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Surface geometry of a rotating black hole in a magnetic field

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We study the intrinsic geometry of the surface of a rotating black hole in a uniform magnetic field, using a metric discovered by Ernst and Wild. Rotating black holes are analogous to material rotating bodies according to Smarr since black holes also tend to become more oblate on being spun up. Our study shows that the presence of a strong magnetic field ensures that a black hole actually becomes increasingly prolate on being spun up. Studying the intrinsic geometry of the black-hole surface also gives rise to an interesting embedding problem. Smarr shows that a Kerr black hole cannot be globally isometrically embedded in $\mathbb{R}^3$ if its specific angular momentum a exceeds $(\sqrt{3}/2)m \sim 0.866\ldots m$. We show that in the presence of a magnetic field of strength B, satisfying $2-\sqrt{3} \leq B^2m^2 \leq 2+\sqrt{3}$, a global isometric embedding is possible in $\mathbb{R}^3$ for all values of the angular momentum.

I. INTRODUCTION

Smarr[1] and Wild and Kerns[2] have investigated the surface geometry of the Kerr black hole and the Schwarzschild black hole in a uniform magnetic field, respectively. The question naturally arises of combining the effects of rotation and magnetic fields. In this paper, we examine the surface geometry of a Kerr black hole in a uniform magnetic field. The metric used was obtained by Ernst and Wild[3] and represents an asymptotically nonflat, stationary, axisymmetric, exact solution of the Einstein-Maxwell equations.

The topology of event horizons is generally that of $S^2 \times \mathbb{R}^1$. A spacelike slice of the horizon then has the topology of the compact manifold S^2, and will be referred to as the black hole. In Sec. II we obtain the metric for the Kerr black hole in a uniform magnetic field. A study of the intrinsic geometry of a black hole provides insight into their nature. It is well known that black holes have many of the properties of material bodies. One may ask: do black holes respond to external forces as material bodies do? For instance, Smarr[1] shows that a black hole becomes more oblate on being spun up, just as a material body would. Wild and Kerns[2] show that a uniform magnetic field makes a static black hole prolate. A natural way to characterize departure from spherical symmetry is to measure the equatorial and polar circumferences, quantities that are intrinsic to the black hole. Rotation and the magnetic field combine to give rise to complicated behavior: we find that for large magnetic field strengths, the black hole actually becomes more prolate as it is spun up and that a correct explanation for this behavior is provided by the concept of surface tension of a black hole due to Bekenstein. All this is the subject of Sec. III.

Section IV discusses the Gaussian curvature, an intrinsic invariant of the black hole. The Gaussian curvature K plays an important role in an embedding problem: Smarr[1] shows that if the specific angular momentum a of a Kerr black hole exceeds a critical value $a_{\rm cr} = (\sqrt{3}/2)m$ ($\sim 0.866\ldots m$) then the black hole cannot be globally isometrically embedded in $\mathbb{R}^3$. He also shows that this is a direct consequence of the fact that for $a > a_{\rm cr}$, the Gaussian curvature is negative at the poles. This may be understood by the following argument: a surface in $\mathbb{R}^3$ with negative Gaussian curvature at a point p is saddle shaped in a neighborhood of p; such a surface cannot be axially symmetric about p. But the Kerr black hole is axially symmetric (about the poles) for all values of the angular momentum. It follows that a neighborhood of the poles cannot be isometrically embedded in $\mathbb{R}^3$ when Gaussian curvature is negative at the poles.

Wild and Kerns,[2] on studying the intrinsic geometry of a Schwarzschild black hole of mass m in a uniform magnetic field of strength B, find that Gaussian curvature is negative at the equator when $B > 1/m$. From this, they infer that the black hole cannot be globally isometrically embedded in $\mathbb{R}^3$ if $B > 1/m$. In Sec. V we show that such an inference is not correct. In general, the condition for global isometric embedding is independent of the existence of negative Gaussian curvature. In the presence of axial symmetry, it is negative Gaussian curvature at the *poles* which implies nonembeddability. The study of Gaussian curvature in Sec. IV emphasizes the fact that rotation and magnetic fields counteract each other: the negative curvature produced by rotation around the poles can be made positive by a suitable magnetic field. And this enables us to prove the following result: The Kerr black hole in a uniform magnetic field can be globally isometrically embedded in $\mathbb{R}^3$ for all values of a for a range of values of the magnetic field strength. We conclude with a general discussion in Sec. VI.

II. THE METRIC

The three-dimensional event horizon of the Kerr spacetime is degenerate in character. This degenerate nature is easily seen when the metric is written in terms of an orthonormal basis of one-forms:

$$ds^2 = -(\omega^t)^2 + (\omega^r)^2 + (\omega^\theta)^2 + (\omega^\varphi)^2 , \tag{2.1}$$

where

$$\omega^t \equiv \frac{\Delta^{1/2}\sin\theta}{f^{1/2}} dt \; ,$$
$$\omega^r \equiv \frac{\rho}{\Delta^{1/2}} dr \; ,$$
$$\omega^\theta \equiv \rho\, d\theta \; , \tag{2.2}$$
$$\omega^\varphi \equiv f^{1/2}(d\varphi - \omega\, dt) \; ,$$

where f is $g_{\varphi\varphi}$ in Boyer-Lindquist coordinates:

$$f \equiv g_{\varphi\varphi} = \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)^2 - \Delta a^2 \sin^2\theta] \; ,$$
$$\Delta \equiv r^2 - 2mr + a^2 \; , \quad \rho^2 \equiv r^2 + a^2\cos^2\theta \; , \tag{2.3}$$
$$\omega \equiv \frac{2mar}{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta} \; .$$

On the submanifold $r = r_+ \equiv m + (m^2 - a^2)^{1/2} = \text{const}$, the metric reduces to

$$^{(3)}ds^2 = (\omega^\theta)^2 + (\omega^\varphi)^2 \; . \tag{2.4}$$

Taking a $t = \text{const}$ slice of this degenerate three-manifold gives the metric on the black hole:

$$^{(2)}ds^2 = g_{\theta\theta} d\theta^2 + g_{\varphi\varphi} d\varphi^2 \; . \tag{2.5}$$

We are free to take any spacelike slice of the horizon, because the objects of our study are invariant under isometries and because of the following theorem:[1] All two-dimensional spacelike slices of the horizon are isometric.

