Natural Poincaré gauge model

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Because it acts on space-time and is not semisimple, the Poincaré group cannot lead to a gauge theory of the usual kind. A candidate model is discussed which keeps itself as close as possible to the typical gauge scheme. Its field equations are the Yang-Mills equations for the Poincaré group. It is shown that there exists no Lagrangian for these equations.

I. INTRODUCTION

Gauge models for the Poincaré group endeavor to bring together three lofty ideas: (i) all interactions should be unified in some all embracing theory; (ii) interactions are mediated by gauge fields; (iii) gravitation is intimately related to the very texture of space-time. Elementary particles are classified by the gauge groups and, in what concerns symmetries in space-time, the same role is played by the Poincaré group.[1] A Poincaré gauge theory will always be kept apart from the usual gauge models by two peculiarities: the nonsemisimple character of the group[2] and the presence of a "kinematical" representation, whose generators are fields on space-time itself. Concerning the first of these features, there are two main consequences: the gauge potentials related to the Abelian subgroup have unusual transformation properties and there is no bi-invariant metric on the group. A purely right-invariant metric may suffice to build up a Lagrangian but the result is an atypical gauge Lagrangian leading to field equations which are not of the Yang-Mills form.[3] As to the second peculiarity, shared by all models involving space-time symmetries, all local transformations in the kinematical representation may ultimately be seen as translations and the trouble is that there is no such thing as "gauging" translations in the usual way: $\exp(a^\mu\partial_\mu)f(x)=f(x+a)$ becomes false as soon as the parameters a^μ become point dependent. The simple idea of gauging by imposing a local symmetry does not apply and at least some of the ideas currently associated to gauge theories will have to be forsaken. Our objective in this paper is to examine a model which remains as close as possible to the general scheme of gauge theories. Poincaré models have been extensively considered,[4,5] but almost always with a Lagrangian as the starting point. In general, their field equations are not the Yang-Mills equations which can be written directly from the group structure constants. These equations will be taken here as the model cornerstone and it will be shown that they cannot come from an action principle.

When looking for a space-time-rooted gauge model for gravitation, it is natural to investigate those features of space-time presenting gaugelike characteristics. On any differentiable manifold there is a naturally defined bundle, the bundle of affine frames,[6] whose structural group is the affine linear group $AL(n,\mathbb{R})=GL(n,\mathbb{R})\otimes T_n$, the semidirect product of the linear group and the translation group. In the case of space-time the restriction to Lorentz frames reduces it to the Poincaré group $P=\mathscr{L}\otimes T_4$. This always-present structure provides the most general gaugelike features related to space-time and justifies the interest in Poincaré gauge models.[7] Of course, in order to accommodate the known elementary particles, $\mathscr{L}$ is to be taken as the covering group of SO(3,1), that is, SL(2, $\mathbb{C}$). The Lie algebra is, as a vector space, a direct sum of the Lorentz and the translation sectors. In a basis with generators $\{Z_{\alpha\beta},I_\gamma\}$, an affine connection $\overline{\Gamma}$ on the bundle decomposes into

$$\overline{\Gamma}=\Gamma+S\ , \tag{1.1}$$

where $\Gamma=\frac{1}{2}Z_{\alpha\beta}\Gamma^{\alpha\beta}{}_\mu dx^\mu$ is a Lorentz connection form and $S=I_\alpha h^\alpha{}_\mu dx^\mu$ is the solder form.[8] The same decomposition affects the curvature of $\overline{\Gamma}$:

$$\overline{F}=F+T\ , \tag{1.2}$$

where F and T are the curvature and the torsion of Γ:

$$F=d\Gamma-i\Gamma\wedge\Gamma\ , \tag{1.3}$$

$$T=dS-i\Gamma\wedge S-iS\wedge\Gamma\ . \tag{1.4}$$

Note that torsion is always present in the bundle of frames. It may be vanishing (as in general relativity) but it has consequences anyhow. Furthermore, the introduction of spinors on a manifold practically enforces (or reveals[5]) its presence. As still another consequence of the Lie-algebra decomposition, two Bianchi identities appear:

$$dF-i[\Gamma,F]=0\ , \tag{1.5}$$

$$dT-i[\Gamma,T]-i[S,F]=0\ . \tag{1.6}$$

These equations summarize the geometry involved, and should correspond to the purely geometrical substratum of any gauge theory. The dynamical content is to be given by the field equations.

The Yang-Mills equations can be written for any group once its structure constants are known. For the Poincaré group, we find that they are

$$d\widetilde{F}-i[\Gamma,\widetilde{F}]=0\ , \tag{1.7}$$

$$d\widetilde{T}-i[\Gamma,\widetilde{T}]-i[S,\widetilde{F}]=0\ , \tag{1.8}$$

where $\widetilde{F}$ and $\widetilde{T}$ are the duals of F and T. These equations have been first proposed by Popov and Daikhin,[9] who pointed out that, if Γ is restricted to be metric preserving and torsionless, they reduce to Einstein's equations $R_{\mu\nu}=0$. Such restrictions are, however, unnatural for other reasons than the desirability of the presence of torsion alluded to above. Important reasons to look for a gauge theory for gravitation are the apparent resistance of Einstein's theory to renormalization and the gauge theories' penchant for it. It is not clear just where such a penchant comes from, but it is a general feeling that conformal invariance is somehow involved in good short-distance behavior.[10,11] The restrictions leading to Einstein's equations break the conformal symmetry of Eqs. (1.7) and (1.8). They also break the discrete duality symmetry of (1.5)—(1.8), another important characteristic of gauge theories.[12]

A problem remains in taking the above equations as gauge field equations: the translational gauge potentials are identified with the tetrad fields $h^{\alpha}{}_{\mu}$. There are many difficulties in such an interpretation. First, the solder form being a canonical attribute of the frame bundle, its components are in a sense given *a priori*[4] and cannot participate in the description of any specific field. A second difficulty appears in the presence of source fields: the $h^{\alpha}{}_{\mu}$ will couple to their kinetic energy and consequently they will have no free propagator. Finally, the absence of a gauge interaction is characterized by the vanishing (up to gauge transformations) of the potential field, an impossibility for the four-leg fields. We shall see below how all these problems can be avoided. The solution will be an old one[13] (the potential being the nontrivial part of the $h^{\alpha}{}_{\mu}$) but in a different context. Equations (1.5)—(1.8) will remain valid, and the field equations will keep the same form, but with T replaced by a new field strength. The validity of these equations is supported by another, independent argument. The Poincaré group P acts on the tangent spaces, each one itself a Minkowski space. Now, P is an Inönü-Wigner contraction of the de Sitter (dS) group. Unlike P, dS is semisimple and it is not difficult to build a gauge theory for it. In the tangent bundle, to replace P by dS corresponds to replacing each tangent Minkowski space by an osculating de Sitter space.[14] Such a space is characterized by a parameter L, a length related to its (constant) curvature. The contraction process corresponds to $L\rightarrow\infty$: in the limit, dS becomes P and each de Sitter space becomes a Minkowski space.[15] The Lorentz group, a common subgroup of P and dS, remains unscathed in the process, but the four remaining dimensionless parameters of dS get multiplied by L and become the translation parameters of P.[16] What happens to gauge fields under contraction has been analyzed some time ago.[7] The point of interest is that the equations taken here as fundamental (both Bianchi and Yang-Mills) come out as the contraction limits of the corresponding equations for the dS model.

