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Brief Reports

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Finite-lattice-spacing corrections to masses and g factors on a lattice

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We suggest an alternative method for extracting masses and g factors from lattice calculations. Our method takes account of more of the infrared and ultraviolet lattice effects. It leads to more reasonable results in simulations of QED on a lattice.

The study of QED on the lattice has led us to reexamine methods for extracting physical quantities from lattice calculations. We will focus on masses and g factors. The meaning of the mass, in particular, in lattice calculations is of great significance because considerable effort is being devoted to extracting mass estimates from QCD calculations.[1] In all these calculations, the mass is obtained by fitting the observed large-time behavior of the space-averaged Euclidean propagator to the form e^{-mt} expected from the continuum propagator.

We believe that this method can be improved in two ways. First, rather than fitting to the continuum behavior, we suggest that the fit be made to the free lattice propagator. This has the hope of incorporating at least some of the infrared and ultraviolet effects due to finite size and finite lattice spacing. Second, we do not believe that, on the lattice, the coefficient of the exponential falloff is the most natural definition of the mass. We are motivated by trying to understand theories that we think of mainly perburbatively, such as QED. The point is that in a perturbative theory the physical mass should approach the bare mass as the coupling goes to zero. But the bare mass is the mass parameter in the Lagrangian.

To illustrate the difference between the bare mass and the rate of falloff of the propagator, consider a spinless particle with free Lagrangian

$$L=\tfrac{1}{2}\phi(\Box-m^2)\phi \ . \tag{1}$$

The simplest lattice version of this is, assuming unit lattice spacing,

$$L=\tfrac{1}{2}\sum_{x,\mu}\phi_x[(\phi_{x+\mu}+\phi_{x-\mu}-2\phi_x)]-\tfrac{1}{2}m^2\sum_x\phi_x\phi_x \ . \tag{2}$$

It is straightforward to compute the space-averaged free lattice propagator. On an infinite lattice it does fall off exponentially in time, but the rate of falloff is e^{-m^*t}, where m^* is related to m by (m^* is really the location of the pole of the propagator in the energy plane)

$$m=2\sinh(m^*/2) \ . \tag{3}$$

This means that if we define the mass by looking at the falloff of the propagator, the mass will be m^*, which differs from the bare mass m even in the absence of interactions. Of course, in the continuum limit, both definitions agree and the disagreement is due to ultraviolet lattice effects. But many lattice calculations are done with the mass parameter m^* of order 1. For bosons the discrepancy between m and m^* in this case is still quite small since $2\sinh(1/2)$ differs from 1 by only 4%.

For fermions the discrepancy is much more serious. Suppose we start with the free Wilson action,[2]

$$S=\frac{i}{2}\left(\sum_{x,\mu}\bar{\psi}_x[(\gamma_\mu-r)\psi_{x+\mu}-(\gamma_\mu+r)\psi_{x-\mu}]+\sum_x(2m+8r)\bar{\psi}_x\psi_x\right) \ . \tag{4}$$

Again, the free propagator is easily computed, and its space-averaged part falls off like e^{-m^*t}, where now m^* is related to m by

$$m=\left(\frac{1+r}{2}\right)e^{m^*}-\left(\frac{1-r}{2}\right)e^{-m^*}-r \ . \tag{5}$$

As Wilson's r parameter varies from -1 to 1, the relation between m and m^* varies between

$$m=1-e^{-m^*} \tag{6}$$

and

$$m=e^{m^*}-1 \ . \tag{7}$$

Again, both formulas agree in the continuum limit. But the ultraviolet effects are much more severe. For example, for $m^*=1$, m varies between 0.63 and 1.7. In QCD calculations, such values of m^* are not at all unusual.

To see the practical effects of our proposals, we analyzed some QCD data which we had available.[3] The space-averaged proton propagator, both upper and lower components, are given for time separations $4n$, n ranging from 0 to 14 on a lattice of dimension $16^3\times 56$. To avoid the effects of the progagation of the negative-parity partner, we restricted ourselves to time separations less than 28. To avoid short-range effects, we considered only time separations of at least 8. The remaining five data points were fit, in a least-squares sense, to a free fermion propagator with m ranging from 0 to 1.5 and r from -1 to 1. We intended to find a best value for m and r, and secretly hoped that the resulting m might be lower than m^*, which in this case was around 0.6. This might alleviate the problem of the anomalously high value of proton-to-ρ mass ratio found in recent QCD calculations.[1] What we found instead was that the χ^2 was completely flat (and acceptable) for a very wide range of m values. As r varied, the "best" value of m ranged from 0.5 to 1.5, and there seemed no way to prefer any one of these solutions. So adopting the philosophy outlined above, we cannot extract an unambiguous prediction for the proton mass from that QCD data. This problem does not occur in the QED data which we shall analyze below.