We now follow the same procedure for the magnetized Kerr metric. Writing the Kerr metric in the form

$$ds^2 = -\frac{\sin^2\theta}{f} dt^2 + \frac{A\sin^2\theta}{f} dr^2 + \frac{A \sin^2\theta}{f} d\theta^2 + f(d\varphi - \omega\, dt)^2 \; , \tag{2.6}$$

where $A \equiv (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$, the prescription to magnetize is as follows:[3] replace f and ω by f' and ω' where

$$f' = |\lambda|^{-2} f \; , \tag{2.7}$$

$$\omega' = |\lambda|^2 \nabla\omega + f^{-1}\Delta^{1/2}\sin\theta(\lambda^*\nabla\lambda - \lambda\nabla\lambda^*) \; . \tag{2.8}$$

Equation (2.8) can be integrated to yield[3]

$$\omega' = \frac{a(1 - B^4 m^2 a^2)}{r_+{}^2 + a^2} + \beta\Delta \; . \tag{2.9}$$

The explicit form of β does not concern us since it does not contribute on the surface $r = r_+$ ($\Delta = 0$). In the above expressions, the field λ is given by

$$\lambda \equiv 1 - \frac{B^2}{4}\epsilon \; ,$$

$$\epsilon \equiv -(r^2+a^2)\sin^2\theta + 2mai\cos\theta(3 - \cos^2\theta) - \frac{2ma^2\sin^4\theta}{r + ia\cos\theta} \; .$$

The parameter B is the strength of the magnetic field and has the dimensions $(\text{mass})^{-1}$. The metric on the black hole is now found to be

$$^{(2)}ds^2 = \rho^2 |\lambda|^2 d\theta^2 + f|\lambda|^{-2} d\varphi^2 \; . \tag{2.10}$$

In a local orthonormal frame,

$$^{(2)}ds^2 = (\omega^\theta)^2 + (\omega^\varphi)^2 \; ,$$
$$\omega^\theta \equiv \rho |\lambda| d\theta \; , \tag{2.11}$$
$$\omega^\varphi \equiv f^{1/2} |\lambda|^{-1} d\varphi \; .$$

In what follows, we will use the metric (2.11) [or (2.10)]. Note that the volume form on the black hole is

$$\Omega = \omega^\theta \wedge \omega^\varphi = \rho f^{1/2} d\theta \wedge d\varphi \; . \tag{2.12}$$

III. CIRCUMFERENCES

A natural way to study the effect of rotation on black holes is to spin it up by injecting particles with nonzero angular momentum. There are two classes of injection processes: those which do not change the irreducible mass of the black hole and those which do; the former are the reversible transformations, the latter, irreversible. When a particle with energy δm and angular momentum δJ is injected into a Kerr black hole, it must necessarily satisfy the inequality

$$\delta m \ge \omega_H \delta J \; , \tag{3.1}$$

$$\omega_H = \frac{a}{r_+{}^2 + a^2} \; , \tag{3.2}$$

where ω_H is the (constant) angular velocity of the horizon. Equality in (3.1) corresponds to making reversible transformations. A measure of the oblateness/prolateness of the black hole is provided by the equatorial and polar circumferences c_e and c_p. The dimensionless number $\delta = (c_e - c_p)/c_e$ determines the degree of oblateness (or prolateness). Smarr[1] shows that a Kerr black hole which is spun up by reversible transformations becomes oblate, just as an ordinary fluid body would. What is the significance of making reversible transformations? There are two factors which change the circumferences of the black hole: the mass (energy) of the particle thrown in, and its angular momentum. We wish to study the effect of adding angular momentum alone—but the relation (3.1) sets a nonzero lower bound on the mass added. To keep this extraneous addition of mass to a minimum, we choose equality in (3.1).

Similarly, to study the effect of spinning up the Kerr black hole in a uniform magnetic field, it would be desirable to make reversible transformations—but there is a problem. A reversible transformation is by definition one which holds fixed the irreducible mass of the black hole. But what is the irreducible mass of the Kerr black hole in a magnetic field? If we think of the surface area as a measure of the irreducible mass, we are led to the conclusion that the irreducible mass is independent of the magnetic field, since the surface area of the magnetized Kerr black hole is independent of the magnetic field, being given by

$$A = 4\pi(r_+{}^2 + a^2) \; . \tag{3.3}$$

This follows trivially from Eq. (2.12). The condition $\delta A = 0$ now gives

$$\delta m = \omega_H \delta J \ . \tag{3.4}$$

where ω_H is given by Eq. (3.2). But this is not the law for reversible transformations that one would expect: The (constant) angular velocity of the horizon of the magnetized black hole is [Eq. (2.9)]

$$\omega'_H = \frac{a(1-B^4J^2)}{r_+{}^2 + a^2} \tag{3.5}$$

and from considerations of particle energetics or otherwise, one would expect the condition for a reversible injection to be

$$\delta m = \omega'_H \delta J \ . \tag{3.6}$$

Note that injections satisfying Eq. (3.6) will decrease the surface area of the black hole since $\omega'_H < \omega_H$ for nonzero B. The source of the problem appears to be the fact that the magnetized Kerr spacetime is not asymptotically flat and empty, and consequently the usual black-hole theorems may not apply to it.