The contraction procedure has been used for a number of years[17] to circumvent difficulties in treating models involving the Poincaré group[18] and will be used in the following as a guide. In Ref. 7, the behavior of fields in the adjoint representation has been examined in detail. In Sec. II below, we analyze briefly what happens in the kinematical representation and give a rather detailed account of the model. The impossibility of getting at the Yang-Mills equations from a Lagrangian is shown in Sec. III as a consequence of Vainberg's theorem[19,20] of functional calculus. In the final section a few comments are made about quantization and the possible interest of a supersymmetric version.

II. GENERAL DESCRIPTION OF THE MODEL

A special characteristic of gauge models involving space-time symmetries is the presence of a "kinematic representation," whose generators are tangent fields. At each point of space-time, one can choose coordinates $\{x^{\alpha}\}$ for the tangent space[21] and realize the P Lie algebra by the well-known generators

$$L_{\alpha\beta}=-i(x_{\alpha}\partial_{\beta}-x_{\beta}\partial_{\alpha})\,, \tag{2.1}$$

$$Z_{\alpha}=-i\partial_{\alpha}\,. \tag{2.2}$$

The important point is that, as these operators act on source fields through their arguments, all fields will respond to their action. Spinor and vector fields will belong also to other representations and their total response to Lorentz transformations will be governed by

$$Z_{\alpha\beta}=L_{\alpha\beta}+S_{\alpha\beta}\,. \tag{2.3}$$

Scalar fields, however, will be singlets in any other representation, and their kinematical response is the only possible explanation for the universality of gravitation in a gauge picture.

The transformations generated by (2.1) and (2.2) will change points in the fiber (i.e., in the tangent space). For an infinitesimal change with parameters $(\delta w^{\alpha\beta},\delta a^{\alpha})$,

$$\delta x^{\gamma}=\frac{i}{2}\delta w^{\alpha\beta}L_{\alpha\beta}x^{\alpha}+i\delta a^{\alpha}Z_{\alpha}x^{\gamma}$$
$$=-\delta w^{\gamma\alpha}x_{\alpha}+\delta a^{\gamma}\,. \tag{2.4}$$

At a fixed point, we shall write the corresponding change of a scalar source field as

$$\delta_0\phi(x)=\left[\frac{i}{2}\delta w^{\alpha\beta}L_{\alpha\beta}+i\delta a^{\alpha}Z_{\alpha}\right]\phi(x)\,. \tag{2.5}$$

For the dS Lie algebra, the kinematic generators L_{ab} (with $a,b=1,\ldots,5$) are[16]

$$L_{\alpha\beta}=-i(x_{\alpha}\partial_{\beta}-x_{\beta}\partial_{\alpha})\,,$$
$$L_{\alpha5}=-iL\left[1+\frac{x^2}{4L^2}\right]\partial_{\alpha}+\frac{x^{\beta}}{2L}L_{\alpha\beta}\,, \tag{2.6}$$

where L is the dS length parameter and $x^2=x_{\alpha}x^{\alpha}$. The ten group parameters can be grouped as $\{w^{ab}=-w^{ba}\}$. They are, of course, dimensionless. The contraction is obtained by redefining

$$\delta a_{\alpha}=L\,\delta w_{\alpha5} \tag{2.7}$$

and proceeding to the limit $L\rightarrow\infty$. The infinitesimal change under a dS transformation,

$$\delta_0\phi(x)=\frac{i}{2}\delta w^{ab}L_{ab}\phi(x) , \quad (2.8)$$

becomes just (2.5). In this process, generators and parameters change their dimensionalities. The gauge potentials $\Gamma^{ab}{}_\mu$ for the dS field will, under contraction, behave in a way analogous to the group parameters:[7] defining

$$\Gamma^{\alpha\beta}{}_\mu=A^{\alpha\beta}{}_\mu \quad (2.9)$$

and

$$\Gamma^{\alpha 5}{}_\mu=L^{-1}B^\alpha{}_\mu , \quad (2.10)$$

the $A^{\alpha\beta}{}_\mu$ and $B^\alpha{}_\mu$ will, after the limit is taken, appear as the P gauge potentials. The covariant derivative for the dS case is

$$D_\mu=\partial_\mu+\frac{1}{2}\Gamma^{ab}{}_\mu\frac{\delta_0}{\delta w^{ab}} . \quad (2.11)$$

For a scalar field,

$$D_\mu\phi=\left[\partial_\mu+\frac{i}{2}\Gamma^{\alpha\beta}{}_\mu L_{\alpha\beta}+i\Gamma^{\alpha 5}{}_\mu L_{\alpha 5}\right]\phi$$

turns into

$$D_\mu\phi=[\partial_\mu-(A^{\alpha\beta}{}_\mu x_\beta-B^\alpha{}_\mu)\partial_\alpha]\phi . \quad (2.12)$$

Note that we have been using $\{x^\alpha\}$ (with the beginning of the Greek alphabet) as coordinates in tangent space, and $\{x^\mu\}$ (with the second half of the Greek alphabet) as coordinates on space-time. So,

$$D_\mu\phi=h^\alpha{}_\mu\partial_\alpha\phi , \quad (2.13)$$

where

$$h^\alpha{}_\mu=\partial_\mu x^\alpha+B^\alpha{}_\mu-A^{\alpha\beta}{}_\mu x_\beta \quad (2.14)$$

can be regarded as a four-leg field. This expression comes out naturally from (2.11) under contraction, using (2.10).