There are several criticisms of our philosophy. First, one can say that the discrepancy between m and m^* is purely a lattice effect which will disappear in the continuum limit, so who cares? We concede, but this view precludes any meaningful lattice calculations. Second, one could define the coefficient of $\bar{\psi}\psi$ in the Lagrangian to be the inverse of

$$\ln\left[\frac{1-r}{(m^2+2mr+1)^{1/2}-m-r}\right] \tag{8}$$

instead of m, so that m and m^* would agree. This is possible, but looks contrived. One might also argue that in QCD, at least, there is no "bare" mass for the proton, so the idea of fitting a free lattice propagator to the full propagator makes no sense. But by going to large distances, one is trying to extract the one-particle piece of the full propagator, so why not match it to a latticized one-particle piece?

Our results are more interesting when we turn to the g factor, and grow out of our study of QED on a lattice. We used the quenched approximation, in which the coupling constant does not run. Here the zero-lattice-spacing limit is clean, at least in perturbation theory.[4]

We used the noncompact formalism of QED for the photon fields, i.e., $F_{\mu\nu}F^{\mu\nu}$, rather than the compact formalism of the lattice gauge theories (the trace of the product of link variables). The advantage of doing this is that in the noncompact formalism in the quenched approximation, the distribution of the virtual photon fields in momentum space is Gaussian.[5] Therefore, we can use the Gaussian distribution to set up the equilibrium configuration for the link fields. Then, for each link configuration, we found the electron propagator by inverting the huge fermion matrix by the conjugate gradient method.[6]

We studied the propagator both in the absence and the presence of an external magnetic field. From the former, we extracted the physical mass and renormalized Wilson parameter r, and from the latter, the g factor. We extracted these using our philosophy rather than just by looking at the exponential falloff of the propagator at large Euclidean time separations. Because the electron can both emit and absorb virtual photons, the mass m and parameter r will get renormalized, and the g factor of the electron will not be exactly equal to 2. The one-particle piece of the full propagator in the absence of an external field should correspond to a free propagator with the physical m and r. The one-particle piece of the full propagator in the presence of an external field should correspond to a modified free propagator in the same external field with the physical m, r, and the appropriate value of g.

To derive such a modified propagator, we recall that in the Dirac equation the g factor of the electron is equal to 2 because the Hamiltonian includes a term $-(e/2m)\boldsymbol{\sigma}\cdot\mathbf{B}$ which describes the interaction energy of the electron spin ($\mathbf{S}=\boldsymbol{\sigma}/2$) with the external field. Therefore, to make the g factor of the electron not equal to 2, we construct a new action which includes the term $(g/2)(e/2m)\boldsymbol{\sigma}\cdot\mathbf{B}$ rather than $(e/2m)\boldsymbol{\sigma}\cdot\mathbf{B}$. Because

$$\frac{g}{2}\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}=\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B}+\frac{g-2}{2}\frac{e}{2m}\boldsymbol{\sigma}\cdot\mathbf{B} \ , \tag{9}$$

and the first term on the right side is included in the usual action, we need only add the extra term $[(g-2)/2]\times(e/2m)\boldsymbol{\sigma}\cdot\mathbf{B}$ to the usual action, which only affects the diagonal elements of the inverse propagator.

To get the renormalized mass, we did a least-squares fit of the large-time, space-averaged, numerically calculated full electron propagator, without an external field, to the analytically derived free lattice propagator of varying mass and parameter r. To get the g factor, we did a similar fit of the numerically calculated full electron propagator in an external field to the modified free lattice propagator with varying g factor in the same external field, but with the renormalized mass and parameter r determined from the above zero-field case.

For the full propagator, the discretized electron action takes the form

$$iS_f=-\sum_{(i,j)}\bar{\psi}_i[M(U^{\text{tot}})]_{ij}\psi_j \ , \tag{10}$$

in which the fermion matrix is

$$[M(U^{\text{tot}})]_{ij}=\frac{a^3}{2}[(\gamma_\mu-r)U^{\mu\,\text{tot}}_{Rij}+(2ma+8r)\delta_{ij} - (\gamma_\mu+r)U^{\mu\,\text{tot}}_{Lij}] \ , \tag{11}$$

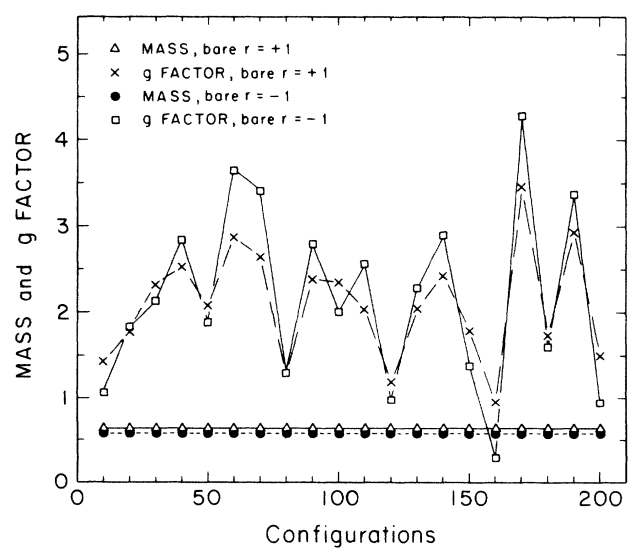

FIG. 1. Scatter of mass and g-factor values obtained in our method by averaging over 10 successive configurations, with bare $r=+1$ and -1.