To avoid the problem of making reversible transformations, we will simply make transformations which keep the surface area constant. To begin, consider a family of Kerr black holes in a magnetic field, each black hole having the same surface area. Inverting Eq. (3.3) provides a relation between the mass and angular momentum of any black hole in this family:

$$m^2 = \frac{A}{16\pi} + \left[\frac{4\pi}{A}\right] J^2 \ . \tag{3.7}$$

Since $J \le m^2$, it immediately follows that the mass and angular momentum of a black hole are bounded:

$$\left[\frac{A}{16\pi}\right]^{1/2} \le m \le \sqrt{2}\left[\frac{A}{16\pi}\right]^{1/2} ,$$
$$0 \le J \le 2\left[\frac{A}{16\pi}\right] . \tag{3.8}$$

The equatorial and polar circumferences are given by

$$c_e = \int \omega^\varphi \quad \text{for } \theta = 0$$
$$= \frac{4\pi m}{1+B^2m^2} ,$$
$$c_p = 2\int \omega^\theta .$$

The latter integral is elliptic and must be integrated numerically.

The dependence of c_e and c_p on the magnetic field strength B is straightforward: c_e decreases and c_p increases as B increases. The net effect is to make the black hole more prolate. Figure 1 shows a graph of δ against B^2 which makes this explicit.

The variation of c_e and c_p with angular momentum is more complicated. Consider two black holes $(m_1,J_1),(m_2,J_2)$ with the same surface area and let $J_1 > J_2$. The black hole (m_1,J_1) can be thought of as being spun up from (m_2,J_2) using Eq. (3.4). Also, Eq. (3.7) implies $m_1 > m_2$. From Eq. (3.9) it now follows that

$$\text{(i) } c_{e1} > c_{e2} \text{ if } B^2 < \frac{1}{m_1m_2} ,$$

$$\text{(ii) } c_{e1} < c_{e2} \text{ if } B^2 > \frac{1}{m_1m_2} ,$$

where c_{ei} refers to the black hole (m_i,J_i). Thus, the effect of spin upon the equatorial circumference depends on the strength of the magnetic field. We find [Eq. (3.8)] for spin up that (i) c_e increases monotonically if $B^2 < \frac{1}{2}(16\pi/A)$, (ii) c_e decreases monotonically if $B^2 > 16\pi/A$, and (iii) if $\frac{1}{2} \le B^2(A/16\pi) \le 1$, c_e possesses a maximum. In case (iii), the maximum occurs at that m where $B^2 = 1/m^2$. Figure 2 illustrates this behavior. For the polar circumference we find numerically (Fig. 3) that (i) c_e decreases monotonically if $B^2 \le (0.25)(16\pi/A)$, (ii) c_p increases monotonically if $B^2 > (0.35)\,(16\pi/A)$, (iii) if $0.25 < B^2(A/16\pi) < 0.35$, c_p has a minimum.

These results have the following consequences for the departure from spherical symmetry. Figure 4 shows a graph of δ against J for various values of B^2. A positive slope indicates evolution into an increasingly oblate configuration and negative slope implies a decrease in oblateness (increase in prolateness). It will be observed that for

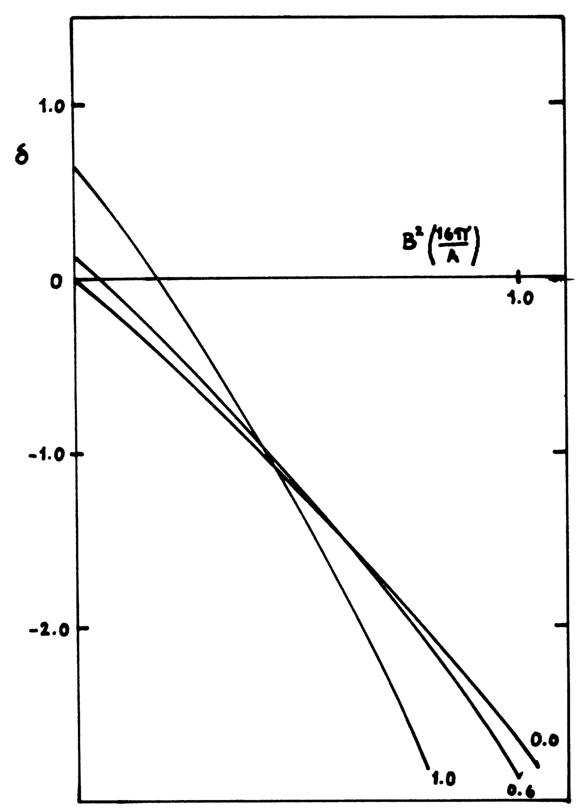

FIG. 1. A plot of δ as a function of B^2. The negative slope indicates increasingly prolate configurations. The point where each curve cuts the B axis corresponds to values of B and J which make c_e equal to c_p. The curves are labeled by a value of J.

magnetic field strengths satisfying $B^2A/16\pi<0.3$, the black hole tends to become more oblate on being spun up, just as an ordinary fluid body would. For $B^2(A/16\pi)\sim 0.4$, the black hole initially does become more oblate, but for large J it is actually less oblate. For larger values of B^2, the situation is completely reversed—the black hole becomes more prolate on being spun up. We are not aware of any classical analogs of such behavior.

