Symmetry breaking and charge operator in SU(9) grand-unification models

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We present a thorough analysis of U(1) generators in all possible symmetry-breaking modes that reduce SU(9) down to $SU_c(3)\times SU_W(2)\times U(1)$. Two types of representations that satisfy a set of grand-unification guidelines are considered. The first type embeds SU(5) trivially and the second type has nontrivial embedding of SU(5). Acceptable representations of both types are found in SU(9). Projection matrices as well as the contribution of U(1) eigenvalues to the weak hypercharge are given in all physically interesting modes of symmetry breaking. An example is explicitly worked out for the evolution of coupling strengths to determine the intermediate mass scales responsible for invisible axions and the delayed decay of the proton while maintaining a satisfactory value of $\sin^2\theta_W$ at low energies. Fermion contents are also discussed.

I. INTRODUCTION

The idea of grand unification was suggested more than a decade ago with a simple gauge group SU(5) to unify the strong, electromagnetic, and weak interactions.[1] Applications as well as generalizations of this model have been quite extensive since then, and the subject has reached a certain maturity in recent years in the sense that a complete list of the anomaly-free, complex, and asymptotic-free representations of all classical Lie groups is now available.[2,3]

One major reason for searching for all such possible grand-unification schemes is to understand the problem of family generations, i.e., to see if it is possible to incorporate the flavor symmetry into a single grand-unified gauge group. The problem of the family generation is one of the long-standing puzzles in particle physics and is one of the major shortcomings of simple grand-unification theories. Another problem is the necessity for a hierarchy of two vastly different energy scales in the Lagrangian. The gauge hierarchy problem has been discussed by many authors and appears to have been solved in the context of supersymmetric grand-unified theories that generate all mass scales below the Planck mass dynamically.[4] However, the problem of flavor generations has not made the same kind of progress even though the flavor dynamics has been persuasively emphasized and studied. It still appears attractive and appealing to find a larger grand-unification group[5,6] into which a multiple family structure of the $SU(3)\times SU(2)\times U(1)$ subgroup or SU(5) can be embedded. The family problem is then to make the family unification in an appropriate grand-unified model.

Another possible approach to the family problem might be the composite models in which quarks and leptons are supposed to be bound states of more fundamental objects, say, preons. Exactly how preons are forming the elementary particles and how many of them are needed to give the desired family structure are still open to solution. However, within the context of composite schemes, one will need to know the representations of the fundamental preons. In this case quarks and leptons then correspond to certain dynamical condensates of preons determined by a certain confining metacolor force among preons and the family generation will be regularity of those condensations. However, no one has found such a composite model yet. Though there are many attempts in this direction, they usually involve additional phenomenological assumptions and we are far from understanding the metacolor group of preons.[7] For example, we do not have a consensus as yet if preons are all fermions.[8] In some sense this is related to the deeply rooted question of the fermion masses, i.e., whether they are generated by spontaneous symmetry breaking or rather by a certain dynamical origin. The search for all possible representations of preons will prove to be useful, anticipating the eventual understanding of the preon gauge symmetry. However, it is fair to say at the moment that the family problem within the context of the ordinary grand-unification scheme is better defined and perhaps more profitable if not more promising. That is to say one updates the criteria and improves the approach originally suggested by Georgi to incorporate the number of families.

In the standard grand-unification approach one family generation of fermions is given by the simplest anomaly-free representation, $\mathbf{5}^*+\mathbf{10}$, of SU(5) or more generally by the 15 chiral states of $SU(3)\times SU(2)\times U(1)$ contained in the minimal SU(5) representation. The representation of one family generation is characteristically complex with respect to $SU(3)\times SU(2)\times U(1)$. This is closely tied to the fact that these particles have survived as the set of ordinary light fermions when the grand-unified gauge symmetry breaks down to $SU(3)\times SU(2)\times U(1)$ at an energy scale M_G. Real representations would have combined to form $SU(3)\times SU(2)\times U(1)$-invariant mass terms of the order M_G. As mentioned already, a complete documentation of such complex representations of classical Lie groups that satisfy the additional conditions of being anomaly free and asymptotic free was presented recently.[2] Obviously, any reasonable and realistic grand-unification representation will have to be anomaly free so as to lead to a renormalizable theory. The requirement of asymptotic freedom implies that all fermions are confined by an infrared slavery and therefore may be reasonable for preon representations. But for ordinary grand unification this requirement can at best be a practical one to limit the dimension of the representations. Since gravity will play a role around the Planck mass scale which is reached beyond the grand-unification scale, there is no strict reason for the single unified coupling constant to decrease forever in high energies. Nevertheless, asymptotic freedom is often required to circumvent the highly reducible nature of particularly SU(N)-type representations.[6]

Another constraint often applied is the condition of the automatic invisible axion. The axion is the pseudo-Goldstone boson arising from the spontaneous breaking of the Peccei-Quinn symmetry $U_{PQ}(1)$ which is introduced to rid the strong CP problem.[9] From the astrophysical consideration of the energy emission of red giant stars one finds that[10] the $U_{PQ}(1)$ symmetry is broken at a mass scale around 10^{11} GeV, which makes the axion invisible[11] both in mass and interaction strength with other particles. In a more natural theory the $U_{PQ}(1)$ should arise automatically[12] from the underlying grand-unification symmetry and undergo a spontaneous symmetry breaking at such a mass scale. In many instances, however, this $U_{PQ}(1)$ leaves a discrete $Z(M)$ symmetry[13] by QCD instantons, M being the color anomalousness of the $U_{PQ}(1)$ symmetry thereby leading to degenerate vacua separated by domain-wall structures in the early Universe. Since such domain-wall structures are in contradiction with modern cosmology, a certain amelioration is desired to remove the degeneracy. Thus the cosmological requirement of no domain walls has been added to the list of necessary constraints that any acceptable theory must obey. A simple way to satisfy this condition without populating the theory with more unknown particles is to embed[14] $Z(M)$ onto the continuous gauge group of the unified theory. As the color-anomalous number M is related to the number of flavors that contribute to the anomaly term and their $U_{PQ}(1)$ charges, such embedding is not always possible and therefore provides a nontrivial consequence to the flavor numbers. In other words, the requirement of no domain walls serves as a useful criterion to choose the acceptable grand-unified models.

Recently we have reported the results of extensive searches for the acceptable representations[15] that have at least three generations of fermions and yet satisfy all of those desired conditions. In addition, all particles were required to be real with respect to $SU_c(3)\times U_{em}(1)$, a generalization[16] of Georgi's first condition[5] of model building. If the representation is SU(3) complex so that the color gauge symmetry is chiral, there will be massless quarks. Otherwise the chiral symmetry associated with confinement will break down spontaneously to SU(2). Reality under $U_{em}(1)$ is also a reasonable requirement because all charged leptons have mass. It turns out that the reality condition under $SU_c(3)\times U_{em}(1)$ is closely related to the counting of the number of generations particularly for the spinor-type representations into which a nontrivial embedding of SU(5) is made. The results of the search[15] have shown that there are only a limited number of representations that satisfy all these constraints. They are mostly in SU(9) though two SU(7) and an SU(8) representations are allowed in the list. In fact there is only one representation[17] that has exactly three low-mass generations, i.e., $4[1^*]+2[2^*]+[3]+[4]$ in SU(9). There are several four-generation models found in SU(9). In this respect, the SU(9) group is of particular interest.

In this paper, we study these SU(9) representations and in particular the various symmetry-breaking patterns that are possible in SU(9) as well as the contribution of U(1)'s to the weak hypercharge in each symmetry-breaking mode. The method we employ is that of the projection matrix which was shown to be both economical and successful for SU(7) models.[18]

In Sec. II we list the constraints imposed on the representations and elaborate on their consequences. Section III deals with the possible symmetry-breaking patterns in SU(9) and the weak hypercharge made of the various U(1) eigenvalues. Consequences to the evolution of coupling constants due to different symmetry-breaking modes are examined in connection with the low-energy value of $\sin^2\theta_W$ and the prolonged proton lifetime. Section IV contains the fermion content of SU(9) models as well as some concluding remarks.

II. CONSTRAINTS OF GRAND UNIFICATION

Although we are interested in SU(N) and in particular SU(9) representations, we will summarize here all the constraints imposed on the representations. Starting with the list of complex, anomaly-free, and asymptotically free representations, we apply systematically the rest of the requirements to narrow down the acceptable ones. The constraints to be satisfied by the *left-handed* fermion representations are as follows.