For fields belonging also to some other representation, $Z_{\alpha\beta}$ instead of $L_{\alpha\beta}$ has to be used. Let us examine the case of spinor fields. The dS Lie algebra has a beautiful representation in terms of the γ matrices: the generators are

$$\sigma_{ab}=-\frac{i}{2}[\gamma_a,\gamma_b] ; \quad (2.15)$$

we see that γ_5 acquires the same status as the other γ's. The complete generators are now

$$Z_{ab}=L_{ab}+\frac{\sigma_{ab}}{2} . \quad (2.16)$$

The covariant derivative (2.11) gets extra terms; the part in $\Gamma^{\alpha 5}{}_\mu\sigma_{\alpha 5}/2\sim L^{-1}B^\alpha{}_\mu\sigma_{\alpha 5}/2$ vanishes on contraction and the resulting Poincaré covariant derivative is

$$D_\mu\psi=\left[h^\alpha{}_\mu\partial_\alpha+\frac{i}{4}A^{\alpha\beta}{}_\mu\sigma_{\alpha\beta}\right]\psi . \quad (2.17)$$

The behavior of the gauge potentials under transformations is obtained from that in the dS theory. There,

$$\Gamma'_\mu=U\Gamma_\mu U^{-1}-iU\,\partial_\mu\,U^{-1} , \quad (2.18)$$

with

$$U=\exp\left[\frac{i}{2}w^{ab}Z_{ab}\right] . \quad (2.19)$$

The double-index notation allows the use of 5×5 matrices in the adjoint representation and leads to a direct contact with the usual notation [as in (2.30) below]. Using

$$(Z_{ab})_{cd}{}^{ef}=if_{ab,cd}{}^{ef}$$

for the matrix elements, a straightforward calculation shows that (2.18) can be written as

$$\Gamma'^{cd}{}_\mu=(\Lambda^{-1})^c{}_a\Gamma^{ab}{}_\mu\Lambda_b{}^d-(\Lambda^{-1})^c{}_a\partial_\mu\Lambda^{ad} , \quad (2.20)$$

where

$$\Lambda_b{}^d=[\exp(w)]_b{}^d=\delta_b{}^d+w_b{}^d+\frac{1}{2!}w_b{}^c w_c{}^d+\cdots ,$$

the indices being raised or lowered by the dS metric. In the same line, the field strength covariance

$$F'_{\mu\nu}=UF_{\mu\nu}U^{-1}$$

can be put in the form

$$F'^{cd}{}_{\mu\nu}=(\Lambda^{-1})^c{}_a F^{ab}{}_{\mu\nu}\Lambda_b{}^d . \quad (2.21)$$

The field strengths are

$$F^{cd}{}_{\mu\nu}=\partial_\mu\Gamma^{cd}{}_\nu-\partial_\nu\Gamma^{cd}{}_\mu-\Gamma^c{}_{e\mu}\Gamma^{ed}{}_\nu+\Gamma^c{}_{e\nu}\Gamma^{ed}{}_\mu . \quad (2.22)$$

The contraction is done by identifying $\Gamma^{\alpha 5}{}_\mu=-\Gamma^{5\alpha}{}_\mu=L^{-1}B^\alpha{}_\mu$; $\Gamma^{\alpha\beta}{}_\mu=A^{\alpha\beta}{}_\mu$; $F^{\alpha 5}{}_{\mu\nu}=L^{-1}\tau^\alpha{}_{\mu\nu}$; after the limit is taken, it remains for the Lorentz sector

$$F^{\alpha\beta}{}_{\mu\nu}=\partial_\mu A^{\alpha\beta}{}_\nu-\partial_\nu A^{\alpha\beta}{}_\mu-A^\alpha{}_{\gamma\mu}A^{\gamma\beta}{}_\nu+A^\alpha{}_{\gamma\nu}A^{\gamma\beta}{}_\mu \quad (2.23)$$

[a term $L^{-2}(B^\alpha{}_\mu B^\beta{}_\nu-B^\alpha{}_\nu B^\beta{}_\mu)$ disappears] and, for the translation sector,

$$\tau^\alpha{}_{\mu\nu}=\partial_\mu B^\alpha{}_\nu-\partial_\nu B^\alpha{}_\mu-A^\alpha{}_{\beta\mu}B^\beta{}_\nu+A^\alpha{}_{\beta\nu}B^\beta{}_\mu . \quad (2.24)$$

This is just the covariant derivative of $B^\alpha{}_\nu$, as defined by the connection A. The same procedure, applied to (2.20) and (2.21), leads to the gauge transformations for the P theory:

$$A'^{\alpha\beta}{}_\mu=(\Lambda^{-1})^\alpha{}_\gamma A^{\gamma\delta}{}_\mu\Lambda_\delta{}^\beta-(\Lambda^{-1})^\alpha{}_\gamma\partial_\mu\Lambda^{\gamma\beta} , \quad (2.25)$$

$$B'^\alpha{}_\mu=(\Lambda^{-1})^\alpha{}_\gamma A^{\gamma\delta}{}_\mu a_\delta+(\Lambda^{-1})^\alpha{}_\gamma B^\gamma{}_\mu-(\Lambda^{-1})^\alpha{}_\gamma\partial_\mu a^\gamma , \quad (2.26)$$

$$F'^{\alpha\beta}{}_{\mu\nu}=(\Lambda^{-1})^\alpha{}_\gamma F^{\alpha\delta}{}_{\mu\nu}\Lambda_\delta{}^\beta , \quad (2.27)$$

$$\tau'^\alpha{}_{\mu\nu}=(\Lambda^{-1})^\alpha{}_\gamma F^{\gamma\beta}{}_{\mu\nu}a_\beta+(\Lambda^{-1})^\alpha{}_\gamma\tau^\gamma{}_{\mu\nu} . \quad (2.28)$$

Here a note of caution: one might wonder about the inversion of the roles of matrices Λ and Λ^{-1}. The reason for it is that we have been considering affine frame transformations in the tangent spaces, given by

$$e'_\alpha=e_\beta\Lambda^\beta{}_\alpha-a_\alpha , \quad (2.29)$$

with the matrices acting on the right. This convention, borrowed from the mathematical literature,[6] corresponds to the coordinate transformations

$$x'^{\alpha}=(\Lambda^{-1})^{\alpha}{}_{\beta}(x^{\beta}+a^{\beta}) , \quad (2.30)$$

of which (2.4) is the small-parameter version. The above rules for $B'^{\alpha}{}_{\mu}$ and $\tau'^{\alpha}{}_{\mu\nu}$ come from (2.20) and (2.21) by using

$$a^{\alpha}=L\Lambda^{\alpha 5} ,$$

of which (2.7) is the infinitesimal case. The above convention of taking the product of Lorentz transformations and translations instead of the exponential (2.19) makes no difference for the above rules but should be taken into account when considering the change of a source field as, for example, the finite version of (2.5).

The torsion $T^{\alpha}{}_{\mu\nu}$ will be the covariant derivative of the tetrad field [the same as (2.24) with $h^{\alpha}{}_{\mu}$ instead of $B^{\alpha}{}_{\mu}$]. Using (2.14) and collecting the terms conveniently, we find that

$$T^{\alpha}{}_{\mu\nu}=\tau^{\alpha}{}_{\mu\nu}-F^{\alpha\beta}{}_{\mu\nu}x_{\beta} . \quad (2.31)$$

The sourceless field equations for the dS case are

$$\partial_{\mu}F^{\alpha\beta\mu\nu}-\Gamma^{\alpha}{}_{\gamma\mu}F^{\gamma\beta\mu\nu}+F^{\alpha}{}_{\gamma}{}^{\mu\nu}\Gamma^{\gamma\beta}{}_{\mu}-\Gamma^{\alpha}{}_{5\mu}F^{5\beta\mu\nu}+F^{\alpha}{}_{5}{}^{\mu\nu}\Gamma^{5\beta}{}_{\mu}=0 \quad (2.32)$$

and

$$\partial_{\mu}F^{\alpha 5\mu\nu}-\Gamma^{\alpha}{}_{\gamma\mu}F^{\gamma 5\mu\nu}+F^{\alpha}{}_{\gamma}{}^{\mu\nu}\Gamma^{\gamma 5}{}_{\mu}=0 . \quad (2.33)$$