The concept of surface tension of a black hole due to Bekenstein[4] provides a natural explanation for the tendency of the black hole to become more prolate on being spun up in the presence of a strong magnetic field. Bekenstein identifies the surface tension of a black hole with its surface gravity. The surface tension is thus inversely proportional to the mass and angular momentum of the black hole. It is well known classically that lowering the surface tension makes it easier to distort a body, the most familiar example being soap bubbles—the larger a soap bubble, the easier it is to distort. This is precisely the effect seen in Fig. 4. As J increases surface tension decreases to a point where the magnetic field can easily distort it—and the magnetic field always distorts a configuration into a more prolate shape. Increasing J beyond this point only lowers the surface tension further; the magnetic field consequently is much more effective, and increasingly prolate configurations are obtained. The role of surface tension for the $J=0$ case is noted by Wild and Kerns.[2]

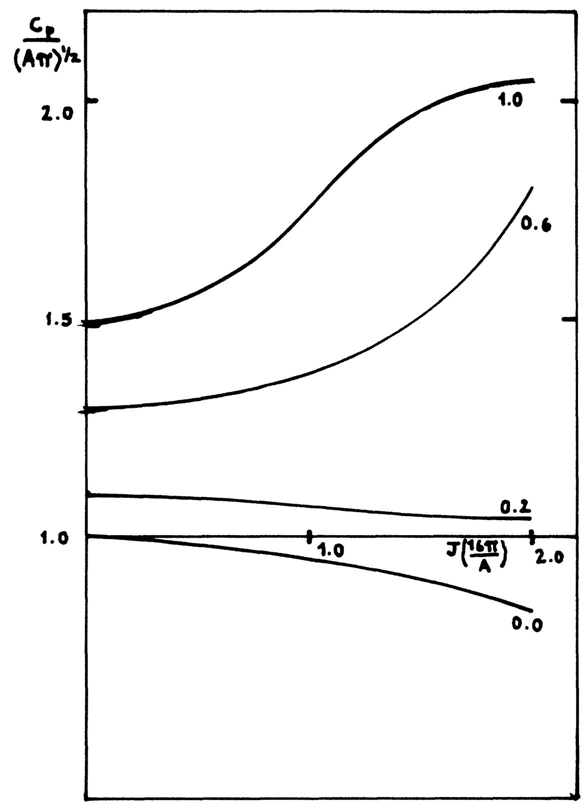

FIG. 3. A plot of c_p as a function of J. Labels are values of B^2 in units of $16\pi/A$.

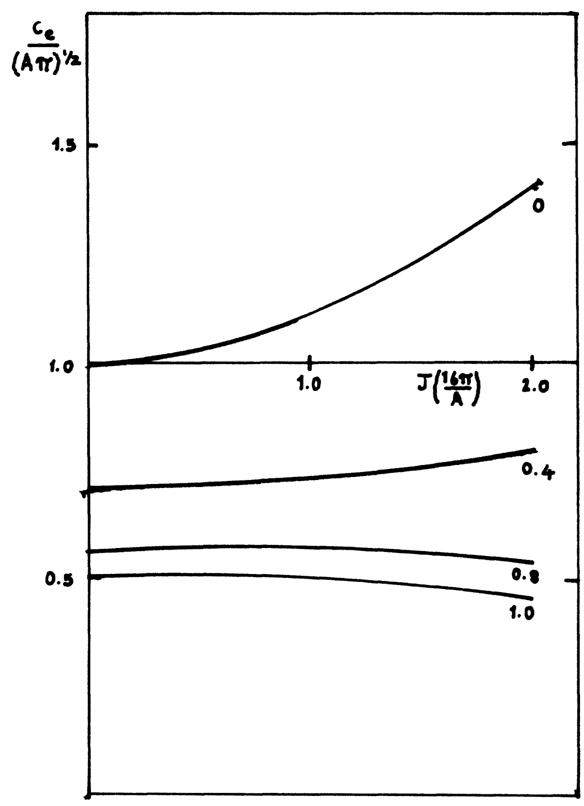

FIG. 2. A plot of c_e as a function of J. Labels are values of B^2 in units of $16\pi/A$.

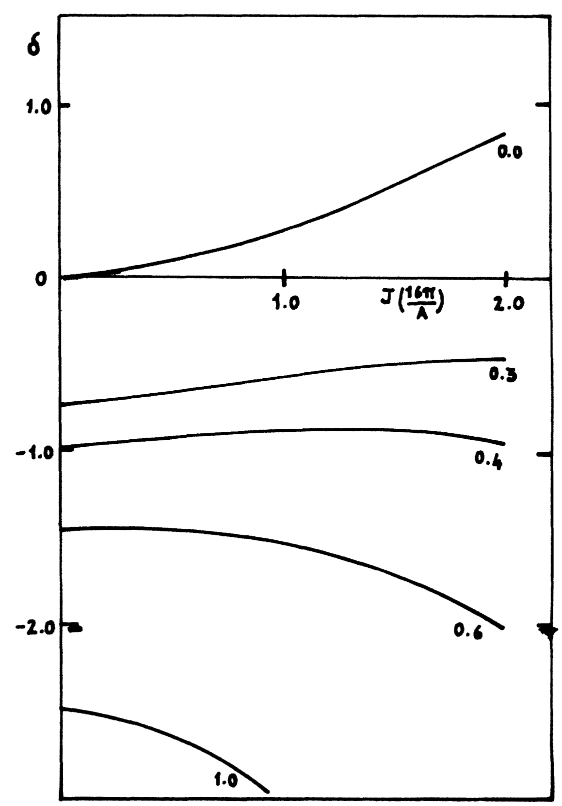

FIG. 4. A plot of δ as a function of J. Labels are values of B^2 in units of $16\pi/A$. Positive (negative) slope indicates increasingly oblate (prolate) configurations.