(a) The representation should be anomaly free.

(b) The representation should be asymptotically free with respect to the grand-unification group G.

(c) The representation must be real with respect to $SU_c(3)\times U_{em}(1)$.

(d) The representation must accommodate at least three generations.

(e) The representation should contain color singlets, triplets, and antitriplets only.

(f) The representation leads to an automatic invisible axion without suffering from the catastrophic domain-wall problem.

As shown in Refs. 15 and 17, all these constraints can be satisfied by reducible representations of SU(N). Although there are other possibilities such as those in SO(10) and E_6, they can overcome the domain-wall difficulty only at the expense of introducing further heavy fermions.[19] However, we choose the domain-wall resolution through the embedding[14] of the unbroken discrete symmetry of $U_{PQ}(1)$ to the continuous gauge group as this gives nontrivial consequence to the flavor numbers. Thus we are left with SU(N) representations only. Constraint (d) is a generalization of Georgi's second rule of model building[5] and requires the representation to be complex with respect to $SU(3)\times SU(2)\times U(1)$.

The defining fundamental representation of an SU(N) group transforms like

$$(3,1,-\tfrac{1}{3})+(1,2,\tfrac{1}{2})+\sum_{i=1}^{N-5}(1,1,q_i)$$

under $SU(3)\times SU(2)\times U(1)$ subject to the traceless condition of the charge operator, $\sum_{i=1}^{N-5} q_i=0$. Constraint (e) can be satisfied by the antisymmetric representations of SU(N), $[M]$, corresponding to the Young diagrams having M boxes in a single column. Thus we write the representation of the left-handed fermions in general as

$$f_L=\sum_{M=1}^{N-1} C_M[M] , \tag{1}$$

where C_M is the multiplicity of the representation $[M]$. In general, not all C_M's will be present because $[M^*]=[N-M]$. Constraint (b) imposes the condition

$$\sum_{M=1}^{N-1} C_M I_2(M)=\sum_{M=1}^{N-1} C_M \frac{(N-2)!}{(M-1)!(N-M-1)!}\le 11N , \tag{2}$$

where $I_2(M)=2\,\mathrm{Tr}(T^2)$ evaluated on $[M]$ with the normalization $I_2(1)=1$. This restricts the complex antisymmetric representations only up to rank 4, i.e., $M\le 4$ and their complex conjugates up to $N\le 12$. Constraint (a) is satisfied if

$$A(N)=\sum_{M=1}^{N-1} C_M \frac{(N-2M)(N-3)!}{(M-1)!(N-M-1)!}=0 \tag{3}$$

in the convention that the anomaly of the defining vector representation of SU(N) is unity. Because $[M^*]=[N-M]$, we will use the notation $C_{N-M}\equiv C_M^*$ and $d_M\equiv C_M-C_M^*$ henceforth. Then Eq. (3) becomes

$$A(N)=d_1+(N-4)d_2+\tfrac{1}{2}(N-3)(N-6)d_3 + \tfrac{1}{6}(N-3)(N-4)(N-8)d_4=0 . \tag{4}$$

This condition is automatically satisfied by constraint (c) in the case of SU(5). In general, the representation must contain exact pairs of (a,q) and $(a^*,-q)$, a and q being the quantum numbers of $SU_c(3)$ and $U_{em}(1)$, in order to meet constraint (c). In particular the number of color triplets must be equal to that of color antitriplets. Also the sum of the charge squares of the particles in the color-triplet representations is equal to that in the color-antitriplet representations. One can easily verify that the former aspect of constraint (c) for Eq. (1) is equivalent to the anomaly-free constraint (a) whereas the latter aspect of constraint (c) of Eq. (1) gives

$$A(N)/9+G(N)\sum_{i=1}^{N-5} {q_i}^2=0 , \tag{5}$$

where

$$G(N)=d_2+(N-6)d_3+\tfrac{1}{2}(N-5)(N-8)d_4 . \tag{6}$$

Note that Eq. (5) reduces to the anomaly-free condition $A(N)=0$ for SU(5). But in general for anomaly-free representations constraint (c) necessarily reduces to

$$G(N)\sum_{i=1}^{N-5} {q_i}^2=0 . \tag{7}$$

For those representations with $G(N)\neq 0$, we must have $q\equiv\sum_{i=1}^{N-5} {q_i}^2=0$, i.e., all $q_i=0$ for the $N-5$ entries in the fundamental representation. For those representations with $G(N)=0$, not all q_i's have to be zero provided $\sum q_i=0$. Thus it is clear that counting the generation number will be different drastically depending on whether or not $q=0$.

In the case of $q=0$, only the usual quarks and leptons appear along with color-singlet neutral components. The net numbers of $\mathbf{5}^*$ and $\mathbf{10}$ representations of SU(5) contained in Eq. (1) are

$$N_{(5^*)}=-d_1-(N-5)d_2-\tfrac{1}{2}(N-5)(N-6)d_3 +[1-\tfrac{1}{6}(N-5)(N-6)(N-7)]d_4 \tag{8}$$

and

$$N_{(10)}=G(N) ; \tag{9}$$

i.e., the coefficient of q in Eq. (5) is precisely the net number of SU(5) **10** representation. Furthermore,

$$N_{(10)}-N_{(5^*)}=A(N) \tag{10}$$

so that

$$N_{(10)}=N_{(5^*)}=G(N) \tag{11}$$

for the representations of our interest. Thus $G(N)$ is the number of generations contained in Eq. (1) when $q=0$, i.e.,

$$f_L\to G(N)(\mathbf{5}^*+\mathbf{10})+\text{neutral singlets} \tag{12}$$

under $SU(N)\to SU(5)$ breaking. Constraint (d) is satisfied as long as $G(N)\ge 3$.

In the case of $q\neq 0$, reality under $SU_c(3)\times U_{em}(1)$ implies $G(N)=0$ for the fermion representations. It turns out that there are only a few anomaly-free representations that satisfy $G(N)=0$ when $q\neq 0$. In fact there are none for SU(6). For SU(7), the representations must satisfy $d_1{:}d_2{:}d_3=-1{:}1{:}-1$, an example of which is the SU(7) representation $[1^*]+[2]+[3^*]$ which can be embedded into the 64-dimensional spinor representation of SO(14) naturally.[20] For SU(8), the representations must satisfy $d_1{:}d_2{:}d_3=-3{:}2{:}-1$. For SU(9) the representations must obey $d_1{:}d_2=6d_3+5d_4{:}-(3d_3+2d_4)$, a typical example of which is $[1^*]+[2]+[3^*]+[4]$ that[21] can be reduced from the complex spinor representation with dimension $2^8=256$ of SO(18). Since $G(N)=0$, these representations become real under SU(5) in the limit of all $q_i=0$ and therefore counting the number of fermion generations should be done by considering the $SU(3)\times SU(2)\times U(1)$ contents directly. On the other hand, this will require specific assignments of q_i's. In other words, the number of generations depends on judicious choice of the electric charge quantum numbers of the SU(5) singlet members in the fundamental representation. In addition, we should make sure to satisfy the reality with respect to $SU_c(3)\times U_{em}(1)$ for a given assignment of q_i. For SU(7), the only possible assignment is $(a,-a)$ and the representation $[1^*]+[2]+[3^*]$ satisfies constraint (c). This SU(7) representation was studied extensively[22] for integral as well as fractional values of a and was shown to accommodate only two generations of quark-lepton family for the choice $a=1$. Thus it is ruled out by constraint (d). For SU(8), there are three different q_i's, the sum of which vanishes. The representation $3[1^*]+2[2]+[3^*]$ is real with respect to $SU_c(3)\times U_{em}(1)$ if and only if some pairs satisfy $q_i+q_j=0$ $(i,j=1,2,3)$. Then the charge assignment has to be of the type $(0,q,-q)$. Again this representation contains only two generations of fermions. For the SU(9) representation $[1^*]+[2]+[3^*]+[4]$, constraint (c) can be satisfied by many different choices of (q_1,q_2,q_3,q_4) subject to $\sum_{i=1}^4 q_i=0$. Under $SU(9)\to SU(3)\times SU(2)\times U(1)$, we have