Redefining the fields prior to contraction, we see that the last two terms in (2.32) acquire factors L^{-2} and vanish in the limit. The resulting equation is

$$\partial_{\mu}F^{\alpha\beta\mu\nu}-A^{\alpha}{}_{\gamma\mu}F^{\gamma\beta\mu\nu}+F^{\alpha}{}_{\gamma}{}^{\mu\nu}A^{\gamma\beta}{}_{\mu}=0 . \quad (2.34)$$

The disappearance of these terms will be responsible for the non-Lagrangian character of the P field equations, as will be seen in the next section. From (2.33) we get

$$\partial_{\mu}\tau^{\alpha\mu\nu}-A^{\alpha}{}_{\gamma\mu}\tau^{\gamma\mu\nu}+F^{\alpha}{}_{\gamma}{}^{\mu\nu}B^{\gamma}{}_{\mu}=0 . \quad (2.35)$$

A direct computation shows that this set of equations is covariant under the transformations (2.25)–(2.28).

Equation (2.34) is just (1.7) in components. As to (2.35), it has the same form as (1.8) when written in components, the difference being that τ is not the torsion, but simply the covariant derivative of the field B. Had we identified $\Gamma^{\alpha 5}{}_{\mu}=L^{-1}h^{\alpha}{}_{\mu}$, just (1.8) would have resulted. Now comes a surprising result: if we take (2.31) into (2.35), we find that

$$\partial_{\mu}T^{\alpha\mu\nu}-A^{\alpha}{}_{\beta\mu}T^{\beta\mu\nu}+F^{\alpha}{}_{\beta}{}^{\mu\nu}h^{\beta}{}_{\mu}=0 . \quad (2.36)$$

This is just (1.8) in components, which is consequently preserved.

The behavior of the tetrad (2.14) under P transformations is obtained by using (2.25), (2.26), and (2.30). One finds that

$$h'^{\alpha}{}_{\mu}=\partial_{\mu}x'^{\alpha}+B'^{\alpha}{}_{\mu}-A'^{\alpha\beta}{}_{\mu}x'^{\beta}=(\Lambda^{-1})^{\alpha}{}_{\beta}h^{\beta}{}_{\mu} \quad (2.37)$$

has an interesting result: the tetrad field ignores translations, behaving (as it should) as a Lorentz vector field. If we use all the above transformation properties in relation (2.31) we find also that, under a P transformation,

$$T'^{\alpha}{}_{\mu\nu}=(\Lambda^{-1})^{\alpha}{}_{\beta}T^{\beta}{}_{\mu\nu} . \quad (2.38)$$

Looking at the equations and transformation properties for the components in the Lorentz sector, we see that it constitutes a gauge subtheory. This is not the case for the translation sector, which clearly is not a subtheory and exhibits rather awkward transformation properties. However, if we look more closely into (2.26) and (2.28), we find that, for pure Lorentz transformations ($a^{\alpha}=0$), both $B^{\alpha}{}_{\mu}$ and $\tau^{\alpha}{}_{\mu\nu}$ behave as vectors in the algebra indices. The awkwardness comes from the coupling between translations and Lorentz transformations and is just what is necessary to endow those quantities possessing clear geometrical meanings, such as $h^{\alpha}{}_{\mu}$ and $T^{\alpha}{}_{\mu\nu}$, with a simple behavior. The set $(F^{\alpha\beta}{}_{\mu\nu},\tau^{\alpha}{}_{\mu\nu})$ can be taken as the field strength despite the strange behavior of $\tau^{\alpha}{}_{\mu\nu}$. In particular, it allows a good, invariant characterization of the vacuum of the model as $F^{\alpha\beta}{}_{\mu\nu}=0$, $\tau^{\alpha}{}_{\mu\nu}=0$, corresponding to gauge transformations of zero potentials in (2.25) and (2.26).

Usual gauge potentials have the dimension of mass and field strengths of (mass)^2 (in units $\hbar=c=1$). The tetrad fields are dimensionless and, because of the redefinition of fields, $B^{\alpha}{}_{\mu}$ and $\tau^{\alpha}{}_{\mu\nu}$ have dimensions zero and one. If we want to get back the normal dimensions, we must add a length factor l to each $B^{\alpha}{}_{\mu}$ (equivalent to a redefinition $\Gamma^{\alpha 5}{}_{\mu}=lL^{-1}B^{\alpha}{}_{\mu}$ instead of that previously adopted). Such a problem was to be expected because translations, unlike other transformations, have dimensional parameters. All current densities have dimension 3, except the Noether current associated to translations: the energy-momentum density has dimension 4 and any theory using it as a source current will have to cope with this fact. We shall here prefer to keep $B^{\alpha}{}_{\mu}$ dimensionless and adopt the (equivalent) rule of adjusting the source terms with l factors whenever necessary.

Taking the covariant derivative (2.17) into the usual free Lagrangian for the spinor field (by the minimal coupling prescription), it is easy to check that the variations with respect to $B^{\alpha}{}_{\mu}$ and $A^{\alpha\beta}{}_{\mu}$ lead to the energy-momentum tensor density $\theta^{\alpha\nu}$ and the total angular momentum density $M^{\alpha\beta\nu}$. The form of (2.14) is enough to ensure the usual relationship between the energy-momentum and the orbital angular momentum, both currents representing the responses of source fields to transformations in the kinematic representation. The equations (2.34), (2.35), and (2.36) have as sources, respectively, $M^{\alpha\beta\nu}$, $l^2\theta^{\alpha\nu}$, and $(l^2\theta^{\alpha\nu}-M^{\alpha\beta\nu}x_{\beta})$.