It will be noticed that for small values of B^2, rotation and magnetic field have precisely opposite effects on the circumferences. Is it possible to choose values of the parameters B and J so that $c_e = c_p$? Figure 1 shows that this is indeed possible. For any given value of J, these exists a value of B such that $\delta=0$—all the curves in Fig. 1 cut the B^2 axis. This does not mean that the black hole is spherically symmetric for such values of B and J. Calculation of the Gaussian curvature K for such configurations shows that K is always a function of θ—a spherically symmetric configuration would have constant Gaussian curvature. A check reveals that spherical symmetry is incompatible with the presence of rotation and magnetic fields. A similar result is true in the classical domain for rotating, conducting fluid balls in a magnetic field.[5]

IV. THE GAUSSIAN CURVATURE

It is convenient to rewrite the metric (2.10) as

$$\begin{aligned} {}^{(2)}ds^2 &= E^2 d\theta^2 + G^2 d\varphi^2 , \\ E &\equiv \rho\,|\lambda| , \\ G &\equiv f^{1/2}\,|\lambda|^{-1} . \end{aligned} \tag{4.1}$$

The Gaussian curvature is given by[6]

$$K(\theta) = -\frac{1}{2EG}\frac{d}{d\theta}\left[\frac{1}{EG}\frac{d}{d\theta}(G^2)\right] . \tag{4.2}$$

We first examine the behavior of the Gaussian curvature at the poles and equator. At the equator ($\theta=\pi/2$)

$$K(\pi/2) = \frac{1}{2{r_+}^4|\lambda|^4}\{2{r_+}^2+2a^2+B^4[\tfrac{1}{8}({r_+}^2+a^2)^2({r_+}^2-a^2)-ma^2r_+({r_+}^2+a^2)+\tfrac{3}{2}m^2a^2(3{r_+}^2+a^2)]\} . \tag{4.3}$$

For the curvature to change sign, K must assume the value zero for some a and B. It is easy to see that $K(\pi/2)$ is strictly positive and never zero unless $a<0.68\ldots m$. For $a>0.68\ldots m$, the Gaussian curvature is strictly positive on the equator for all values of B.

At the poles ($\theta=0$)

$$K(0) = \frac{1}{\rho^4|\lambda|^2}m^2a^2({r_+}^2-3a^2)B^4+2\rho 4B^2+({r_+}^2-3a^2). \tag{4.4}$$

Clearly, if ${r_+}^2-3a^2>0$, this has no zeros. We conclude that for $a<(\sqrt{3}/2)m$ [$a=(\sqrt{3}/2)m \Longleftrightarrow {r_+}^2=3a^2$] curvature at the poles is always positive for all B (this is the value that Smarr obtains for the $B=0$ case). If ${r_+}^2<3a^2$, note that $K(0)$ can have two positive real roots. Thus for $a>0.866\ldots m$, there are two values of the magnetic field which give zero curvature at the poles; one of these is less than $1/m$, the other greater. We also find that for $2-\sqrt{3}<B^2m^2<2+\sqrt{3}$, $K(0)$ has no zeros. As a matter of fact, for such a range of values of B^2, a numerical check shows that the Gaussian curvature is strictly positive all over the black hole.

To understand these features of the Gaussian curvature, it is instructive to examine the limits $a=0$ and $B=0$. If $a=0$, there is negative Gaussian curvature at the equator for $B>1/m$ and this region of negative curvature can spread over the entire surface except the poles as B is increased. (The entire surface cannot have negative Gaussian curvature.) If $B=0$, there is negative Gaussian curvature in a neighborhood of the poles for $a>(\sqrt{3}/2)m$. The maximum extent of negative curvature is for $a=m$, when the "cap" $0\le\theta\lesssim 54°$ has negative curvature at each point. Negative Gaussian curvature due to rotation cannot spread beyond $\theta\sim 54°$ (we are considering only the "upper hemisphere," the situation for the "lower hemisphere" being symmetric). Our results for the combined effect of rotation and magnetic field may now be understood as follows: The magnetic field produces negative Gaussian curvature near the equator, and rapid rotation ($a>0.68\ldots m$) can make this positive. (See Fig. 5.) Next, we consider an example to illustrate the situation at the poles. Let $a=m$ and $B^2=0$ initially. (See Fig. 6.) The Gaussian curvature K at $\theta=0$ then is -0.5. As B^2 is increased to $(2-\sqrt{3})/m^2\sim 0.26/m^2$, $K(0)$ is zero. To summarize, rapid rotation produces negative Gaussian curvature at the poles, and it is always possible to make the curvature positive with an appropriate magnetic field. But increasing B further produces negative Gaussian curvature again, in this instance for $B^2>(2+\sqrt{3})/m^2\sim 3.73/m^2$. It is clear from Fig. 6 that Gaussian curvature at the poles is always positive when B^2 satisfies $2-\sqrt{3}<B^2m^2<2+\sqrt{3}$, a result that is important for the next section.

