Time-energy Heisenberg-type relations for nonlinear classical fields

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Two time-energy Heisenberg-type relations are obtained for the solitary waves of two nonlinear Klein-Gordon and Dirac fields. Both relations define an "ultraquantum" region for the studied models. Such a region is characterized by actions far smaller than $\hbar$ as well as by large values of the self-coupling constant.

I. INTRODUCTION

In previous works[1-4] the compatibility of the position-momentum uncertainty Heisenberg relation, in the Lieb form,[5] and the nonlinear classical theories of extended particles were studied. It was found that the Heisenberg relation also is valid for the nonlinear classical scalar fields even if the interpretation cannot be the same as in the linear theory. Nevertheless, for the solitary waves (SW's) of two nonlinear Dirac fields the position-momentum relation is violated for large enough values of the self-coupling constant. This violation defines an "ultraquantum" region in which $\hbar$ is larger than a quantity of the order of the norm of the SW.

In this paper, we discuss two time-energy (TE) Heisenberg-type relations for the SW's of two nonlinear Klein-Gordon and Dirac fields. Both relations define an "ultraquantum" region for the studied models. It is found that such a region can be characterized by actions smaller than $\hbar$ as well as by large values of the nonlinear constant, in agreement with the previous results in which the position-momentum relation is examined.

In Sec. II, the nonlinear scalar field is studied while the nonlinear spinor field is considered in Sec. III.

II. A NONLINEAR SCALAR FIELD

Let us consider the classical nonlinear scalar field described by the Lagrangian density

$$\mathscr{L}=\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)\left(\frac{\partial\phi^*}{\partial t}\right) -(\nabla\phi)\cdot(\nabla\phi^*)-\frac{m^2c^2}{\hbar^2}\phi\phi^*+\frac{\alpha}{2\hbar c}(\phi^*\phi)^2 \quad , \tag{1}$$

where $\alpha>0$ is the dimensionless self-coupling constant.

The field equation is

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\Delta\phi+\frac{m^2c^2}{\hbar^2}\phi-\frac{\alpha}{\hbar c}(\phi^*\phi)\phi=0 \quad . \tag{2}$$

This equation admits[6,7] spherically symmetric SW's

$$\phi=e^{-i(\omega/\hbar)t}\phi(r) \quad , \tag{3}$$

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}-\left(\frac{m^2c^2}{\hbar^2}-\frac{\omega^2}{\hbar^2c^2}\right)\phi+\frac{\alpha}{\hbar c}\phi^3=0 \quad , \tag{4}$$

where ω is a free parameter, such that $0\leq\omega^2<m^2c^4$ in order to guarantee the existence of SW's.

Making the transformation

$$r=\rho/\mu,\quad \phi=\mu\left(\frac{\hbar c}{\alpha}\right)^{1/2}\varphi \quad \mu=\frac{1}{\hbar c}(m^2c^4-\omega^2)^{1/2} \quad , \tag{5}$$

we obtain the dimensionless equation

$$\frac{d^2\varphi}{d\rho^2}+\frac{2}{\rho}\frac{d\varphi}{d\rho}-\varphi+\varphi^3=0 \quad . \tag{6}$$

The SW's associated to (1) satisfy the following integral relations:[8]

$$I=4\pi\int_0^\infty\rho^2\varphi^2d\rho=\frac{4\pi}{3}\int_0^\infty\rho^2\left(\frac{d\varphi}{d\rho}\right)^2d\rho=\pi\int_0^\infty\rho^2\varphi^4d\rho \quad . \tag{7}$$

By using (7) the energy of the SW can be written as

$$E=\frac{2I}{\alpha[1-(\omega/mc^2)^2]^{1/2}}mc^2 \quad . \tag{8}$$

In order to establish the TE relations, we define two characteristic times associated with every SW.

(a) For a SW we can define a characteristic action S as

$$S=-\int_0^{\hbar/\omega}dt\int\mathscr{L}d^3x=-\frac{\hbar}{\omega}\int\mathscr{L}d^3x \quad , \tag{9}$$

and by using the relation (7) we get

$$S=\left\{\frac{2Imc^2}{\alpha\omega}\left[1-\left(\frac{\omega}{mc^2}\right)^2\right]^{1/2}\right\}\hbar \quad . \tag{10}$$

In this context we may interpret

$$\tau_S=\frac{\hbar}{\omega} \tag{11}$$

as a possible characteristic time of the SW. Thus, we are able to define the first TE relation

$$\tau_SE=\left(\frac{2Imc^2}{\alpha\omega[1-(\omega/mc^2)^2]^{1/2}}\right)\hbar \quad . \tag{12}$$

For a given value of the self-coupling constant α, the minimum value of the product τ_SE occurs for $\omega=mc^2/\sqrt{2}$, i.e.,

$$(\tau_SE)_{\min}=\frac{4I}{\alpha}\hbar \quad . \tag{13}$$

According to this fact we can define an "ultraquantum" region for large enough values of the self-coupling constant α. From (10) we also obtain that in such a region the action S is smaller than $\hbar$.

(b) There is another characteristic time which can be associated to a SW:

$$\tau_R = \frac{R}{c} \ , \tag{14}$$

where R is the root of the mean-square-radius value of the SW at rest.