The equations remain covariant under (2.25)–(2.28), but the coupling between translations and angular momentum imposes on $\theta^{\alpha\nu}$ a peculiar transformation law: for a transformation corresponding to (2.30),

$$l^2\theta'^{\alpha\nu}=(\Lambda^{-1})^{\alpha}{}_{\beta}(M^{\beta\gamma\nu}a_{\gamma}+l^2\theta^{\beta\nu}) . \quad (2.39)$$

By taking derivatives of the field equation and combining conveniently the terms, we arrive at the invariant conservation laws

$$\partial_{\mu}M^{\alpha\beta\mu}-A^{\alpha}{}_{\gamma\mu}M^{\gamma\beta\mu}+M^{\alpha}{}_{\gamma}{}^{\mu}A^{\gamma\beta}{}_{\mu}=0 , \quad (2.40)$$

$$\partial_{\mu}(l^2\theta^{\alpha\mu})-A^{\alpha}{}_{\beta\mu}(l^2\theta^{\beta\mu})+M^{\alpha}{}_{\gamma}{}^{\mu}B^{\gamma}{}_{\mu}=0 . \quad (2.41)$$

The angular momentum $M^{\alpha\beta\nu}=\mathscr{L}^{\alpha\beta\nu}+S^{\alpha\beta\nu}$ contains the orbital part $\mathscr{L}^{\alpha\beta\nu}$, which is inconvenient for a field theory. The coordinate x_β appears explicitly in both the field equations and the Lagrangian (note that the source Lagrangians are always well defined and it is no problem to obtain the currents, even after contraction). We can follow here the usual procedure[13] to get around this problem: it is enough to use, as the point-dependent Poincaré parameters, the set $\delta w^{\alpha\beta}$ and δx^α given by (2.4), instead of the ten original parameters $\delta w^{\alpha\beta}$ and δa^α. This stratagem is used without much ado by most authors but it has some consequences deserving discussion even at the price of repeating some apparently trivial things. Of course, the new parameters are to be considered as functionally independent so that now

$$\frac{\delta_0\psi}{\delta x^\alpha}=\frac{\delta_0\psi}{\delta a^\alpha}$$

from (2.4). As

$$\delta_0\psi=\frac{i}{2}\delta w^{\alpha\beta}\frac{\sigma_{\alpha\beta}}{2}+\delta x^\alpha\partial_\alpha\psi\ ,$$

the covariant derivative

$$D_\mu\psi=\partial_\mu\psi+\frac{1}{2}A^{\alpha\beta}{}_\mu\frac{\delta_0\psi}{\delta w^{\alpha\beta}}+B^\alpha{}_\mu\frac{\delta_0\psi}{\delta a^\alpha}$$

becomes

$$\widetilde{D}_\mu\psi=(\partial_\mu x^\alpha+B^\alpha{}_\mu)\partial_\alpha\psi+\frac{i}{2}A^{\alpha\beta}{}_\mu\sigma_{\alpha\beta}\psi\ , \tag{2.42}$$

with a new tetrad field

$$\widetilde{h}^\alpha{}_\mu=\partial_\mu x^\alpha+B^\alpha{}_\mu\ . \tag{2.43}$$

For a scalar field, of course, the last term in (2.42) is absent. With the transformations described in terms of the new parameters, the fields $B^\alpha{}_\mu$ will exhibit a behavior different from that given by (2.26). The simplest way to find the new rule is to notice that, if (2.42) is to be covariant, $B^\alpha{}_\mu$ must behave now in the same way the expression $B^\alpha{}_\mu-A^{\alpha\beta}{}_\mu x_\beta$ behaved in terms of the old parameters. For the infinitesimal case

$$B'^\alpha{}_\mu=B^\alpha{}_\mu-\delta w^\alpha{}_\gamma h^\gamma{}_\mu-\partial_\mu\delta x^\alpha\ . \tag{2.44}$$

In this parametrization, a pure translation ($\delta w^\alpha{}_\gamma=0$) changes $B^\alpha{}_\mu$ in a simpler way:

$$B'^\alpha{}_\mu=B^\alpha{}_\mu-\partial_\mu\delta x^\alpha\ , \tag{2.45}$$

whose finite version is

$$B'^\alpha{}_\mu=B^\alpha{}_\mu-\partial_\mu(x'^\alpha-x^\alpha)\ . \tag{2.46}$$

A direct calculation shows that $\widetilde{h}^\alpha{}_\mu$ keeps its behavior (2.37) and that (2.38) still holds for $\widetilde{T}^\alpha{}_{\mu\nu}$. However, with (2.43) the relation between the torsion and the field strength τ becomes

$$\widetilde{T}^\alpha{}_{\mu\nu}=t^\alpha{}_{\mu\nu}+\tau^\alpha{}_{\mu\nu}\ , \tag{2.47}$$

where

$$t^\alpha{}_{\mu\nu}=A^\alpha{}_{\beta\nu}\partial_\mu x^\beta-A^\alpha{}_{\beta\mu}\partial_\nu x^\beta \tag{2.48}$$

is a contribution to torsion coming from the Lorentz sector. If now we identify the coordinate systems, so that $\partial_\mu x^\alpha=\delta^\alpha{}_\mu$, we see that $t^\alpha{}_{\mu\nu}$ measures the asymmetry of the connection A or, in other words, its noninertial character. Formally $t^\alpha{}_{\mu\nu}$ is the covariant derivative of the trivial frame $\partial_\mu x^\alpha$ in the connection A. One would expect a noninertial effect in the presence of an angular momentum field density, but the gauge nonlinearity may create it even in the absence of sources in Eq. (2.34). Another effect of the reparametrization is to hide the duality symmetry for the torsion: Eq. (2.36) is no more valid when $T=t+\tau$. Note, however, that the reparametrization, which is essential for a future quantization, keeps $B^\alpha{}_\mu$ as the fundamental field with the same relation to $\tau^\alpha{}_{\mu\nu}$ and furthermore, preserves the duality symmetry for the dynamical equations. All explicit dependence on the coordinates disappears. The sources in (2.34) and (2.35) become, respectively, $S^{\alpha\beta\nu}$ and $L^2\widetilde{\theta}^{\alpha\nu}$, where $\widetilde{\theta}^{\alpha\nu}$ is the new energy momentum obtained when the new covariant derivatives are used in the source Lagrangians.

The reparametrization brings forth a problem in the characterization of the vacuum. Before the change of parameters, the vacuum is given by a gauge transformation of vanishing fields, $\mathring{B}^\alpha{}_\mu=-(\Lambda^{-1})^\alpha{}_\gamma\partial_\mu a^\gamma$ or, for infinitesimal transformations, $\mathring{B}^\alpha{}_\mu=-\partial_\mu\delta a^\alpha$. This should not change by a reparametrization, but (2.44) tells us that the gauge transformation of $B^\alpha{}_\mu=0$ is now $\mathring{B}^\alpha{}_\mu=-(\delta w^\alpha{}_\gamma\partial_\mu x^\gamma+\partial_\mu\delta x^\alpha)$ which gives $\mathring{\tau}^\alpha{}_{\mu\nu}\neq 0$. In reality, let us recall that, to obtain (2.44), we used the fact that $B^\alpha{}_\mu$ should have, in terms of the new parameters, the same transformation properties of $(B^\alpha{}_\mu-A^{\alpha\beta}{}_\mu x_\beta)$ in terms of the old. This is not to say that $B^\alpha{}_\mu$ has been changed to absorb the term $A^{\alpha\beta}{}_\mu x_\beta$, it is simply a way to fix its transformation properties. If we want to recover the vacuum via the transformation rules, we have to add to it the piece we had extracted: $\mathring{A}^{\alpha\gamma}{}_\mu x_\gamma=-(\partial_\mu\delta w^{\alpha\gamma})x_\gamma$. Once this is done, we obtain the same vacuum as before (although written in terms of the new parameters). An interesting consequence of the change of parameters is that the tetrad of the vacuum fields becomes integrable: the absence of the gravitational field is signaled by its holonomy.