The counteracting effects of rotation and the magnetic field near the poles and equator follow as a consequence of the Gauss-Bonnet theorem which states that for a compact two-manifold

$$\int K\Omega = 2\pi\chi , \tag{4.5}$$

where Ω is the volume form [Eq. (2.12)] and χ is the Euler characteristic of the black hole. A computation reveals that $\chi=2$ (the black hole has the topology of S^2). If such a compact two-manifold were continuously deformed so as to produce negative Gaussian curvature at one point, the Gaussian curvature at some other point must increase since the right-hand side of Eq. (4.5) is constant.

V. THE EMBEDDING PROBLEM

For an axially symmetric black hole, it is negative Gaussian curvature at the poles that is incompatible with an isometric embedding in $\mathbb{R}^3$. The argument given in the Introduction is obviously not valid for a neighborhood of the equator. Indeed, we show below that the black hole considered by Wild and Kerns[2] can be globally isometrically embedded in $\mathbb{R}^3$ for all values of the magnetic field strength.

For $a=0$, the case considered by Wild and Kerns, the metric (2.10) simplifies to

$$\begin{aligned} {}^{(2)}ds^2 &= r_+{}^2 |\lambda|^2 d\theta^2 + r_+{}^2 \sin^2\theta |\lambda|^{-2} d\varphi^2 , \\ |\lambda| &\equiv 1 + B^2 m^2 \sin^2\theta , \\ r_+ &\equiv 2m . \end{aligned} \tag{5.1}$$

It is convenient to rewrite the metric (5.1) by defining a new coordinate $\mu \equiv \cos\theta$ and a function $h(\mu) \equiv (1-\mu^2)/\lambda^2$. Note that μ is single valued since $0 \leq \theta \leq \pi$. The recast metric is

$${}^{(2)}ds^2 = r_+{}^2 \left[\frac{1}{h(\mu)} d\mu^2 + h(\mu) d\varphi^2 \right] . \tag{5.2}$$

In this coordinate system, the Gaussian curvature is simply

$$K(\mu) = -\frac{h''(\mu)}{2r_+{}^2} , \tag{5.3}$$

where a prime denotes differentiation with respect to μ.

A Riemann two-manifold H with metric g_{ik} is said to be isometrically embeddable in $\mathbb{R}^3$ if g_{ik} is the metric that the embedding induces (from $\mathbb{R}^3$) on H. In practice, this is verified as follows: Let $\phi: H \to \mathbb{R}^3$ be an embedding and let u^1, u^2 denote local coordinates on H. The question then is of the solvability of the differential equations[7]

$$\left\langle \frac{\partial\phi}{\partial u^i}, \frac{\partial\phi}{\partial u^k} \right\rangle = g_{ik} , \tag{5.4}$$

where the inner product on the left is the Euclidean inner product of $\mathbb{R}^3$. Since H (the black hole) is axially symmetric [Eq. (5.1)], we may attempt to embed it as a surface of revolution in $\mathbb{R}^3$. There is a standard way of doing this.[8] Define $\phi: H \to \mathbb{R}^3$ by

$$\phi(\mu,\varphi) = (F(\mu)\cos\varphi, F(\mu)\sin\varphi, G(\mu)) ,$$

where F and G are to be determined. Solving Eq. (5.4)

$$F(\mu) = r_+ h^{1/2} , \tag{5.5}$$

$$G(\mu) = r_+ \int_0^\mu h^{-1/2} \left[1 - \frac{h'^2}{4} \right] d\mu . \tag{5.6}$$

But G is the z coordinate in $\mathbb{R}^3$ and must be real and so Eq. (5.6) demands that

$$|h'| \leq 2 . \tag{5.7}$$

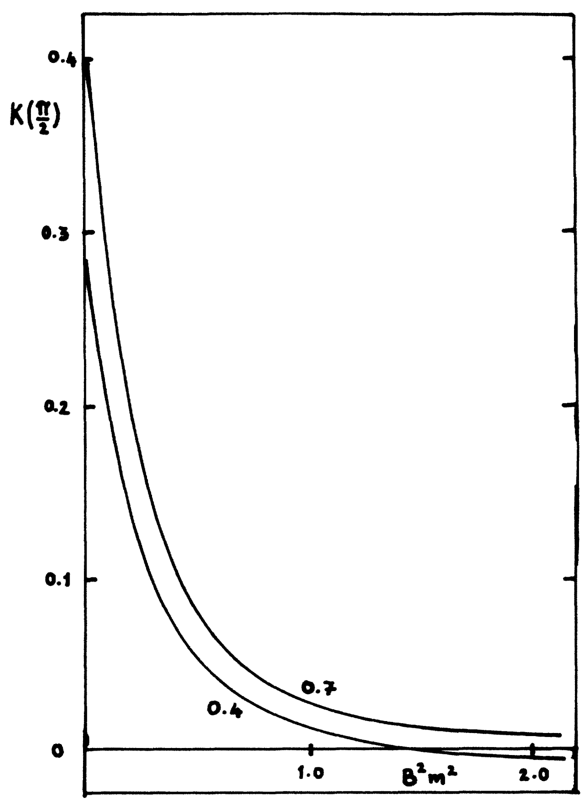

FIG. 5. Gaussian curvature at the equator. Labels are values of the specific angular momentum a. For $a > 0.68\ldots$, the Gaussian curvature at the equator is always positive.