TABLE I. m, r, ang g obtained by different methods, averaging over the same 200 configurations generated with bare mass $=0.6$. The electron charge has its physical value. Method exp 23 means the exponential method based on time separations of 2 and 3, etc.

Method	Bare r	Output m	Output r	Output g
Our method	1.0000	0.6355 ± 0.0004	1.0029 ± 0.0021	2.08 ± 0.14
exp 23	1.0000	0.4923 ± 0.0002	$\cdots$	0.0051 ± 0.0013
exp 34	1.0000	0.4911 ± 0.0002	$\cdots$	-0.5439 ± 0.0031
exp 45	1.0000	0.4872 ± 0.0005	$\cdots$	-2.5498 ± 0.0098
Our method	-1.0000	0.5757 ± 0.0004	-0.9994 ± 0.0006	2.17 ± 0.23
exp 23	-1.0000	0.8595 ± 0.0006	$\cdots$	1.879 ± 0.012
exp 34	-1.0000	0.8540 ± 0.0009	$\cdots$	-0.063 ± 0.019
exp 45	-1.0000	0.8507 ± 0.0034	$\cdots$	-10.43 ± 0.09

where the γ matrices are in Euclidean space, and (i,j) stands for the space-time indices of the lattice sites, only the nearest neighbors of which are allowed. If $j=i+1$ in the μ direction we take the term $(\gamma_\mu - r)U^{\mu\,\mathrm{tot}}_{Rij}$, while if $j=i-1$ in the μ direction we take the term $-(\gamma_\mu + r)U^{\mu\,\mathrm{tot}}_{Lij}$, and $U_{Rij}=U^\dagger_{Lji}$ is enforced.

In the absence of an external field, the element U^{tot} in (11) is simply given by

$$U^{\mathrm{tot}}(n,n+\mu)=e^{ig_0 aA_\mu(x_n)} , \qquad (12)$$

where n and $n+\mu$ stand for i and j; $A_\mu(x_n)$ is obtained by Fourier transforming the Gaussian distributed photon fields in momentum space, and g_0 is the coupling constant. In the presence of a constant external field B in the z direction, one adds to $A_\mu(x_n)$ an extra term which can be chosen to be $A_2(x)=aBx_1$, so the U^{tot} operators along the y direction should include a factor $e^{ig_0 a^2 Bx_1}$. We preserve the periodicity of U^{tot} and choose a small external field so as to keep the linearity relation between the energy and the external field.

The bosonic fields are periodic along the time direction, while the fermionic fields are antiperiodic. We can choose periodic boundary conditions in space for both bosons and fermions. This will allow us to set the fermion momentum equal to zero.

We worked on a 6^4 lattice, fixed the QED coupling g_0 at $\sqrt{4\pi/137}$ and the external field at 0.02, and investigated Wilson's parameter r at both $+1$ and -1. We picked the mass $m=0.6$, which is within the region $1/N<m<1$. We took the average of the propagators over 200 independent link configurations which is more than in the usual QCD lattice calculations.

We extracted the statistical error as follows: We took the average propagators for each 10 successive individual propagators. Then we extracted the mass m, Wilson parameter r, and the g factor from these 20 average propagators, respectively. Finally, from these 20 values for each m, r, and g, we calculated the uncertainty in the mean of m, r, and g, correspondingly, by using $\sigma_{\mathrm{mean}}=\sigma/\sqrt{20}$, in which σ is the standard deviation of the above 20 values. The results are shown in Fig. 1 and Table I. It is noteworthy that r is essentially unrenormalized in all cases.

The results using our method show large fluctuations in g. The final values (see Table I) are insensitive to whether we form m, r, and g from the average over 200 propagators, or whether we take the average of the 20 values we obtained, each based on 10 propagators. The results are also insensitive to whether, on our lattice with six time slices, we called a time separation of 3 large, or whether we called time separations of ± 2, and 3 large.

The data were also analyzed in the more conventional manner in which we exploit the exponential falloff of the propagators. Even on our small lattice, the exponential behavior is observed. In this case we applied the formula[7]

$$g=2[m_+(E_+-m_+)-m_-(E_--m_-)]/g_0B \qquad (13)$$

for the g factor of the electron, where E_+ (E_-) and m_+ (m_-) are the energy in the external field and renormalized mass of spin-up (spin-down) electrons, respectively. There is no mention of Wilson's parameter r. For both values of bare r, the exponential method shows much smaller fluctuations than the method we are proposing. But it depends critically on whether we use time separations of 2 and 3, 3 and 4, or 4 and 5 to extract the g factor. The point is that E_+ (E_-) is close to m_+ (m_-) and the g factor becomes very sensitive to finite lattice distortions (see Table I). This confirms our view that lattice effects must be considered in the fit to the parameters.

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