Concavity of the quarkonium potential

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The heavy-quark–antiquark potential is shown to be a monotone nondecreasing and concave function of the separation. This property holds independent of the gauge group and the details of the matter sector.

Some interesting theorems relating the order of energy levels of quarkonium to concavity properties of the heavy-quark potential $V(R)$ were recently derived by Baumgartner, Grosse, and Martin.[1] In particular, these authors showed that if V is a concave function of R^2, then

$$E_{n,l} \leqslant E_{n-1,l+2} \ , \tag{1}$$

where the energy levels are labeled by the angular momentum l and the number of nodes of the wave function n. In this note I would like to point out that the concavity of V as a function of R^α, for all $\alpha \geqslant 1$, is a general property of gauge theories, independent of the choice of gauge group and the details of the matter (light fermion and scalar) sectors.

I should state right at the outset that this paper makes no claim to originality. The concavity of V as a function of R^α means that

$$\frac{d}{d(R^\alpha)}\frac{d}{d(R^\alpha)}V = \frac{R^{2-2\alpha}}{\alpha^2}\frac{d^2V}{dR^2} - \frac{(\alpha-1)}{\alpha^2}R^{1-2\alpha}\frac{dV}{dR} \leqslant 0$$

for all $\alpha \geqslant 1$. This is equivalent to the combined statements

$$\frac{d^2V}{dR^2} \leqslant 0 \ , \tag{2a}$$

$$\frac{dV}{dR} > 0 \ , \tag{2b}$$

i.e., that the quark-antiquark force is everywhere attractive and a monotone nonincreasing function of their separation. That this is indeed so is known to physicists who have worked on rigorous aspects of lattice gauge theories,[2] but given the simplicity of its proof, it has been considered unworthy of particular emphasis. It has thus escaped the attention of a wider audience, which I believe it actually deserves, since (a) combined with the result of Baumgartner *et al.*, it elevates the ordering (1) of the levels of charmonium (ψ), b quarkonium (Υ), t quarkonium, etc., to a most general rigorous theorem of quantum field theory, and (b) it provides a subtle and apparently forgotten consistency check for Monte Carlo simulations of quarkonia. In this note I will thus present the simple derivation of this result.

Let us start with a pure gauge theory on a hypercubic lattice[3] with sites $\mathbf{s}=(s^1,s^2,s^3,s^4) \in Z^4$. The role of the lattice is only technical, and sets the stage for a rigorous proof; the inclusion of dynamical matter will be discussed later. The fields $U(b)$ are as usual defined on the directed bonds $b=(\mathbf{s},\mathbf{s}')$ of the lattice and take values in the gauge group G. A group element can more generally be assigned to every directed path $\omega=(\mathbf{s}_0 \to \mathbf{s}_1 \to \cdots \to \mathbf{s}_f)$ on the lattice:

$$U(\omega)=U(\mathbf{s}_0,\mathbf{s}_1)U(\mathbf{s}_1,\mathbf{s}_2)\cdots U(\mathbf{s}_{f-1},\mathbf{s}_f) \ .$$

Traversing the same path in the opposite direction induces Hermitian conjugation:

$$U(-\omega)=U^\dagger(\omega) \ , \tag{3}$$

with $-\omega=(\mathbf{s}_f \to \mathbf{s}_{f-1} \to \cdots \to \mathbf{s}_1 \to \mathbf{s}_0)$. For the action we take

$$S=\frac{1}{g^2}\sum_{\text{plaquettes } p} \mathrm{Re}\,\mathrm{tr}U(p) \ , \tag{4}$$

with the trace in, say, the fundamental representation, but any other one-plaquette action would do. Finally, the