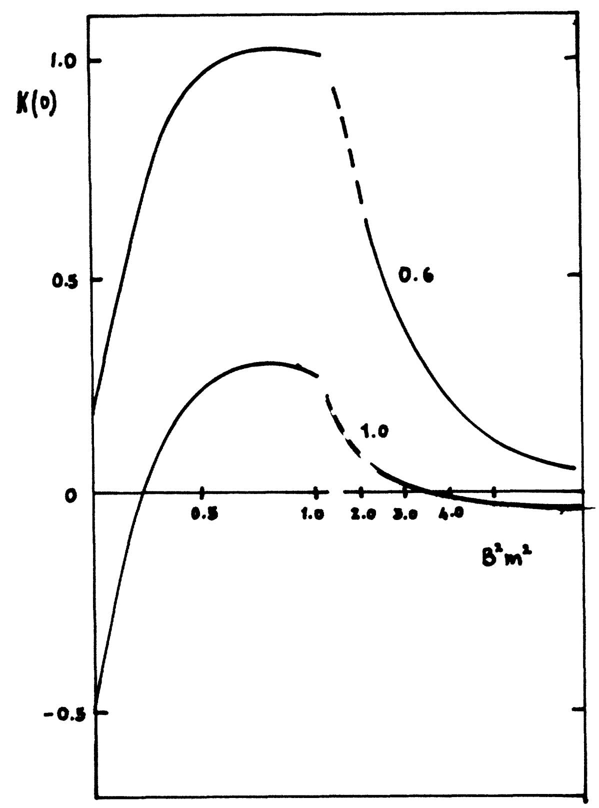

FIG. 6. Gaussian curvature at the poles. Labels indicate the value of the specific angular momentum a in units of m. For $2-\sqrt{3} \leq B^2m^2 \leq 2+\sqrt{3}$, the Gaussian curvature is always positive.

This is the condition for global isometric embedding. But it is easy to check that the condition (5.7) is true for all μ and B. Figure 7 pictorially shows the behavior of $h'(\mu)$. The portion of the curves with positive slope indicates the existence of negative Gaussian curvature by virtue of Eq. (5.3). Thus, we have shown that the Schwarzschild black hole in a uniform magnetic field can be globally isometrically embedded in $\mathbb{R}^3$ for all values of the magnetic field strength.[9]

We now turn to Smarr's result concerning the nonembeddability of the Kerr black hole in $\mathbb{R}^3$. In the previous section, we saw that for a Kerr black hole in a magnetic field there cannot be negative Gaussian curvature at the poles for any J if $2-\sqrt{3}<B^2m^2<2+\sqrt{3}$. Again, using the reasoning given in the Introduction, we would expect such black holes to be globally isometrically embeddable in $\mathbb{R}^3$; and solving Eq. (5.4) for the metric (2.10) (repeating the procedure above) shows this expectation to be true. We have proved that the Kerr black hole in a uniform magnetic field is globally isometrically embeddable in $\mathbb{R}^3$ for all values of the angular momentum if the magnetic field strength B satisfies $2-\sqrt{3}<B^2m^2<2+\sqrt{3}$. This result naturally concludes the investigations of Smarr and Wild and Kerns.

It is possible to draw embedded diagrams of these black holes by integrating Eq. (5.6) and Figs. 8 and 9 show the results. Figure 8 shows the Wild and Kerns result ($J=0$). (Compare with their Fig. 1.) Note that the black hole is slightly nonconvex at the equator when $B^2=2/m^2$, a consequence of the Gaussian curvature being negative. Figure 9 shows our result. The innermost configuration is the $a=m$ hole (in a $B=1/m$ field). The apparent cusp at the poles is only due a finite-step integration.

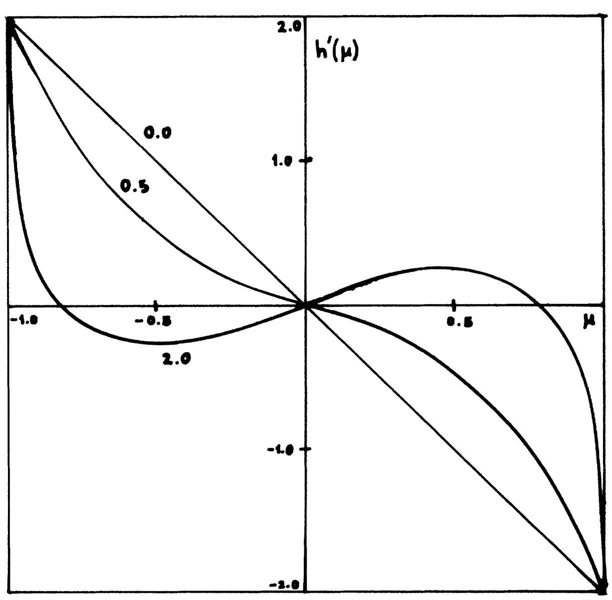

FIG. 7. A schematic to indicate a proof that $|h'|\leq 2$. The figure is not drawn to correct scale. The essential point is that the maximum of h' never exceeds 2. Labels indicate values of B^2. As $B^2\rightarrow\infty$, the maximum moves to the μ axis. Positive slope indicates negative Gaussian curvature.

