuss possible ways of resolving this problem in Sec. V.

V. CONCLUSION

We have argued that it is necessary to include the tensor $T_C{}^{\mu\nu}$ when approximating the renormalized stress tensor for the electromagnetic field. In this way we have been able to obtain exact results for all spins in all our flat space-time examples—Rindler space, the Casimir wedge (and hence also Casimir parallel plates), and for thermal states in Minkowski space (and hence also for the Hartle-Hawking state at infinity). Moreover, in conformally flat space-times we reproduce the zero-temperature results of Brown and Cassidy,[25] as well as obtaining a thermal form which yields the correct answer in all known cases. Unfortunately, not all curved space-time results are so good. While we have agreement with the exact result for scalar fields in Nariai space-time we have disagreement for higher spins. In addition, in correcting the Hartle-Hawking result at infinity we have destroyed the agreement we had previously with the exact result on the horizon for the electromagnetic field.

There are two ways one might hope to improve the approximation: either make a conformal transformation to a space-time in which the trace anomaly vanishes or else add conformal invariants to the effective action in some systematic way [see the discussion after Eq. (2.37)]. The advantages of the first approach will be clearer when one has a better understanding of the connection between the renormalized stress tensor and the Green's functions of the theory for the electromagnetic field.[17] Instead here we briefly investigate the second approach.

For clarity we limit the discussion to the case of the Hartle-Hawking state on a space-time with a Killing horizon. So far the content of our approximation can be summarized by the equation

$$W_R\approx(2a+2b-c)\kappa_0{}^4\int g^{1/2}d^4x\,(K^2)^{-2}+(\tfrac{1}{3}a+b-\tfrac{1}{2}c)\kappa_0{}^2\int g^{1/2}d^4x\,(-K^2)^{-1/2}(\Box-\tfrac{1}{6}R)(-K^2)^{-1/2}+(aA[\omega]+bB[\omega]+cC[\omega])\ . \tag{5.1}$$

To improve on this approximation it is natural to add to the right-hand side of Eq. (5.1) conformally invariant actions constructed from K^μ and the geometry. Such an action if local can be written schematically as

$$(\kappa_0{}^2)^{2-q-r}\int g^{1/2}d^4x\,(K^2)^{q+r-2}\left[\frac{K^\mu K_\nu}{K^2}\right]^p\times(C^{\rho\tau}{}_{\lambda\sigma})^q(\Box-\tfrac{1}{6}R)^r\ , \tag{5.2}$$

where p, q, and r are assumed to be non-negative integers. $S_4[\kappa_0]$ corresponds to the choice $p=q=r=0$, while $S_2[\kappa_0]$ corresponds to the choice $p=q=0, r=1$. There are no other actions one can construct in this way containing a positive power of $\kappa_0{}^2$.

In the action (5.2) decreasing powers of $\kappa_0{}^2$ correspond to increasing powers of K^2 so that higher-order terms can be expected to contribute less to the stress tensor near the horizon. In addition, for an asymptotically flat space-time the geometrical terms will damp the action at infinity. Hence, here again, the higher-order terms should make only a small contribution to the stress tensor.

There are only a limited number of actions of the form (5.2) for each power of $\kappa_0{}^2$ and one can list them systematically. One can then attempt to fix their coefficients to give exact agreement with known results such as Nariai. This is not trivial since the form of the correction term given by Eq. (4.14) is very specific. For example, the action

$$\kappa_0{}^{-2}\int g^{1/2}d^4x\, K^2 C_{\mu\nu}{}^{\rho\tau}C_{\rho\tau}{}^{\lambda\sigma}C_{\lambda\sigma}{}^{\mu\nu} \tag{5.3}$$

can be used to correct the value of the higher-spin stress tensors in Nariai and Schwarzschild space-times on the horizon; however, it does not give a tensor of the required form everywhere in Nariai space-time.

It is not clear how far it is worth pursuing this expansion. The usefulness of having an analytic approximation decreases with its increasing complexity. It may be that a nonlocal expression for the action will eventually be needed to get a better approximation.