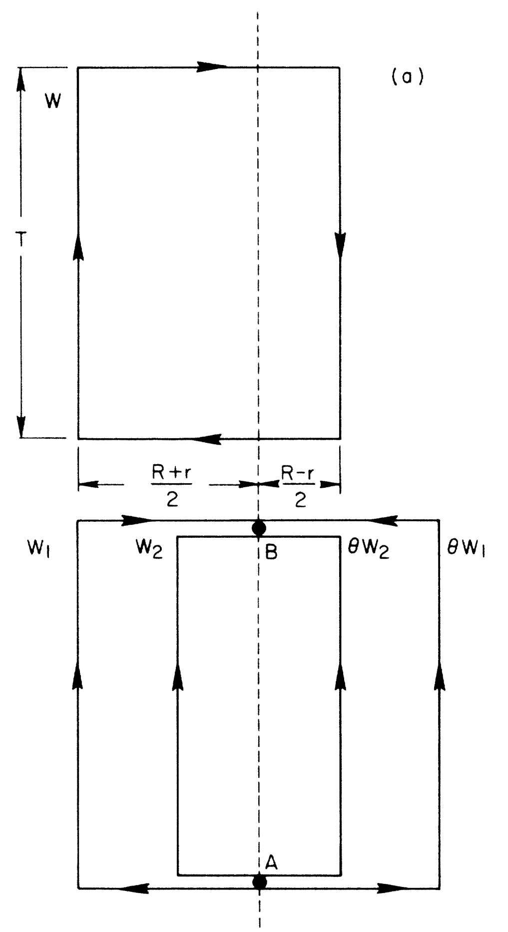

FIG. 1. (a) The large Wilson loop W, with sides of length T and R. The dashed line is its intersection with the reflection hyperplane. (b) The paths W_1 and W_2, going from A to B, used in inequality (8), and their reflections. Note that W is the combination of W_1 and $-\theta W_2$.

heavy-quark–antiquark potential can be extracted from the expectation value of long rectangular Wilson loops[3] W, with sides of length T and R [see Fig. 1(a)] as

$$V(R)=\lim_{T\to\infty}\left[-\frac{1}{T}\ln\langle \mathrm{tr}U(W)\rangle+\mathrm{const}\right] , \quad (5)$$

where

$$\langle \mathrm{tr}U(W)\rangle=\frac{\int \prod_b [dU(b)]e^{-S}\mathrm{tr}U(W)}{\int \prod_b [dU(b)]e^{-S}} ,$$

and $[dU(b)]$ is the invariant group measure. [Strictly speaking, the physical potential is obtained by taking the appropriate scaling limit $g^2(a)\to 0$ as the lattice spacing a shrinks to zero. Since, however, we will establish concavity for any value of g^2, it will also hold in this limit.] We should emphasize that the potential, so defined, is the vacuum energy in the presence of infinitely heavy external $q\bar{q}$ sources, and does not include spin-dependent interactions.

The action (4) enjoys a remarkable property, reflection positivity, which guarantees the existence of a positive-metric Hilbert space, and of a transfer (time-evolution) matrix.[4] Indeed, take any three-dimensional hyperplane normal to a principal axis of the lattice, for instance, the hyperplane $s^1=0$, and denote collectively the set of sites, links, plaquettes, etc., that lie above, on, or below this hyperplane by L_+, L_0, or L_-, respectively. Define a reflection θ on all functionals of bond variables

$$\theta f(U(b))=f^*(U(\theta b)) , \quad (6a)$$

where the reflection on sites, bonds, paths, etc., is, of course,

$$\theta \mathbf{s}=\theta(s^1,s^2,s^3,s^4)=(-s^1,s^2,s^3,s^4) ,$$

$$\theta b=\theta(\mathbf{s},\mathbf{s}^{1'})=(\theta\mathbf{s},\theta\mathbf{s}') , \quad (6b)$$

$$\theta\omega=\theta(\mathbf{s}_0\to\mathbf{s}_1\to\cdots\to\mathbf{s}_f)=(\theta\mathbf{s}_0\to\theta\mathbf{s}_1\to\cdots\to\theta\mathbf{s}_f) .$$