$$[1^*]\to(3^*,1,\tfrac{1}{3})+(1,2,-\tfrac{1}{2})+\sum_{i=1}^{4}(1,1,-q_i)\,, \tag{13a}$$

$$[2]\to(3^*,1,-\tfrac{2}{3})+(3,2,\tfrac{1}{6})+(1,1,1)+\sum_{i=1}^{4}(3,1,-\tfrac{1}{3}+q_i)+\sum_{i=1}^{4}(1,2,\tfrac{1}{2}+q_i)+\sum_{i\neq j}^{4}(1,1,q_i+q_j)\,, \tag{13b}$$

$$\begin{aligned}[3^*]\to{}&(3,2,\tfrac{1}{6})+(3^*,1,-\tfrac{2}{3})+(1,1,+1)+\sum_{i=1}^{4}(3,1,+\tfrac{2}{3}-q_i)+\sum_{i=1}^{4}(3^*,2,-\tfrac{1}{6}-q_i)\\&+\sum_{i\neq j=1}^{4}(3^*,1+\tfrac{1}{3}-q_i-q_j)+\sum_{i=1}^{4}(1,1,-1-q_i)+\sum_{i\neq j}^{4}(1,2,-\tfrac{1}{2}-q_i-q_j)+\sum_{i\neq j\neq k}^{4}(1,1,-q_i-q_j-q_k)\,,\end{aligned} \tag{13c}$$

$$\begin{aligned}[4^*]\to{}&(1,2,-\tfrac{1}{2})+(3^*,1,\tfrac{1}{3})+\sum_{i=1}^{4}(1,1,-1+q_i)+\sum_{i=1}^{4}(3^*,2,-\tfrac{1}{6}+q_i)+\sum_{i=1}^{4}(3,1,\tfrac{2}{3}+q_i)\\&+\sum_{i\neq j}^{4}(3^*,1,-\tfrac{2}{3}+q_i+q_j)+\sum_{i\neq j}^{4}(3,2,\tfrac{1}{6}+q_i+q_j)+\sum_{i\neq j}^{4}(1,1,1+q_i+q_j)\\&+\sum_{i\neq j\neq k}^{4}(3,1,-\tfrac{1}{3}+q_i+q_j+q_k)+\sum_{i\neq j\neq k}(1,2,\tfrac{1}{2}+q_i+q_j+q_k)+(1,1,0)\,.\end{aligned} \tag{13d}$$

Noting that one family generation consists of $(5^*)+(10)$ of SU(5) or

$$(5^*)+(10)\to(3,2,\tfrac{1}{6})+(3^*,1,\tfrac{1}{3})+(3^*,1,-\tfrac{2}{3})+(1,2,-\tfrac{1}{2})+(1,1,0) \tag{14}$$

in the usual $SU(3)\times SU(2)\times U(1)$ contents, we need $q_i+q_j=0$ $(i\neq j)$ for the quark doublets $(3,2,\frac{1}{6}+q_i+q_j)$ in [4]. Then the charge assignment has to be of the form $(a,-a,b,-b)$. In this case, there are four generations in $[1^*]+[2]+[3^*]+[4]$. If we choose $a=b$ in addition, two more generations can be unified in this SU(9) model.

Now we come to constraint (f). As we discussed in Sec. I, the requirement of an automatic invisible axion with no domain walls is desired. Of the several resolutions suggested, the prescription to embed the remnant discrete symmetry left unbroken by QCD instantons onto the center group of continuous gauge symmetry is most interesting and yet nontrivial so far as the number of flavors is concerned. Lie groups possess invariant discrete center groups (subgroups), i.e., $Z(2)$ for $SO(2N+1)$ and $Sp(2N)$, $Z(4)$ and $Z(2)\times Z(2)$ for $SO(2N)$ with odd and even N, respectively, $Z(3)$ for E_6, and $Z(N)$ for $SU(N)$. All other classical groups have trivial centers. We have recently reported[15] the results of searches for the grand-unified models that satisfy all of the constraints (a)—(f). In fact, the no-domain-wall requirement is so restrictive that there are a small number of models allowed: There are only two models that possess a *natural* automatic $U_{PQ}(1)$ both of which are in SU(9). They are

$$f_L=4[1^*]+2[2^*]+[3]+[4]\,, \tag{15}$$

$$f_L=26[1^*]+7[2]+[3^*]\,. \tag{16}$$

The first model contains three flavor generations while the second has $G(9)=4$. By relaxing the naturalness restriction so as to allow an $U_{PQ}(1)$ to exist through particular choices of Higgs scalars or by considering a further intricacy of the embedding mechanism concerning the general congruence class of representations, we found five more four-generation models, three of which are in SU(9). All of these are of the type of Eq. (12), i.e., all $q_i=0$ for the $N-5$ entries in the fundamental representations. It was also shown that[21] that SU(9) model

$$f_L=[1^*]+[2]+[3^*]+[4] \tag{17}$$

with the charge assignment $(a,-a,b,-b)$ for the additional members in the fundamental representation, could satisfy the requirement (f) with particular choice of Higgs scalars. Though $G(9)=0$, this model accommodates four fermion generations as pointed out before. In fact, in the case of $G(N)=0$ this model is the only one satisfying all of the constraints (a)—(f). We note again that this model is equivalent to the 256-dimensional complex spinor representation of SO(18) revived recently by Bagger and Dimopoulos.[23] The charge assignment $(a,-a,b,-b)$ activates abundant particles with color. Consequently the color SU(3) is not asymptotically free in this model though it has asymptotic freedom with respect to the full SU(9) gauge group. One needs to split the model and give those particles with exotic quantum numbers a mass of the order of the grand-unified scale M_G leaving four left-handed families light in the low-energy sector. Such splitting of heavy from light particles was possible in SO(18) and a similar scheme can be applied to SU(9), which has SU(5)×SU(4) breaking with an extra U(1) symmetry.

All in all we see that SU(9) is of special interest. This is the only group that admits representations satisfying all the necessary constraints of model building regardless of the charge assignments of the SU(5) singlet components in the fundamental representation. In the case of all $q_i=0$ for the single components, the number of families is given by $G(9)$. Equations (15) and (16) are two interesting models in this case. On the other hand, Eq. (17) is the only acceptable model when all $q_i\neq 0$ and can accommodate four families.

In the remainder of the paper, we concentrate on and study SU(9) models. In the next section we examine all symmetry-breaking patterns that are possible in SU(9) as well as the contribution of each U(1) to the weak hypercharge in each mode of symmetry breaking.

III. SYMMETRY-BREAKING PATTERNS IN SU(9) AND THE WEAK HYPERCHARGE

The structure of hypercharge operators will depend on the mode of symmetry breaking. We first study all patterns of symmetry breaking that reduce SU(9) to SU(3)×SU(2)×U(1). We choose to trace down the maximal subgroups at each stage of breaking. There are five U(1) factors in the end that contribute to the charge operator

$$Q=T_3+Y\ , \tag{18}$$

where T_3 is the third generator of SU(2) subgroup and Y is the weak hypercharge operator. The fundamental representation of SU(9) embeds that of SU(5) so that **9** decomposes into (3,1) + (1,2) + 4(1,1) under $\mathrm{SU}_c(3)\times\mathrm{SU}(2)$. Since we are interested in SU(9) breakings to $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\prod_{i=1}^{5}\mathrm{U}(1)_i$ eventually, the weak hypercharge can be written as

$$Y=\sum_{i=1}^{5}a_iY_i\ , \tag{19}$$

where Y_i is the diagonal operator corresponding to the ith U(1) rotation and

$$a_i=\mathrm{Tr}(YY_i)/\mathrm{Tr}Y_i^2\ . \tag{20}$$

Since the charge operator is a generator of SU(9), the charge assignment of the fundamental representation is

$$Q=\mathrm{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},1,0,q_1,q_2,q_3\ ,-q_1-q_2-q_3)\ . \tag{21}$$

The hypercharge operator as given by Eq. (18) is not properly normalized. To ensure the proper normalization, we set $Y=CY_0$ with the convention $\mathrm{Tr}T_3{}^2=\mathrm{Tr}Y_0{}^2=\frac{1}{2}$. Then we obtain from Eq. (18) and Eq. (21) that

$$Q=T_3+\{\tfrac{5}{3}+2[q_1{}^2+q_2{}^2+q_3{}^2+(q_1+q_2+q_3)^2]\}^{1/2}Y_0\ . \tag{22}$$