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APPENDIX

In this appendix we list a number of useful tensor identities valid for static Einstein space-times. First, we recall the definitions of χ and $\Pi^{\mu\nu}$:

$$\chi = -\frac{K^2{}_{;\mu}K^{2;\mu}}{4K^2} - \frac{\Lambda}{3}K^2 , \tag{A1}$$

$$\Pi^{\mu\nu} = \frac{K_\rho C^{\rho\mu\nu\tau}K_\tau}{K^2} . \tag{A2}$$

We now have the following results:

$$R_{\mu\nu\rho\tau} = C_{\mu\nu\rho\tau} + \frac{\Lambda}{3}(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho}) , \tag{A3}$$

$$R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} = C_{\mu\nu\rho\tau}C^{\mu\nu\rho\tau} + \tfrac{8}{3}\Lambda^2 , \tag{A4}$$

$$K_{\mu;\nu\rho} = R_{\mu\nu\rho\tau}K^\tau = C_{\mu\nu\rho\tau}K^\tau + \frac{\Lambda}{3}(g_{\mu\rho}K_\nu - g_{\nu\rho}K_\mu) , \tag{A5}$$

$$K^2K_{\mu;\nu} = \tfrac{1}{2}(K_\mu K^2{}_{;\nu} - K_\nu K^2{}_{;\mu}) , \tag{A6}$$

$$K^2{}_{;\mu\nu} = 2K^2\Pi_{\mu\nu} - 2\chi\frac{K_\mu K_\nu}{K^2} - \frac{2}{3}\Lambda K^2 g_{\mu\nu} + \frac{K^2{}_{;\mu}K^2{}_{;\nu}}{2K^2} . \tag{A7}$$

Furthermore, we can decompose the Weyl tensor according to the equation

$$\begin{aligned} C_{\mu\nu\rho\tau} = {} & \Pi_{\mu\rho}\left[g_{\nu\tau} - 2\frac{K_\nu K_\tau}{K^2}\right] + \Pi_{\nu\tau}\left[g_{\mu\rho} - 2\frac{K_\mu K_\rho}{K^2}\right] \\ & - \Pi_{\mu\tau}\left[g_{\nu\rho} - 2\frac{K_\nu K_\rho}{K^2}\right] - \Pi_{\nu\rho}\left[g_{\mu\tau} - 2\frac{K_\mu K_\tau}{K^2}\right] . \end{aligned} \tag{A8}$$

With this decomposition the Bianchi identities take the form

$$\Pi_{\mu\rho}K^2{}_{;\nu} - \Pi_{\nu\rho}K^2{}_{;\mu} + K^2(\Pi_{\mu\rho;\nu} - \Pi_{\nu\rho;\mu}) = \tfrac{1}{2}(\chi_{;\nu}g_{\mu\rho} - \chi_{;\mu}g_{\nu\rho}) . \tag{A9}$$

In addition, $\Pi^{\mu\nu}$ and χ have the following properties:

$$\Pi_\mu{}^\mu = 0 , \tag{A10}$$

$$\Pi_{\mu\nu}K^\nu = 0 , \tag{A11}$$

$$\Pi_{\mu\nu}K^{2;\nu} = -\chi_{;\mu} , \tag{A12}$$

$$\Pi_\mu{}^\nu\Pi_\nu{}^\mu = \tfrac{1}{8}C_{\mu\nu}{}^{\rho\tau}C_{\rho\tau}{}^{\mu\nu} \equiv \tfrac{1}{8}C^2 , \tag{A13}$$

$$\Pi_\mu{}^\nu\Pi_\nu{}^\rho\Pi_\rho{}^\mu = -\tfrac{1}{16}C_{\mu\nu}{}^{\rho\tau}C_{\rho\tau}{}^{\lambda\sigma}C_{\lambda\sigma}{}^{\mu\nu} \equiv -\tfrac{1}{16}C^3 , \tag{A14}$$

$$\Pi_{\mu\nu}{}^{;\nu} = -\frac{1}{2K^2}\chi_{;\mu} , \tag{A15}$$

$$(\Pi_\mu{}^\rho\Pi_{\rho\nu})^{;\nu} = \tfrac{1}{16}C^2{}_{;\mu} - \frac{1}{K^2}\Pi_\mu{}^\rho\Pi_\rho{}^\tau K^2{}_{;\tau} + \frac{1}{8K^2}C^2K^2{}_{;\mu} , \tag{A16}$$

and

$$\left[\frac{\chi^{;\mu}}{K^2}\right]_{;\mu} = -\tfrac{1}{4}C^2 . \tag{A17}$$

Finally, we note the identities

$$C_{\mu\lambda}{}^{\rho\tau}C_{\rho\tau}{}^{\nu\lambda} = \tfrac{1}{4}C^2 g_\mu{}^\nu , \tag{A18}$$

$$C_{\mu\kappa}{}^{\rho\tau}C_{\rho\tau}{}^{\lambda\sigma}C_{\lambda\sigma}{}^{\nu\kappa} = \tfrac{1}{4}C^3 g_\mu{}^\nu . \tag{A19}$$

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