The minimum requirement for a candidate theory for gravitation is that, under the due conditions, Newton's law be obtained. Suppose a spinless point source of mass M in the simplest case where $F=0$ and the potential A can be gauged out. The static solution in the frame attached to the source will exhibit rotational symmetry, $B^\alpha=B^\alpha{}_\mu dx^\mu=B^\alpha{}_0(r)dx^0+B^\alpha{}_r(r)dr$, and in this case, $\partial_i B^\alpha{}_j-\partial_j B^\alpha{}_i=0$ for $i,j=1,2,3$. Consequently, $B^\alpha{}_j=\partial_j\varphi^\alpha$ for some φ^α and these components can be gauged out by choosing $a^\alpha=\varphi^\alpha$ in (2.26). Equation (2.35) with source reduces to $\Delta B^{j0}=0$ and $\Delta B^{00}=-l^2M\delta(\mathbf{r})$. If we look for solutions vanishing at some spherical boundary at infinity, we find $B^{j0}=0$ and $B^{00}=l^2M/(4\pi r)=-V(r)$. This shows also that $l=\sqrt{4\pi G}$ is simply related to the Planck length.

III. NON-LAGRANGIAN CHARACTER

Despite the fact that some important equations do not come from a Lagrangian (Navier-Stokes,[22] Burgers, Korteweg—de Vries[20]), there is a widespread belief that the fundamental equations of physics should be related to an extremal principle.[23] As a consequence of Feynman's picture of quantum mechanics, it has even become a matter of common acceptance that the action is, in some sense, more "fundamental" than the equations of motion, not the least because it takes into account the global characteristics of the system. There are difficulties in this point of view,[24] but we shall not discuss this subject. The model above is the (contraction) limit of a nice Lagrangian dS theory. We shall show that, once the limit is taken, it is no longer a Lagrangian theory. Without any pretense to real mathematical rigor, we shall simply state the fundamental Vainberg's theorem involved,[19] suitably adapted to the language of field theory, and show how it works for gauge theories. In particular, it will become evident that a de Sitter model does satisfy the requirements for a Lagrangian theory. These requirements, however, fail to be observed by (2.34) and (2.35). To find that a field equation complies with the Lagrangian conditions is, in general, an easy task. To be sure that it does not is often very difficult. In our case, we shall be able, first, to suspect that the conditions are violated and then, to show indeed that they are ruined by the contraction process.

Suppose that we have an equation

$$\mathscr{D}\varphi(x)=0 , \tag{3.1}$$

where $\mathscr{D}$ is a differential operator and $\varphi(x)$ a field belonging to some functional space. The Fréchet derivative of $\mathscr{D}$ along some field $\eta(x)$ at the point $\varphi(x)$ of the functional space can be calculated by

$$\mathscr{D}'_\varphi\eta=\lim_{\epsilon\to 0}\frac{1}{\epsilon}[\mathscr{D}(\varphi+\epsilon\eta)-\mathscr{D}(\varphi)] = \left[\frac{d}{d\epsilon}\mathscr{D}(\varphi+\epsilon\varphi)\right]_{\epsilon=0} , \tag{3.2}$$

and is itself a linear operator acting on $\eta(x)$. Given this operator $\mathscr{D}'_\varphi$, its adjoint is the operator $\tilde{\mathscr{D}}'_\varphi$ such that

$$\int d^4x\, \lambda(x)\mathscr{D}'_\varphi\eta(x)=\int d^4x\, \eta(x)\tilde{\mathscr{D}}'_\varphi\eta(x) \tag{3.3}$$

for any two fields $\lambda(x)$, $\eta(x)$.

Vainberg's theorem says that[20] the necessary and sufficient condition for (3.1) to come by variation from some action functional is that

$$\mathscr{D}'_\varphi=\tilde{\mathscr{D}}'_\varphi \tag{3.4}$$

in a ball around φ. Such self-adjointness, taken in (3.3), corresponds to a symmetry of the Fréchet derivative along any two directions $\lambda(x)$ and $\eta(x)$ around $\varphi(x)$,

$$\int d^4x\, \lambda(x)\mathscr{D}'_\varphi\eta(x)=\int d^4x\, \eta(x)\mathscr{D}'_\varphi\lambda(x) \tag{3.5}$$

and is reminiscent of the integrability condition of calculus.[25]

Once this symmetry condition is satisfied, the action functional can be obtained as

$$S[\varphi]=\int d^4x\, \varphi(x)\int_0^1 d\alpha\, \mathscr{D}(\alpha\varphi(x)) . \tag{3.6}$$

It is an easy exercise to check the statements above for the simplest cases in field theory: for linear equations, they are rather trivial. For a sourceless gauge field, the Yang-Mills equations state the vanishing of

$$\mathscr{D}A^{a\nu}=(\delta^a{}_c\partial_\mu+f^a{}_{bc}A^b{}_\mu)(\partial^\mu A^{c\nu}-\partial^\nu A^{c\mu}+f^c{}_{de}A^{d\mu}A^{e\nu}) . \tag{3.7}$$

The Fréchet derivative of $\mathscr{D}$ is

$$(\mathscr{D}'_A\Gamma)^{a\nu}=\left[\frac{d}{d\epsilon}\mathscr{D}(A^{a\nu}+\epsilon\Gamma^{a\nu})\right]_{\epsilon=0} =(DD\Gamma)^{a\nu}+f^a{}_{bc}\Gamma^b{}_\mu F^{c\mu\nu} , \tag{3.8}$$

where D is the covariant derivative fitted to each case:

$$(D\Gamma)^{a\mu\nu}=\partial^\mu\Gamma^{a\nu}-\partial^\nu\Gamma^{a\mu}+f^a{}_{bc}(A^{b\mu}\Gamma^{c\nu}-A^{b\nu}\Gamma^{c\mu}) , \tag{3.9}$$

$$(D\psi)^{a\nu}=(\delta^a{}_c\partial_\mu+f^a{}_{bc}A^b{}_\mu)\psi^{c\mu\nu} \tag{3.10}$$

for a field $\psi^{c\mu\nu}$ $(=-\psi^{c\nu\mu})$ in the adjoint representation. Now, for any such ψ and any $\varphi_{a\nu}$,

$$\int d^4x\, \varphi_{a\nu}(D\psi)^{a\nu}=-\tfrac{1}{2}\int d^4x\, \psi^{a\mu\nu}(D\varphi)_{a\mu\nu} . \tag{3.11}$$

This can be found by using (3.10), performing an integration by parts and antisymmetrizing to obtain the covariant derivative $D\varphi$, which has the form (3.9). It follows that

$$\int d^4x\, \varphi_{a\nu}[D(D\Gamma)]^{a\nu}=-\tfrac{1}{2}\int d^4x\, (D\Gamma)^{a\mu\nu}(D\varphi)_{a\mu\nu} . \tag{3.12}$$