This gives the SU(2)×U(1) mixing angle at the grand-unification mass scale M_G to be

$$\sin^2\theta_W(M_G)=\mathrm{Tr}T_3{}^2/\mathrm{Tr}Q^2 =\{\tfrac{8}{3}+2[q_1{}^2+q_2{}^2+q_3{}^2+(q_1+q_2+q_3)^2]\}^{-1}\ . \tag{23}$$

Altogether there are 60 different ways of breaking the SU(9) symmetry, all of which reduces SU(9) to SU(3)×SU(2)×U(1). Intermediate stages of symmetry breaking are of the form $\prod_i \mathrm{SU}(n_i)\prod_j \mathrm{U}(1)_j$. There are four classes of SU(9) breaking distinguished by the first stages: i.e., SU(9)→SU(8), SU(7)×SU(2), SU(6)×SU(3), and SU(5)×SU(4). When we trace down the maximal subgroups systematically and consider different possibilities of splitting the color $\mathrm{SU}_c(3)$ and weak $\mathrm{SU}_W(2)$, these four classes have 23, 17, 11, and 9 subpatterns, respectively. However, a more interesting classification may be to make use of how $\mathrm{SU}_c(3)$ and $\mathrm{SU}_W(2)$ emerge.

Case (A). When $\mathrm{SU}_c(3)$ and $\mathrm{SU}_W(2)$ appear at the same time from an intermediate subgroup $\mathrm{SU}(n_I)$, the unification mass M_G is determined by the energy scale below which the symmetry is broken to

$$\mathrm{SU}(n_I)\to\mathrm{SU}_c(3)\times\mathrm{SU}_W(2)\times\prod_{j=1}^{n_I-4}\mathrm{U}(1)_j\ .$$

There are eight such cases, all of which have an intermediate SU(5) reducing to $\mathrm{SU}_c(3)\times\mathrm{SU}_W(2)\times\mathrm{U}(1)$ and therefore have the same M_G as in the standard SU(5) model.

Case (B). When SU(9) has an intermediate subgroup $\mathrm{SU}(n_1)$ breaking at the scale M_1 down to $\mathrm{SU}(n_2)\times\mathrm{SU}_W(2)\times\cdots$ followed by $\mathrm{SU}(n_2)\to\mathrm{SU}_c(3)\times\cdots$ at M_2, the color coupling strength $\alpha_3(\mu)$ changes the slope and decreases faster for $\mu>M_2$. Thus the unification of the strong and electroweak interactions that occurs above $\mu=M_1$ will come earlier than in the standard SU(5) model. There are 19 patterns belonging to this case.

Case (C). If SU(9) goes through an intermediate breaking $\mathrm{SU}(n_1)\to\mathrm{SU}(n_2)\times\mathrm{SU}_c(3)\times\cdots$ at M_1 followed by $\mathrm{SU}(n_2)\to\mathrm{SU}_W(2)\times\cdots$ at M_2, the weak-coupling strength $\alpha_2(\mu)$ evolves rapidly above $\mu=M_2$ with a slope larger or equal to that of $\alpha_3(\mu)$. In order for a grand unification to occur at M_1, α_2 must be larger than α_3 in the interval $M_2<\mu<M_1$. Furthermore, if this $\alpha_2(\mu)$ is to be extrapolated to the low-energy region realistically, it must cross $\alpha_2(\mu)$ at some value $\mu<M_2$; i.e., there is a premature false unification of the two interactions in the low-energy region, the meaning of which is not clear. On the other hand, if one wants to avoid such a premature unification and adjusts $\alpha_2(\mu)$ and $\alpha_3(\mu)$ from low-energy data, their evolution will not lead to a grand unification at M_1 unless many more fermions start to contribute to $\alpha_2(\mu)$ above $\mu=M_2$ preventing a faster decrease. However, it is hard to imagine how $\alpha_3(\mu)$ can maintain the same slope

above $\mu = M_2$ in that case. There are 20 subpatterns belonging to this case.

Case (D). The final situation is when $SU_c(3)$ and $SU_W(2)$ split off the different subgroups $SU(n_2)$ and $SU(n_3)$ of an intermediate stage $SU(n_1) \to SU(n_2) \times SU(n_3) \times \cdots$. Then there are at least three different mass scales that influence $\alpha_3(\mu)$ and $\alpha_2(\mu)$. They are M_1 associated with $SU(n_1) \to SU(n_2) \times SU(n_3) \times \cdots$, M_2 of $SU(n_2) \to SU_c(3) \times \cdots$, and M_3 of $SU(n_3) \to SU_W(2) \times \cdots$. As long as $n_2 - n_3 \geq 0$, there will be a grand unification above $\mu = M_1$. Depending on where $SU(n_1)$ and $SU(n_2)$ breakings occur, the unification scale M_1 can be reached later or earlier than that of the standard SU(5) model. In fact, one may determine the hierarchical ratios M_2/M_1 and M_3/M_1 from the experimental inputs such as $\sin^2\theta_W$ and the proton lifetime. Thus this case provides rich possibilities. There are 12 subpatterns of symmetry breaking belonging to this case.

Of the 60 patterns we find 20 symmetry-breaking modes belonging to either case (A) or case (D) to be potentially interesting and meaningful. While case (B) can give rise to a meaningful grand unification, models with this class of symmetry breaking are likely to run into difficulty with the proton lifetime since the best limit presently available, $\tau/B(p \to e^+\pi^0) > 2\times 10^{32}$ yr, may imply a delay of grand unification compared to the standard SU(5) model. We have already elaborated on the physical inaptitude of the symmetry-breaking patterns of case (C). The 20 symmetry-breaking patterns that are more relevant to the physics of SU(9) grand-unification models are summarized in Table I. Obviously, different patterns of symmetry breaking entail different evolutions of coupling constants and $\sin^2\theta_W$. In particular, different combinations of U(1) generators will appear in the charge and hypercharge operators, Eqs. (18) and (19). Thorough investigation of the five U(1) generators in each and every interesting case of symmetry breaking is needed, but this is not a simple task. For this purpose, we give also the projection operators of

$$SU(9) \to SU(3) \times SU(2) \times \prod_{i=1}^{5} U(1)_i$$

in Table I. The projection operator method is the most economic way of tracing the branchings of a representation directly on its Dynkin weight basis. Here we use the convention that the highest height $(a_1, a_2, \ldots, a_n)$ of a Dynkin basis in $SU(n+1)$ denotes the representation whose Young tableau has a_i columns with i boxes $(i = 1, 2, \ldots, n)$. By applying a projection operator P from the right on SU(9) weight basis, W_9, we obtain $SU(3) \times SU(2) \times \prod_{i=1}^{5} U(1)_i$ weights, i.e.,

$$W_9 P = W_3 W_2 Y_1 Y_2 Y_3 Y_4 Y_5 \ . \tag{24}$$

In other words, the first two columns on the right-hand side represent the SU(3) projection whereas the third column is the SU(2) weights. The remaining five columns are eigenvalues of the U(1) generators. Note that W_9 is an $8 \times D$ matrix in which each row represents a weight component in descending order from the top. An operator of P matrix on W_9 from the right then decomposes the $8 \times D$ matrix into the form from which the weights of SU(3), SU(2), and U(1) generators can readily be read. For example, the fundamental representation [1] of SU(9) with any one of the projection matrices given in Table I is decomposed to $([1],\cdot) + (\cdot,[1]) + 4(\cdot,\cdot)$ in (SU(3),SU(2)). We can then identify $(SU(3), SU(2), U(1)_Y)$ quantum numbers from Eq. (21) as

$$[1] \to (3, 1, -\tfrac{1}{3}) + (1, 2, \tfrac{1}{2}) + \sum_{i=1}^{4} (1, 1, q_i) \ , \tag{25}$$

where Eq. (18) is used and $\sum_{i=1}^{4} q_i = 0$. Finally the contributions of each U(1) operator to the hypercharge can be determined from Eq. (19). Table I shows the results of charge combinations in each symmetry-breaking case. Having obtained the U(1) contributions to the total charge operator, the hypercharge quantum number of the $(SU(3), SU(2), U(1)_Y)$ decompositions of higher SU(9) representations can readily be read off from Eq. (25) with the help of Table I. In particular, this procedure confirms the results of Eq. (13) based on tensor decompositions. We note in addition that Table I contains a_i's of each U(1) generator for general values of charges q_i. Obviously, a different choice of q_i gives rise to a different charge operator.