Then for all functionals f that only depend on the bond variables in $L_+\cup L_0$ we have

$$\langle f\theta f\rangle=Z^{-1}\int\prod_{b\in L_0}[dU(b)]\exp\left(-\frac{1}{g^2}\sum_{p\in L_0}\mathrm{tr}U(p)\right)\int\prod_{b\in L_+}[dU(b)]f(U(b))\exp\left(-\frac{1}{g^2}\sum_{p\in L_+}\mathrm{tr}U(p)\right)$$
$$\times\int\prod_{b\in L_-}[dU(b)]f^*(U(\theta b))\exp\left(-\frac{1}{g^2}\sum_{p\in L_-}\mathrm{tr}U(p)\right)$$
$$=Z^{-1}\int\prod_{b\in L_0}[dU(b)]\exp\left(-\frac{1}{g^2}\sum_{p\in L_0}\mathrm{tr}U(p)\right)\left|\int\prod_{b\in L_+}[dU(b)]f(U(b))\exp\left(-\frac{1}{g^2}\sum_{p\in L_+}\mathrm{tr}U(p)\right)\right|^2\geq 0 ,$$

which implies the Schwarz-type inequality:

$$\langle f_1\theta f_2\rangle^2\leq\langle f_1\theta f_1\rangle\langle f_2\theta f_2\rangle . \quad (7)$$

Using this inequality for a reflection about a hyperplane parallel to the (long) time axis, and normal to the plane of the Wilson loop of Fig. 1(a), and recalling Eqs. (3) and (6), we obtain

$$\begin{aligned}\langle \mathrm{tr}U(W)\rangle&=\sum_{i,j}\langle U(W_1)_{ij}\theta U(W_2)_{ij}\rangle\\&\leq\sum_{ij}\langle U(W_1)_{ij}\theta U(W_1)_{ij}\rangle^{1/2}\langle U(W_2)_{ij}\theta U(W_2)_{ij}\rangle^{1/2}\\&\leq\langle \mathrm{tr}[U(W_1)U(-\theta W_1)]\rangle^{1/2}\langle \mathrm{tr}[U(W_2)U(-\theta W_2)]\rangle^{1/2} .\end{aligned} \quad (8)$$

Here i,j are indices in the fundamental representation of the group G, the paths W_1 and W_2 are defined in Fig. 1(b), and the last step is the conventional Schwarz inequality. From the definition of the heavy-quark potential Eq. (5) we then deduce immediately for all $0<r<R$ that

$$V(R)\geq\tfrac{1}{2}V(R-r)+\tfrac{1}{2}V(R+r) ;$$

i.e., V is indeed a concave function of R.

It remains to show that V is also monotone nondecreasing. In view of its concavity, we need only prove this asymptotically, i.e., show that no finite repulsive force can survive at infinite separation. But this follows immediately from the fact, established by Simon and Yaffe,[5] that large Wilson loops can be bounded from above by a perimeter-law decaying exponential, so that the potential is bounded from below by a constant.

Let us finally discuss the inclusion of dynamical matter. Adding light fermions will not destroy concavity, even though it drastically modifies the shape of the potential (in particular one loses heavy-quark confinement due to screening). The reason is that a gauge theory with light fermions is still reflection positive (provided one uses only nearest-neighbor fermion interactions,[4] by universality this is, presumably, not an essential restriction) and reflection positivity was the only ingredient in our proof of concavity. The same is true in the presence of (Higgs) scalars $\phi(\mathbf{s})$, except that the definition of the heavy-quark potential, Eq. (5), should now be modified to take into account the direct Yukawa couplings of the scalars to the external sources:

$$V(R)=\lim_{T\to\infty}\left[-\frac{1}{T}\ln\left\langle \mathrm{tr}U(W)\exp\left(\lambda\sum_{\mathbf{s}\in W}\phi(\mathbf{s})\right)\right\rangle+\mathrm{const}\right] ,$$

where λ is the Yukawa coupling constant and the summation runs over all lattice sites on the Wilson loop W. The proof of concavity then goes through as before.

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