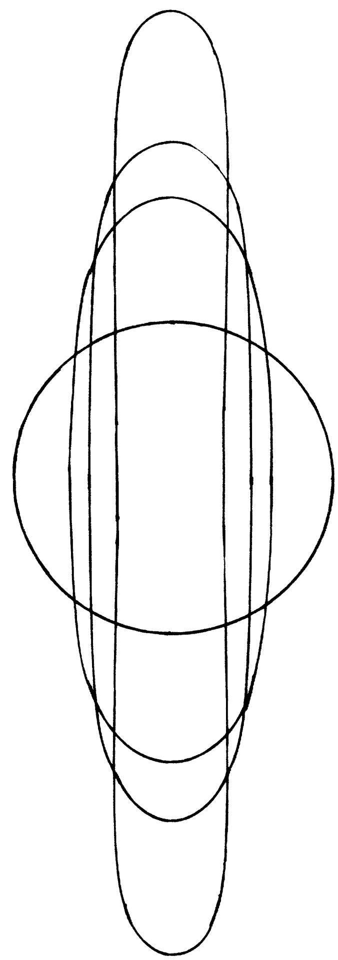

FIG. 8. An embedded diagram of a Schwarzschild black hole in a magnetic field. The spherically symmetric figure is the Schwarzschild black hole ($B=0$). The other figures in order of increasing prolateness are for $B^2=0.6, 1.0, 2.0$. For $B^2=2.0$, the configuration is slightly nonconvex at the equator.

VI. CONCLUSION

The metric given by Ernst and Wild is not very successful in describing magnetic black-hole spacetimes.[3,2] This can mainly be traced to the asymptotic nonflat character of the spacetime. The magnetic field strengths described by the parameter B are rather high. One may use the following formula[10] to compute the magnetic field strength in Gauss. Setting $\beta=Bm$,

$$\beta=8.5\times10^{-9}\left[\frac{m}{m_0}\right]\frac{B}{10^{12}\ \mathrm{G}}$$

for a $10m_0$ black hole, $\beta^2=0.1$ corresponds to a magnetic field of 10^{18} G. For a $10^8 M_\odot$ black hole, a reasonable field of 10^6 G implies $\beta\sim10^{-6}$. Thus, this metric would describe reasonable black-hole situations only when $\beta\ll 1$.

The fact that magnetizing the Kerr spacetime changes the angular velocity of the horizon but does not change the area has curious consequences as noted in Sec. III.

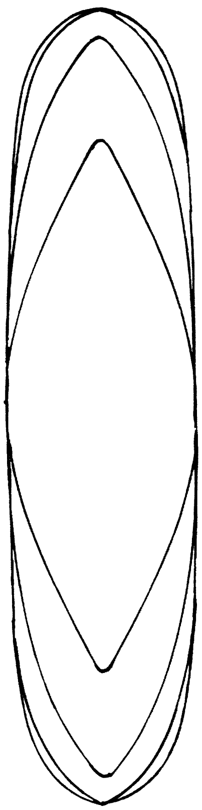

FIG. 9. Embedded diagrams of a Kerr black hole in a magnetic field $B = 1/m$. The outermost figure has $a = 0$. Going inwards, $a = 0.4m, 0.8m, m$. Note that it is possible to embed the $a = m$ black hole because the magnetic field strength B satisfies $2-\sqrt{3} \le Bm \le 2+\sqrt{3}$.

The problem of asymptotic nonflatness can apparently only be circumvented by considering nonstationary solutions.[11] If the quantity

$$\omega'_H = \frac{a(1-B^4J^2)}{r_+{}^2+a^2}$$

is interpreted as the angular velocity of the horizon, one must set an upper bound on the parameter B in the Ernst and Wild metric: $B^2 < 1/J$; otherwise ω'_H is negative. As to black holes becoming prolate on being spun up in the presence of a magnetic field, note that it would take very large field strengths to achieve this, but Fig. 4 shows that such strong fields still satisfy $B^2 < 1/J$. Whether material bodies behave thus remains to be seen. For a fluid body in rotation in a magnetic field, there is a classical result[5] which states that it is possible to achieve a spherically symmetric configuration when the energy of the magnetic field equals the kinetic energy of rotation (the equipartition value). We find numerically that it is possible to achieve configurations that deviate very little from spherical symmetry by suitably choosing B and J. Whether this has any significance as in the classical case is being investigated.

We believe we have successfully demonstrated that axial symmetry and negative Gaussian curvature at the poles are incompatible with a global isometric embedding in $\mathbb{R}^3$. Thus, the presence of negative Gaussian curvature at any point other than the poles does not imply the impossibility of a global isometric embedding. Having observed this, the fact that the magnetic field can remove negative Gaussian curvature at the poles allows us to generalize Smarr's result.

Note added in proof. B. R. Iyer has brought to our notice that Wild, Kerns, and Drish[12] have also considered the problem of the Kerr black hole in a uniform magnetic field. We believe our results to be more general, however. We thank B. R. Iyer for this information.

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[1] L. Smarr, Phys. Rev. D **7**, 289 (1973).
[2] W. J. Wild and R. M. Kerns, Phys. Rev. D **21**, 332 (1980).
[3] F. J. Ernst and W. J. Wild, J. Math. Phys. **17**, 182 (1976).
[4] J. Bekenstein, Bull. Am. Phys. Soc. **17**, 450 (1972).
[5] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Dover, New York, 1981), Sec. 117.
[6] D. Laugwitz, *Differential and Riemannian Geometry*, (Academic, New York, 1965).
[7] Laugwitz, Ref. 6, p. 117.
[8] Laugwitz, Ref. 6, p. 69.
[9] The authors thank Nitin Nitsure for useful discussions on this section.
[10] R. Wald, Phys. Rev. D **10**, 1680 (1974).
[11] J. Ipser, Phys. Rev. Lett. **27**, 529 (1971).
[12] W. J. Wild, R. M. Kerns, and W. F. Drish, Jr., Phys. Rev. D **23**, 829 (1981).