We have seen in Sec. II that the number of flavor generations is counted differently depending on the choice of q_i. When all $q_i = 0$, the flavor number is given by the number of $(\mathbf{5}^* + \mathbf{10})$ of SU(5) contained in the model. Two such models are given in Eqs. (15) and (16), which have the same charge operators in each mode of symmetry breaking. When none of the q_i's are zero, Eq. (17) is the only acceptable model which contains four generations for $q_1 = -q_2 = a$ and $q_3 = -q_4 = b$ and six generations if $a = b$ in addition. Different choices of q_i give different charge operators for a given symmetry-breaking mode. In particular the various U(1)'s contribute differently to the evolution of $\sin^2\theta_W$ and also to the production of monopoles. From Eq. (23), we get $\sin^2\theta_W(M_G) = \frac{3}{8}$ for the models of Eqs. (15) and (16), the same as in the standard SU(5), while $\sin^2\theta_W(M_G)$ is much smaller for the models of Eq. (17) for all $q_i \neq 0$. Obviously one needs to increase (decrease) $\sin^2\theta_W$ in the case all $q_i \neq 0$ (all $q_i = 0$) when it is extrapolated to the low-energy region following the flow of the renormalization-group equations for the coupling constants. This will provide *a posteriori* preferences of certain symmetry-breaking modes in each case. The renormalization corrections of the coupling constants will usually depend on two or three parameters coming from intermediate mass scales.

Suppose that SU(9) is broken down to $SU(3) \times SU(2) \times U(1)$ in several steps as in the case (13) of Table I: $SU(9) \to SU(5) \times SU(4) \times U(1)_A$ at M_0 followed by $SU(4) \to SU(3) \times U(1)_B$ and $SU(5) \to SU(4) \times U(1)_C$ at M_1, and $SU(3) \to SU_W(2) \times U(1)_D$ and $SU(4) \to SU_c(3) \times U(1)_E$ at M_2, and finally at $M_3 = M_W$, $SU(9) \to SU_c(3) \times U(1)_{em}$. Assuming that there is a large region between the successive mass scales where an effective perturbative theory with the specified intermediate gauge symmetry is applicable we can write[24] at $M_3 = M_W < M_2$

$$\alpha_i^{-1}(M_W) = \alpha_9^{-1}(M_0) + 2\sum_{j=1}^{3} d_j^i \ln(M_{j-1}/M_j) \ , \tag{26}$$

TABLE I. Symmetry-breaking patterns, projection matrices, and $U(1)_i$ contributions to the hypercharge.

Symmetry-breaking pattern	Projection matrix	$U(1)_i$ Contributions to Y
(1)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 1 1 1 2	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(7)\times U(1)_2$	0 1 0 2 2 2 2 4	$a_2=\frac{1}{56}(q_1+q_2)-\frac{1}{8}q_3$
$SU(7)\to SU(6)\times U(1)_3$	0 0 0 3 3 3 3 6	$a_3=\frac{1}{42}q_1-\frac{1}{7}q_2$
$SU(6)\to SU(5)\times U(1)_4$	0 0 1 4 4 4 4 3	$a_4=-\frac{1}{6}q_1$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 5 5 5 5 0	$a_5=-\frac{1}{6}$
	0 0 0 6 6 6 0 0	
	0 0 0 7 7 0 0 0	
	0 0 0 8 0 0 0 0	
(2)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 1 2 0 2	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(7)\times U(1)_2$	0 1 0 2 2 4 0 4	$a_2=\frac{1}{56}(q_1+q_2)-\frac{1}{8}q_3$
$SU(7)\to SU(5)\times SU(2)\times U(1)_3$	0 0 0 3 3 6 0 6	$a_3=-\frac{1}{14}(q_1+q_2)$
$SU(2)\to U(1)_4$	0 0 1 4 4 8 0 3	$a_4=\frac{1}{2}(q_1-q_2)$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 5 5 10 0 0	$a_5=-\frac{1}{6}$
	0 0 0 6 6 5 1 0	
	0 0 0 7 7 0 0 0	
	0 0 0 8 0 0 0 0	
(3)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 1 3 0 1	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(7)\times U(1)_2$	0 1 0 2 2 6 0 2	$a_2=\frac{1}{56}(q_1+q_2)-\frac{1}{8}q_3$
$SU(7)\to SU(4)\times SU(3)\times U(1)_3$	0 0 0 3 3 9 0 3	$a_3=\frac{1}{28}q_1-\frac{1}{21}q_2-\frac{1}{12}$
$SU(3)\to SU(2)_W\times U(1)_4$	0 0 0 4 4 12 0 0	$a_4=-\frac{1}{3}q_2+\frac{1}{6}$
$SU(4)\to SU(3)_c\times U(1)_5$	0 0 1 5 5 8 1 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 0 6 6 4 2 0	
	0 0 0 7 7 0 0 0	
	0 0 0 8 0 0 0 0	
(4)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 2 0 1 2	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(6)\times SU(2)\times U(1)_2$	0 1 0 2 4 0 2 4	$a_2=\frac{1}{48}q_1-\frac{1}{16}(q_2+q_3)$
$SU(2)\to U(1)_3$	0 0 0 3 6 0 3 6	$a_3=\frac{1}{2}(q_2-q_3)$
$SU(6)\to SU(5)\times U(1)_4$	0 0 1 4 8 0 4 3	$a_4=-\frac{1}{6}q_1$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 5 10 0 5 0	$a_5=-\frac{1}{6}$
	0 0 0 6 12 0 0 0	
	0 0 0 7 6 1 0 0	
	0 0 0 8 0 0 0 0	
(5)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 3 0 0 2	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(5)\times SU(3)\times U(1)_2$	0 1 0 2 6 0 0 4	$a_2=-\frac{1}{24}(q_1+q_2+q_3)$
$SU(3)\to SU(2)\times U(1)_3$	0 0 0 3 9 0 0 6	$a_3=\frac{1}{6}(q_1+q_2)-\frac{1}{3}q_3$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_4$	0 0 1 4 12 0 0 3	$a_4=\frac{1}{2}(q_1-q_2)$
$SU(2)\to U(1)_5$	0 0 0 5 15 0 0 0	$a_5=-\frac{1}{6}$
	0 0 0 6 10 1 1 0	
	0 0 0 7 5 2 0 0	
	0 0 0 8 0 0 0 0	

TABLE I. (*Continued*).