This would be enough to show the symmetry of the first term in (3.8) but we can go a step further. We reverse the roles by setting $\psi_{a\mu\nu}=[D\varphi]_{a\mu\nu}$ and using again (3.11), arriving at

$$\int d^4x\, \varphi_{a\nu}[D(D\Gamma)]^{a\nu}=\int d^4x\, \Gamma_{a\nu}[D(D\varphi)]^{a\nu} . \tag{3.13}$$

The condition for the existence of a Lagrangian for the equation $\mathscr{D}A^{a\nu}=0$ is

$$\int d^4x\, \varphi_{a\nu}[(DD\Gamma)^{a\nu}+f^a{}_{bc}\Gamma^b{}_\mu F^{c\mu\nu}] =\int d^4x\, \Gamma_{a\nu}[(DD\varphi)^{a\nu}+f^a{}_{bc}\varphi^b{}_\mu F^{c\mu\nu}] . \tag{3.14}$$

That the first terms on each side are equal is guaranteed by (3.13). For the remaining terms, it is enough to exchange the indices and use the cyclic property of the structure constants to show that

$$\int d^4x\, \varphi_{a\nu}f^a{}_{bc}\Gamma^b{}_\mu F^{c\mu\nu}=\int d^4x\, \Gamma_{a\nu}f^a{}_{bc}\varphi^b{}_\mu F^{c\mu\nu} . \tag{3.15}$$

So, the requirements are more than satisfied, as the symmetry conditions (3.13) and (3.15) hold separately. Note that, in gauge theories, it is the summation on the components that makes the symmetrization possible. The Yang-Mills Lagrangian is obtained from (3.6) in the form

$$\mathscr{L}=\tfrac{1}{2}A_{a\nu}(\delta^a{}_c\partial_\mu+f^a{}_{bc}A^b{}_\mu)F^{c\mu\nu} . \tag{3.16}$$

Let us now consider the Yang-Mills equations for the P group (2.34) and (2.35). Applied to (2.34) alone, the above treatment would lead to the existence of a good Lagrangian like (3.16), still a manifestation of the fact that the Lorentz sector constitutes a gauge theory by itself. The problem concerns the whole set of equations. Consider the Fréchet derivative of the differential operator in (2.35) along $\Gamma=(\Gamma^{\alpha}{}_{\beta\mu},\eta^{\gamma}{}_{\nu}=L\Gamma^{\gamma 5}{}_{\nu})$ at the point $(A^{\alpha}{}_{\beta\mu},B^{\gamma}{}_{\nu})$ in the functional space:

$$\mathscr{D}'^{\alpha\nu}_{A,B}[\Gamma,\eta]=-\Gamma^{\alpha}{}_{\epsilon\mu}\tau^{\epsilon\mu\nu}+(\delta^{\alpha}{}_{\epsilon}\partial_{\mu}-A^{\alpha}{}_{\epsilon\mu})[(D\eta)^{\epsilon\mu\nu}-(\Gamma^{\epsilon}{}_{\gamma}{}^{\mu}B^{\gamma\nu}-\Gamma^{\epsilon}{}_{\gamma}{}^{\nu}B^{\gamma\mu})]$$
$$+[(D\Gamma)^{\alpha}{}_{\epsilon}{}^{\mu\nu}-(\Gamma^{\alpha}{}_{\gamma}{}^{\mu}A^{\gamma}{}_{\epsilon}{}^{\nu}-\Gamma^{\alpha}{}_{\gamma}{}^{\nu}A^{\gamma}{}_{\epsilon}{}^{\mu})]B^{\epsilon}{}_{\mu}+F^{\alpha}{}_{\epsilon}{}^{\mu\nu}\eta^{\epsilon}{}_{\mu}\ . \quad (3.17)$$

For (2.34), $\mathscr{D}'^{\alpha\beta\nu}{}_{A,B}[\Gamma,\eta]$ is of the form (3.8), the only differences coming from our use of double indices for the Lie-algebra components. We take then another direction in the functional space, say, $\varphi=(\varphi^{\alpha}{}_{\beta\mu},w^{\gamma}{}_{\nu})$ and check to see whether or not

$$\int d^4x\, \varphi_{\alpha\beta\nu}\mathscr{D}'^{\alpha\beta\nu}[\Gamma,\eta]+\int d^4x\, w_{\gamma\nu}\mathscr{D}'^{\gamma\nu}[\Gamma,\eta]$$
$$=\int d^4x\, \Gamma_{\alpha\beta\nu}\mathscr{D}'^{\alpha\beta\nu}[\varphi,w]+\int d^4x\, \eta_{\gamma\nu}\mathscr{D}'^{\gamma\nu}[\varphi,w]\ . \quad (3.18)$$

As expected, the Lorentz sector alone satisfies the symmetry condition. Neither η nor w really appear in the first terms in each side in (3.18). These terms exactly cancel each other, and we have to verify if the second terms, which come from the translational sector, coincide or not. We find that (i) some pieces do allow for symmetrization, such as the last term in (3.17), which contributes with

$$w_{\alpha\nu}F^{\alpha}{}_{\epsilon}{}^{\mu\nu}\eta^{\epsilon}{}_{\mu} \quad (3.19)$$

to the left-hand side, and (ii) some other pieces are not symmetrizable. It is always very difficult to be sure that a certain term is not somehow canceled or symmetrized by some other. The first term in (3.17) is a good suspect: $w_{\gamma\nu}\Gamma^{\alpha}{}_{\epsilon\mu}\tau^{\epsilon\mu\nu}$ is not symmetrical by itself. The best way to see that it is not symmetrized by any other is to go back to the dS theory and trace what happens during the contraction process. Let us write (3.15) for the dS case: the left-hand side will be

$$\int d^4x\big(\tfrac{1}{8}\varphi_{\alpha\beta\nu}f^{\alpha\beta}{}_{\gamma\delta,\epsilon\varphi}\Gamma^{\gamma\delta}{}_{\mu}F^{\epsilon\varphi\mu\nu}+\tfrac{1}{2}\varphi_{\alpha\beta\nu}f^{\alpha\beta}{}_{\gamma5,\epsilon5}\Gamma^{\gamma5}{}_{\mu}F^{\epsilon5\mu\nu}$$
$$+\tfrac{1}{2}\varphi_{\alpha5\nu}f^{\alpha5}{}_{\gamma\delta,\epsilon5}\Gamma^{\gamma\delta}{}_{\mu}F^{\epsilon5\mu\nu}$$
$$-\tfrac{1}{2}\varphi_{\alpha5\nu}f^{\alpha5}{}_{\gamma\delta,\epsilon5}F^{\gamma\delta\mu\nu}\Gamma^{\epsilon5}{}_{\mu}\big)\ . \quad (3.20)$$