For the SW's of the model (1) it has been proved[8] that its radius R has a lower bound

$$R_0 = \frac{1}{\sqrt{12}[1-(\omega/mc^2)^2]^{1/2}} \frac{\hbar}{mc} \ . \tag{15}$$

So we can define the characteristic time

$$\tau_R = \frac{R_0}{c} = \frac{1}{\sqrt{12}[1-(\omega/mc^2)^2]^{1/2}} \frac{\hbar}{mc^2} \ , \tag{16}$$

and the second TE relation is

$$\tau_R E = \left(\frac{I}{\sqrt{3}} \frac{1}{\alpha[1-(\omega/mc^2)^2]} \right) \hbar \ . \tag{17}$$

For fixed α, the minimum value of the product $\tau_R E$ occurs for $\omega = 0$, i.e.,

$$(\tau_R E)_{\min} = \frac{I}{\sqrt{3}\alpha} \hbar \ . \tag{18}$$

As before, we can define the "ultraquantum" region for large values of α.

III. A NONLINEAR DIRAC FIELD

Let us study the TE relations for the classical nonlinear Dirac field described by the Lagrangian density[9,10]

$$\mathcal{L} = \frac{i}{2c}[\bar{\psi}\gamma^0\partial_t\psi - (\partial_t\bar{\psi})\gamma^0\psi] + \frac{i}{2}[\bar{\psi}\gamma^K\partial_K\psi - (\partial_K\bar{\psi})\gamma^K\psi] - \frac{mc}{\hbar}\bar{\psi}\psi + \frac{\lambda\hbar}{m^2c^3}(\bar{\psi}\psi)^2 \ , \tag{19}$$

where $\lambda > 0$ is the dimensionless self-coupling constant.

The field equation is

$$\frac{i}{c}\gamma^0\partial_t\psi + i\gamma^K\partial_K\psi - \frac{mc}{\hbar}\psi + \frac{2\lambda\hbar}{m^2c^3}(\bar{\psi}\psi)\psi = 0 \ . \tag{20}$$

This equation admits SW's of the form

$$\psi = e^{-i(\omega/\hbar)t} \left(\frac{m^3c^4}{2\lambda\hbar^2} \right)^{1/2} \begin{pmatrix} G(\rho) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ iF(\rho) \begin{pmatrix} \cos\theta \\ \sin\theta e^{-\varphi} \end{pmatrix} \end{pmatrix} \ , \tag{21}$$

$$\rho = \frac{mc}{\hbar} r \ .$$

The existence of such solutions has been proved recently by Cazenave and Vázquez,[11] while they have been calculated numerically by Soler,[10] finding that their energy has a minimum for $\omega = 0.936mc^2$ (ground state). The energy of the SW's is

$$E = mc^2 \frac{2\pi}{\lambda} \left[\frac{\omega}{mc^2} \int_0^\infty (F^2+G^2)\rho^2 d\rho + \frac{1}{2} \int_0^\infty (F^2-G^2)^2 \rho^2 d\rho \right] . \tag{22}$$

Let us study the TE relations for the ground state. Numerically we have

$$E = \frac{23.59}{\lambda} mc^2 \tag{23}$$

and the mean-square radius is

$$R_0 = \frac{\hbar}{mc} \left(\frac{\int_0^\infty (F^2+G^2)\rho^4 d\rho}{\int_0^\infty (F^2+G^2)\rho^2 d\rho} \right)^{1/2} = 3.26 \frac{\hbar}{mc} \ . \tag{24}$$

On the other hand, as in the scalar field, we can define the characteristic action S:

$$S = -\frac{\hbar}{\omega} \int \mathcal{L} d^3x \ .$$

By using the field equation (20) we get $\mathcal{L} = -\lambda(\bar{\psi}\psi)^2$. Thus, we obtain

$$S = 4\pi \frac{\hbar mc^2}{\lambda\omega} \int \rho^2 (F^2-G^2)^2 \rho^2 d\rho \ , \tag{25}$$

and for the ground state, numerically,

$$S = \frac{8.90}{\lambda} \hbar \ . \tag{26}$$

With these numerical estimations we can establish the following TE relations for the ground state of (20).

(i) The characteristic time associated to the action is

$$\tau_S = \frac{\hbar}{0.936mc^2} = 1.07 \frac{\hbar}{mc^2} \ . \tag{27}$$

Thus, we obtain

$$\tau_S E = \frac{25.20}{\lambda} \hbar \ . \tag{28}$$

(ii) From the width of the ground state we can define the characteristic time

$$\tau_F = 3.26 \frac{\hbar}{mc^2} \ , \tag{29}$$

so we get

$$\tau_R E = \frac{76.90}{\lambda} \hbar \ . \tag{30}$$

As we can see, both TE relations define an "ultraquantum" region for large values of the self-coupling constant λ, as well as for the nonlinear scalar field.

Finally, we must stress that from the analysis of the position-momentum Heisenberg relation for the ground state[1] it follows that the "ultraquantum" behavior takes place for $\lambda > 5.07$, which is consistent with the "ultraquantum" region obtained from the two TE relations (28) and (30).

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