Symmetry-breaking pattern	Projection matrix	$U(1)_i$ Contributions to Y
(6)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 4 0 0 1	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(4)\times SU(4)\times U(1)_2$	0 1 0 2 8 0 0 2	$a_2=\frac{1}{32}(q_1-q_2-q_3)-\frac{1}{16}$
{ $SU(4)\to SU(3)\times U(1)_3$	0 0 0 3 12 0 0 3	$a_3=\frac{1}{12}(q_2+1)-\frac{1}{4}q_3$
{ $SU(4)\to SU(3)_c\times U(1)_4$	0 0 0 4 16 0 0 0	$a_4=-\frac{1}{3}q_2+\frac{1}{6}$
$SU(3)\to SU(2)_W\times U(1)_5$	0 0 1 5 12 1 1 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 0 6 8 2 2 0	
	0 0 0 7 4 3 0 0	
	0 0 0 8 0 0 0 0	
(7)		
$SU(9)\to SU(8)\times U(1)_1$	1 0 0 1 4 0 0 1	$a_1=\frac{1}{8}(q_1+q_2+q_3)$
$SU(8)\to SU(4)\times SU(4)\times U(1)_2$	0 1 0 2 8 0 0 2	$a_2=\frac{1}{32}(q_1-q_2-q_3)-\frac{1}{16}$
{ $SU(4)\to SU(2)\times SU(2)_W\times U(1)_3$	0 0 0 3 12 0 0 3	$a_3=-\frac{1}{8}(q_2+q_3-1)$
{ $SU(4)\to SU(3)_c\times U(1)_4$	0 0 0 4 16 0 0 0	$a_4=\frac{1}{2}(q_2-q_3)$
$SU(2)\to U(1)_5$	0 0 1 5 12 2 0 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 0 6 8 4 0 0	
	0 0 0 7 4 2 1 0	
	0 0 0 8 0 0 0 0	
(8)		
$SU(9)\to SU(7)\times SU(2)\times U(1)_1$	1 0 0 2 0 1 1 2	$a_1=\frac{1}{14}(q_1+q_2)$
{ $SU(2)\to U(1)_2$	0 1 0 4 0 2 2 4	$a_2=\frac{1}{2}(q_1+q_2)+q_3$
{ $SU(7)\to SU(6)\times U(1)_3$	0 0 0 6 0 3 3 6	$a_3=\frac{1}{42}q_1-\frac{1}{7}$
$SU(6)\to SU(5)\times U(1)_4$	0 0 1 8 0 4 4 3	$a_4=-\frac{1}{6}q_1$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 10 0 5 5 0	$a_5=-\frac{1}{6}$
	0 0 0 12 0 6 0 0	
	0 0 0 14 0 0 0 0	
	0 0 0 7 1 0 0 0	
(9)		
$SU(9)\to SU(7)\times SU(2)\times U(1)_1$	1 0 0 2 0 2 0 2	$a_1=\frac{1}{14}(q_1+q_2)$
{ $SU(2)\to U(1)_2$	0 1 0 4 0 4 0 4	$a_2=\frac{1}{2}(q_1+q_2)+q_3$
{ $SU(7)\to SU(5)\times SU(2)\times U(1)_3$	0 0 0 6 0 6 0 6	$a_3=-\frac{1}{14}(q_1+q_2)$
{ $SU(2)\to U(1)_4$	0 0 1 8 0 8 0 3	$a_4=\frac{1}{2}(q_1-q_2)$
{ $SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 10 0 10 0 0	$a_5=-\frac{1}{6}$
	0 0 0 12 0 5 1 0	
	0 0 0 14 0 0 0 0	
	0 0 0 7 1 0 0 0	
(10)		
$SU(9)\to SU(7)\times SU(2)\times U(1)_1$	1 0 0 2 0 3 0 1	$a_1=\frac{1}{14}(q_1+q_2)$
{ $SU(2)\to U(1)_2$	0 1 0 4 0 6 0 2	$a_2=\frac{1}{2}(q_1+q_2)+q_3$
{ $SU(7)\to SU(4)\times SU(3)\times U(1)_3$	0 0 0 6 0 9 0 3	$a_3=\frac{1}{28}q_1-\frac{1}{2}q_2-\frac{1}{12}$
{ $SU(3)\to SU(2)_W\times U(1)_4$	0 0 0 8 0 12 0 0	$a_4=-\frac{1}{3}q_2+\frac{1}{6}$
{ $SU(4)\to SU(3)_c\times U(1)_5$	0 0 1 10 0 8 1 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 0 12 0 4 2 0	
	0 0 0 14 0 0 0 0	
	0 0 0 7 1 0 0 0	

TABLE I. (*Continued*).

Symmetry-breaking pattern	Projection matrix	$U(1)_i$ Contributions to Y
(11)		
$SU(9)\to SU(6)\times SU(3)\times U(1)_1$	1 0 0 3 0 0 1 2	$a_1=\frac{1}{18}q_1$
$SU(3)\to SU(2)\times U(1)_2$	0 1 0 6 0 0 2 4	$a_2=\frac{1}{3}q_1+\frac{1}{2}(q_2+q_3)$
$SU(6)\to SU(5)\times U(1)_2$	0 0 0 9 0 0 3 6	$a_3=\frac{1}{2}(q_2-q_3)$
$SU(2)\to U(1)_4$	0 0 1 12 0 0 4 3	$a_4=-\frac{1}{6}q_1$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_5$	0 0 0 15 0 0 5 0	$a_5=-\frac{1}{6}$
	0 0 0 18 0 0 0 0	
	0 0 0 12 1 1 0 0	
	0 0 0 6 2 0 0 0	
(12)		
$SU(9)\to SU(5)\times SU(4)\times U(1)_1$	1 0 0 4 0 0 0 2	$a_1=-\frac{1}{3}q_1$
$SU(4)\to SU(3)\times U(1)_2$	0 1 0 8 0 0 0 4	$a_2=\frac{1}{3}(q_1+q_2+q_3)$
$SU(5)\to SU(3)_c\times SU(2)_W\times U(1)_3$	0 0 0 12 0 0 0 6	$a_3=\frac{1}{6}(q_1+q_2)-\frac{1}{3}q_3$
$SU(3)\to SU(2)\times U(1)_4$	0 0 1 16 0 0 0 3	$a_4=\frac{1}{2}(q_1-q_2)$
$SU(2)\to U(1)_5$	0 0 0 20 0 0 0 0	$a_5=-\frac{1}{6}$
	0 0 0 15 1 1 1 0	
	0 0 0 10 2 2 0 0	
	0 0 0 5 3 0 0 0	
(13)		
$SU(9)\to SU(5)\times SU(4)\times U(1)_1$	1 0 0 4 0 0 1 1	$a_1=\frac{1}{20}(q_1+q_2-1)$
$SU(4)\to SU(3)\times U(1)_2$	0 1 0 8 0 0 2 2	$a_2=\frac{1}{4}(q_1+q_2)+\frac{1}{3}q_3+\frac{1}{12}$
$SU(5)\to SU(4)\times U(1)_4$	0 0 0 12 0 0 3 3	$a_3=-\frac{1}{3}q_3+\frac{1}{6}$
$SU(3)\to SU(2)_W\times U(1)_3$	0 0 0 16 0 0 4 0	$a_4=\frac{1}{20}q_1-\frac{1}{5}q_2-\frac{1}{20}$
$SU(4)\to SU(3)_c\times U(1)_5$	0 0 0 20 0 0 0 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 1 15 1 1 0 0	
	0 0 0 10 2 2 0 0	
	0 0 0 5 3 0 0 0	
(14)		
$SU(9)\to SU(5)\times SU(4)\times U(1)_1$	1 0 0 4 0 0 1 1	$a_1=\frac{1}{20}(q_1+q_2-1)$
$SU(4)\to SU(2)\times SU(2)_W\times U(1)_2$	0 1 0 8 0 0 2 2	$a_2=\frac{1}{8}(q_1+q_2+1)$
$SU(5)\to SU(4)\times U(1)_4$	0 0 0 12 0 0 3 3	$a_3=\frac{1}{2}(q_1+q_2)+q_3$
$SU(2)\to U(1)_3$	0 0 0 16 0 0 4 0	$a_4=\frac{1}{20}(q_1-1)-\frac{1}{5}q_2$
$SU(4)\to SU(3)_c\times U(1)_5$	0 0 0 20 0 0 0 0	$a_5=-\frac{1}{4}q_1-\frac{1}{12}$
	0 0 1 15 2 0 0 0	
	0 0 0 10 4 0 0 0	
	0 0 0 5 2 1 0 0	
(15)		
$SU(9)\to SU(5)\times SU(4)\times U(1)_1$	1 0 0 4 0 0 2 0	$a_1=\frac{1}{20}(q_1+q_2-1)$
$SU(4)\to SU(3)\times U(1)_2$	0 1 0 8 0 0 4 0	$a_2=\frac{1}{4}(q_1+q_2)+\frac{1}{3}q_3+\frac{1}{12}$
$SU(5)\to SU(3)_c\times SU(2)\times U(1)_4$	0 0 0 12 0 0 6 0	$a_3=-\frac{1}{3}q_3+\frac{1}{6}$
$SU(3)\to SU(2)_W\times U(1)_3$	0 0 0 16 0 0 3 1	$a_4=-\frac{1}{10}(q_1+q_2)-\frac{1}{15}$
$SU(2)\to U(1)_5$	0 0 0 20 0 0 0 0	$a_5=\frac{1}{2}(q_1-q_2)$
	0 0 1 15 1 1 0 0	
	0 0 0 10 2 2 0 0	
	0 0 0 5 3 0 0 0	

TABLE I. (*Continued*).