The f's are the structure constants for the dS group, written in a hopefully clear antisymmetric-double-index notation, and the numerical factors account for double counting. The first term above is obviously $\varphi\leftrightarrow\Gamma$ symmetrical; it is a contribution related only to the Lorentz sector. Also the last term is symmetrical: by contraction, with $L\varphi_{\alpha5\nu}=w_{\alpha\nu}$, $L\Gamma^{\epsilon5}{}_{\mu}=\eta^{\epsilon}{}_{\mu}$ it gives (3.19), related to the last term in (3.17). Now, the second and the third terms are not, each one, symmetrical: they are "symmetrizing companions:" they symmetrize each other when we substitute φ for Γ and vice versa. The third one is precisely that giving by contraction our suspect first term in (3.17), once multiplied by $w_{\alpha\nu}$. So, the suspect term would be symmetrized by the term coming from the second term in (3.20). That is where the asymmetry comes from: there is no such term. If we examine the equations in detail we see that the symmetrizing second term in (3.20) comes from the Fréchet derivative of the last two terms in (2.32). We had called attention to the fact that, in the contraction process leading to (2.32), these terms vanish. Summing up: the term, present in the dS theory, which symmetrizes the first term in (3.17), disappears during the contraction process. In this way we can pinpoint how contraction spoils the symmetry necessary for the theory to be Lagrangian: some terms in the field equations disappear and terms like the first one in (3.17) no longer find a symmetrizing companion in the contracted theory.[26] We could still think that some miracle might occur: we have been analyzing terms of (3.17) which correspond to (3.15); there are other nonderivative terms, corresponding to (3.13), which could eventually symmetrize or compensate just the above "offending" terms. That this is not the case may be verified by a direct term by term comparison.

Consequently, there is no Lagrangian leading to the dynamical equations (2.34) and (2.35). The argument above can be line-by-line adapted to equations (1.7) and (1.8), with the same result.

IV. FINAL COMMENTS

The considerations presiding over the elaboration of the model, however valid, are no guarantee that it does describe gravitation. Comparison of a large number of solutions with experiment only can do that. The model gives Newton's law in the appropriate conditions but other solutions should be examined. Being as gaugelike as possible, it has better chances to exhibit good short-distance behavior than most models, except possibly those of the supersymmetric type. Supersymmetry must anyhow be broken at present-day attainable energies, leading to some simpler theory, and the Poincaré model is a good candidate for that. Being already conformally invariant, its supersymmetric extension would be worth considering as a possible alternative to conformal gravity[27] models. The quantization of the model is still under study. Its eventual failure with respect to renormalization would signal the interest of such an extension. In the absence of a Lagrangian, quantization can be approached in two ways: directly from the field equations by the Källén-Yang-Feldman[28] method, and indirectly by the path-integral procedure using the dS model as an intermediate step. Both have been pursued with reasonable success, but the question of the renormalizability is as yet unsettled.

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[1]Larger symmetries, as the conformal group, could be considered but they are not classifying for massive particles and only the Poincaré group allows for the usual causality properties in Minkowski space. See A. J. Briginshaw, Int. J. Theor. Phys. **19**, 899 (1980).

[2]See, for instance, G. Basombrio, Gen. Relativ. Gravit. **12**, 109 (1980).

[3]M. O. Katanaev, Teor. Mat. Fiz. **54**, 381 (1983) [Theor. Math. Phys. **54**, 248 (1983)].

[4]D. Ivanenko and G. Sardanashvily, Phys. Rep. **94**, 1 (1983).

[5]F. W. Hehl, Found. Phys. **15**, 451 (1985).

[6]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963), Vol. I.

[7]R. Aldrovandi and E. Stedile, Int. J. Theor. Phys. **23**, 301 (1984).

[8]In reality, $\bar{\Gamma}$ can be defined in many different ways, with S being horizontal forms of a more general type (see Ref. 6, Chap. III). The solder form is particularly convenient because, when written in a frame given by the four-leg field $h_\alpha{}^\mu$, its components are just those of the dual basis.

[9]D. A. Popov and L. I. Daikhin, Dok. Akad. Nauk SSSR **225**, 790 (1975) [Sov. Phys. Dokl. **20**, 818 (1975)].

[10]G. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).

[11]C. Fronsdal, Phys. Rev. D **30**, 2081 (1984).

[12]The role of discrete duality in the short-distance behavior of gauge fields is not clear. The continuous duality symmetry has been used [S. Deser *et al.*, Phys. Lett. **58B**, 355 (1975)] to improve renormalizability in the Einstein-Maxwell theory, but it does not exist at a fundamental level in the non-Abelian case [S. Deser and C. Teitelboim, Phys. Rev. D **13**, 1592 (1976)].

[13]T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[14]Minkowski space is a homogeneous space under the action of P, the quotient space P/SO(3,1). There are two dS groups, SO(3,2) and SO(4,1), under the action of which dS spaces, their quotients by the Lorentz group, are homogeneous.

[15]R. Gilmore, *Lie Groups, Lie Algebras and Some of their Applications* (Wiley, New York, 1974).

[16]F. Gürsey, *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Gordon and Breach, New York, 1964), p. 365.

[17]S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. **38**, 739 (1977).

[18]E. Angelopoulos, M. Flato, C. Fronsdal, and D. Sternheimer, Phys. Rev. D **23**, 1278 (1981).

[19]M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators* (Holden-Day, San Francisco, 1964).

[20]R. W. Atherton and G. M. Homsy, Stud. Appl. Math. **54**, 31 (1975).

[21]We prefer, for the time being, to keep separate coordinates $\{x^\alpha\}$ on each tangent space, understanding that they are functions of the corresponding point of space-time. In this way the analogy with the fibers in gauge theories is more evident. The coordinates will be unified when convenient. One could, alternatively, work directly with a local representation of the Lie algebra in terms of the tangent fields, as is done in Ref. 11 for the conformal case.

[22]B. A. Finlayson, Phys. Fluids **15**, 963 (1972).

[23]We are considering here a direct relation, as that found usually in field theory. By conveniently transforming the fields and their derivatives, it is always possible to find some variational principle.

[24]See, for instance, S. Okubo, Phys. Rev. D **22**, 919 (1980), and references therein.

[25]The theorem is valid in terms of the far more general Gateau derivative, which only coincides with the Fréchet derivative when it is linear and meets some requirements of uniform continuity. We shall suppose it to be the case here.

[26]First-order derivative terms are the most frequent sources of Vainberg's symmetry violation. It is one more miracle of gauge theories that each such term does find a symmetrizing companion. We can trace back the offending term in (3.17) to the term $A^\alpha_{\gamma\mu}\tau^{\gamma\mu\nu}$ in (2.35), which plays here the same role as the "spoiling" term $u\times(\nabla\times u)$ in the Navier-Stokes equation.

[27]F. Mansouri, J. Math. Phys. **24**, 890 (1983), and references therein.

[28]G. Källén, Ark. Fys. **2**, 187 (1950); **2**, 371 (1950); C. N. Yang and D. Feldman, Phys. Rev. **79**, 972 (1950).