Symmetry-breaking pattern	Projection matrix	$U(1)_i$ Contributions to Y
(16)		
$SU(9) \to SU(5) \times SU(4) \times U(1)_1$	1 0 0 4 0 0 2 0	$a_1 = \frac{1}{20}(q_1+q_2-1)$
$SU(4) \to SU(2)_W \times SU(2) \times (1)_2$	0 1 0 8 0 0 4 0	$a_2 = \frac{1}{8}(q_1+q_2+1)$
$SU(5) \to SU(3)_c \times SU(2) \times U(1)_4$	0 0 0 12 0 0 6 0	$a_3 = \frac{1}{2}(q_1+q_2)+q_3$
$SU(2) \to U(1)_3$	0 0 0 16 0 0 3 1	$a_4 = -\frac{1}{10}(q_1+q_2)-\frac{1}{15}$
$SU(2) \to U(1)_5$	0 0 0 20 0 0 0 0	$a_5 = \frac{1}{2}(q_1-q_2)$
	0 0 1 15 2 0 0 0	
	0 0 0 10 4 0 0 0	
	0 0 0 5 2 1 0 0	
(17)		
$SU(9) \to SU(5) \times SU(4) \times U(1)_1$	0 0 1 4 0 1 1 1	$a_1 = \frac{1}{20}(q_1+q_2+q_3+1)$
$SU(4) \to SU(3)_c \times U(1)_2$	0 0 0 8 0 2 2 2	$a_2 = \frac{1}{4}(q_1+q_2+q_3)-\frac{1}{12}$
$SU(5) \to SU(4) \times U(1)_3$	0 0 0 12 0 3 3 0	$a_3 = \frac{1}{20}(q_1+q_2+1)-\frac{1}{5}q_3$
$SU(4) \to SU(3) \times U(1)_4$	0 0 0 16 0 4 0 0	$a_4 = \frac{1}{12}(q_1+1)-\frac{1}{4}q_2$
$SU(3) \to SU(2)_W \times U(1)_5$	0 0 0 20 0 0 0 0	$a_5 = -\frac{1}{3}q_1+\frac{1}{6}$
	1 0 0 15 1 0 0 0	
	0 1 0 10 2 0 0 0	
	0 0 0 5 3 0 0 0	
(18)		
$SU(9) \to SU(5) \times SU(4) \times U(1)_1$	0 0 1 4 0 1 2 0	$a_1 = \frac{1}{20}(q_1+q_2+q_3+1)$
$SU(4) \to SU(3)_c \times U(1)_2$	0 0 0 8 0 2 4 0	$a_2 = \frac{1}{4}(q_1+q_2+q_3)-\frac{1}{12}$
$SU(5) \to SU(4) \times U(1)_3$	0 0 0 12 0 3 2 1	$a_3 = \frac{1}{20}(q_1+q_2+1)-\frac{1}{5}q_3$
$SU(4) \to SU(2)_W \times SU(2) \times U(1)_4$	0 0 0 16 0 4 0 0	$a_4 = -\frac{1}{8}(q_1+q_2-1)$
$SU(2) \to U(1)_5$	0 0 0 20 0 0 0 0	$a_5 = \frac{1}{2}(q_1-q_2)$
	1 0 0 15 1 0 0 0	
	0 1 0 10 2 0 0 0	
	0 0 0 5 3 0 0 0	
(19)		
$SU(9) \to SU(5) \times SU(4) \times U(1)_1$	0 0 0 4 0 2 1 1	$a_1 = \frac{1}{20}(q_1+q_2+q_3+1)$
$SU(4) \to SU(3)_c \times U(1)_2$	0 0 0 8 0 4 2 0	$a_2 = \frac{1}{4}(q_1+q_2+q_3)-\frac{1}{12}$
$SU(5) \to SU(3) \times SU(2)_W \times U(1)_3$	0 0 0 12 0 6 0 0	$a_3 = \frac{1}{15}(q_1+q_2+q_3)-\frac{1}{10}$
$SU(3) \to SU(2) \times U(1)_4$	0 0 1 16 0 3 0 0	$a_4 = \frac{1}{6}(q_1+q_2)-\frac{1}{3}q_3$
$SU(2) \to U(1)_5$	0 0 0 20 0 0 0 0	$a_5 = \frac{1}{2}(q_1-q_2)$
	1 0 0 15 1 0 0 0	
	0 1 0 10 2 0 0 0	
	0 0 0 5 3 0 0 0	
(20)		
$SU(9) \to SU(5) \times SU(4) \times U(1)_1$	0 0 1 4 0 2 0 1	$a_1 = \frac{1}{20}(q_1+q_2+q_3+1)$
$SU(4) \to SU(3)_c \times U(1)_2$	0 0 0 8 0 4 0 2	$a_2 = \frac{1}{4}(q_1+q_2+q_3)-\frac{1}{12}$
$SU(5) \to SU(3) \times SU(2) \times U(1)_3$	0 0 0 12 0 6 0 0	$a_3 = \frac{1}{15}(q_1+1)-\frac{1}{10}(q_2+q_3)$
$SU(2) \to U(1)_4$	0 0 0 16 0 3 1 0	$a_4 = \frac{1}{2}(q_2-q_3)$
$SU(3) \to SU(2)_W \times U(1)_5$	0 0 0 20 0 0 0 0	$a_5 = -\frac{1}{3}q_1+\frac{1}{6}$
	1 0 0 15 1 0 0 0	
	0 1 0 10 2 0 0 0	
	0 0 0 5 3 0 0 0	

where α_1, α_2, and α_3 denote $U(1)_Y$, $SU_W(2)$, and $SU_c(3)$ coupling strengths, respectively, and

$$d_1^3=b_5,\ d_1^2=b_4,\ d_1^1=b_5P_{1,5}^1+b_4P_{1,4}^1+b_1P_{1,A}^1\ ,$$
$$d_2^3=b_4,\ d_2^2=b_3\ ,$$
$$d_2^1=b_4P_{1,4}^2+b_3P_{1,3}^2+b_1(P_{1,A}^2+P_{1,B}^2+P_{1,C}^2)\ ,$$
$$d_3^3=b_3,\ d_3^2=b_2\ ,$$
$$d_3^1=b_3P_{1,3}^3+b_2P_{1,2}^3+b_1(P_{1,A}^3+P_{1,B}^3+P_{1,C}^3+P_{1,D}^3+P_{1,E}^3)\ .$$

Here $P_{1,i}^j$ denotes the probability that the $U(1)_Y$ subgroup exists in the $SU(i)$ or $U(1)_i$ subgroup in the jth step and b_n is the constant appearing in the one-loop renormalization equation for α_n,

$$\mu\frac{d}{d\mu}\alpha_n=2b_n\alpha_n^2\ . \tag{27}$$

For $SU(n)$, $b_n=-(1/12\pi)(11n-2N_f)$, N_f being the number of fermions. Then it can be easily shown that

$$\frac{\sin^2\theta_W(M_W)}{\alpha_{\rm em}(M_W)}-\frac{1}{\alpha_3(M_W)}=\frac{11}{6\pi}\ln\left[\frac{M_0}{M_W}\right] \tag{28}$$

and

$$\frac{\sin^2\theta_W(M_W)}{\sin^2\theta_W(M_0)}=1+\frac{11}{6\pi}\alpha_{\rm em}(M_W)\cot^2\theta_W(M_0)\times\left[(5P_{1,5}^1+4P_{1,4}^1-4)\ln\frac{M_0}{M_1}+(4P_{1,4}^2+3P_{1,3}^2)-3\ln\frac{M_1}{M_2}-2\ln\frac{M_2}{M_W}\right], \tag{29}$$

where we have used

$$\frac{1}{\alpha_2(M_W)}=\frac{\sin^2\theta_W(M_W)}{\alpha_{\rm em}(M_W)},\quad \frac{1}{\alpha_1(M_W)}=\frac{\cos^2\theta_W(M_W)}{C^2\alpha_{\rm em}(M_W)}\ , \tag{30}$$
$$\cot^2\theta_W(M_0)=C^2=\tfrac{5}{3}+4(a^2+b^2)\ .$$

Furthermore, the various probabilities of finding $U(1)_Y$ in the subgroups of intermediate stages satisfy

$$P_{1,5}^1=P_{1,4}^2+P_{1,C}^2=P_{1,E}^3+P_{1,C}^3\ ,$$
$$P_{1,4}^1=P_{1,3}^2+P_{1,B}^2=P_{1,D}^3+P_{1,B}^3\ , \tag{31}$$
$$P_{1,A}^1=P_{1,A}^2=P_{1,A}^3,\ P_{1,B}^2=P_{1,B}^3,\ P_{1,C}^2=P_{1,C}^3\ ,$$

subject to the condition $P_{1,A}^3+P_{1,B}^3+P_{1,C}^3+P_{1,D}^3+P_{1,E}^3=1$. In order to evaluate the probabilities, note from Eq. (19) that after the third stage of symmetry breaking

$$\mathrm{Tr}\,Y^2=\sum_{i=A}^{E}a_i^2\mathrm{Tr}Y_i^2 \tag{32}$$

so that we identify

$$P_{1,i}^3=a_i^2\mathrm{Tr}Y_i^2/\mathrm{Tr}Y^2=\frac{2a_i^2}{C^2}\mathrm{Tr}Y_i^2 \tag{33}$$

which gives from Table I for $q_1=-q_2=a$ and $q_3=-q_4=b$

$$P_{1,A}^3=\frac{9}{10C^2},\ P_{1,B}^3=\frac{1}{6C^2}(3b+1)^2\ ,$$
$$P_{1,C}^3=\frac{1}{10C^2}(a-4b-1)^2,\ P_{1,D}^3=\frac{1}{3C^2}(1-2b)^2\ , \tag{34}$$
$$P_{1,E}^3=\frac{1}{6C^2}(1+3a)^2\ .$$

One can use Eqs. (28) and (29) to determine M_0, M_1, and M_2 for given values of $\sin^2\theta_W(M_W)$, $\alpha_{\rm em}(M_W)$, and $\alpha_3(M_W)$ and for each choice of the charges q_i. Currently accepted values are

$$\sin^2\theta_W(M_W)=0.217\pm0.014\ ,$$
$$\alpha_{\rm em}(M_W)=(127.7)^{-1}$$

and

$$\alpha_3(M_W)=\left[-2b_3\ln\frac{M_W}{\Lambda_{\overline{\rm MS}}}\right]^{-1}=0.1-0.2$$

for

$$\Lambda_{\overline{\rm MS}}=0.1-0.2\ \mathrm{GeV}\ .$$

It can be seen that there are no solutions for the intermediate mass scales M_1 and M_2 for the models with all $q_i=0$ with the symmetry-breaking mode under consideration. Thus the spinor-type representation Eq. (17) with all $q_i\neq0$ is favored by this gauge hierarchy. In particular, one finds $M_0=5.2\times10^{15}$ GeV, $M_1=4\times10^7$ GeV, and $M_2=2.4\times10^2$ GeV for $\sin^2\theta_W(M_W)=0.215$, $\alpha_3(M_W)=0.12$, and $\alpha_{\rm em}(M_W)=128$, and for the charge assignments $a=1$ and $b=-\frac{1}{3}$. With this M_0, the proton lifetime can easily be larger than 10^{33} yr.

Renormalization corrections for other symmetry-breaking modes can similarly be calculated. For each mode, one can determine the mass scales of symmetry breaking for the choices of the charges q_i. In general, for the models of all $q_i=0$, the symmetry-breaking class (A) is preferred and the proton lifetime is more or less comparable to the prediction of the standard SU(5) model. On the other hand, the symmetry-breaking class (D) favors the charge assignments $q_i\neq0$. Delay in proton decay is easily accomplished by the presence and judicious choice of various mass scales of gauge hierarchy, but there will be many more exotic particles, such as fractionally charged color singlets in this class of model as we will see in the next section.

IV. FERMION CONTENTS AND CONCLUDING REMARKS

We have seen in Sec. II that there are two types of SU(9) representations that can meet all of the constraints for grand unification. The first type embeds the SU(5) fundamental representation into that of SU(9) together with electrically neutral four SU(5) singlets. In this case, the number of fermion generations unified is just the number of 5^*+10 of SU(5) contained in the representation. Equations (15) and (16) are models belonging to this class having three and four generations each. All other fermions are superheavy in this type. The second type embeds the SU(5) fundamental representation together with electrically charged SU(5) singlets into the SU(9) fundamental representation. In this case, the reality constraint under $SU_c(3)\times U_{em}(1)$ implies the same as under SU(5) and the number of fermion generations counts the number of $SU_c(3)\times SU(2)\times U(1)$ contents of Eq. (14) contained in the representation. There is only one model, Eq. (17), belonging to this type, which can unify four fermion generations with the charge assignment $(a,-a,b,-b)$ with $ab\neq 0$ for the four SU(5) singlets. Unlike the first type of representation, Eq. (23) shows that $\sin^2\theta_W$ at the grand-unification scale is much smaller than $\frac{3}{8}$ in this case and should increase upon interpolation to the low-energy region. For the model of Eq. (17), it is essential to go through an intermediate stage of symmetry breaking before arriving at the low-energy symmetry realm. The existence of such an intermediate mass hierarchy is in fact complementary to the concept of an invisible axion. In Sec. III we treated the example of the symmetry-breaking mode (13) in Table I for the choice of $a=1$ and $b=-\frac{1}{3}$ and determined the intermediate mass scales that could explain the low-energy value of $\sin^2\theta_W$ and the prolonged lifetime of the proton at the same time.

We note that the models having a trivial embedding of SU(5) are the anomaly-free combinations allowing repetition of the same irreducible representations, while a nontrivial embedding of SU(5) can be possible only for the anomaly-free combination in which no irreducible representation is repeated. This is consistent with previous speculation[25] based on the study of SU(7) models that satisfied the reality condition under $SU_c(3)$, which is a special case of our constraint (c).

The fermion content of both types of models can be studied from the particle content of Eq. (13) in each case and for the specific assignment of charges q_i. The two types of representations have a rather different fermionic content: while there are no exotic particles with unusual charges in the models of the first type, exotic particles are naturally appearing in the model of Eq. (17) depending on the choice of q_i. Exotic lepton doublets with fractional charges can appear from $(1,2,\frac{1}{2}+q_1)$, $(1,2,-\frac{1}{2}-q_i-q_j)$, and $(1,2,\frac{1}{2}+q_i+q_j+q_k)$ terms, and exotic quarks with integral or unusual charges can be present from $(3,1,-\frac{1}{3}+q_i)$, $(3,1,\frac{2}{3}-q_i)$, $(3^*,2,-\frac{1}{6}-q_i)$, etc., terms. For the choice $q_1=-q_2=a$ and $q_3=-q_4=b$, there are two combinations for which $q_i+q_j=0$ and the 126-dimensional representation of Eq. (13d) can give two more ordinary quark doublets with quantum numbers $(3,1,\frac{1}{6})$ making a total of four quark doublets in Eq. (17), which are complex under $SU_c(3)\times SU(2)\times U(1)$. Similarly, we can see that there are four singlets each of the type $(3^*,1,\frac{1}{3})$ and of the type $(3^*,1,-\frac{2}{3})$ along with $(3,1,-\frac{1}{3}+q_i)$, $(3,1,\frac{2}{3}-q_i)$, $(3,1,\frac{2}{3}+q_i)$, and $(3,1,-\frac{1}{3}-q_i)$. Thus with the usual survival hypothesis, two in each type of singlet are expected to be substantially heavier than the other two if we choose $q_1=-q_2=1$ and $q_3=-q_4=-\frac{1}{3}$. There are six ordinary lepton doublets which are all complex with respect to $SU_c(3)\times SU(2)\times U(1)$. In addition there are many exotic quarks and leptons in Eq. (17) some of which are complex under $SU_c(3)\times SU(2)\times U(1)$ and therefore expected to be light. Experimental confirmation on the existence of fractionally charged leptons and hadrons will determine the usefulness of this model. The complete weight systems of the 9-, 36-, 84-, and 126-dimensional irreducible representations of SU(9) with their $SU_c(3)\times SU(2)\times \prod_{j=1}^{5} U(1)_j$ properties can also be worked out with the aid of projection operators given in Table I.

We have discussed SU(9) models which satisfy a set of six criteria of grand unification. Symmetry-breaking patterns are examined thoroughly along with U(1) eigenvalues and the weak hypercharge generator is determined in each case of symmetry breaking. A case of symmetry-breaking mode is explicitly treated to determine intermediate mass scales that can be responsible for invisible axions while supporting the experimental value of $\sin^2\theta_W$ and prolonging the proton lifetime. Finally, we have given the fermion content of SU(9) models.

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