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U(1) problem on a lattice: Strong-coupling limit

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We investigate the mass difference between the π and η mesons by using lattice QCD with the Wilson fermion formulation. The calculation of the effective potential is done by the $1/N$ expansion in the strong-coupling limit. From a tree-level analysis of the effective potential we obtain the result that there is no mass difference in the parity-conserving phase to all orders in the $1/N$ expansion and that there exists a mass difference in the parity-violating phase.

I. INTRODUCTION

The U(1) problem is that the η meson (flavor singlet) is much heavier than the π meson (flavor nonsinglet) although both mesons are Nambu-Goldstone bosons associated with the spontaneous breakdown of chiral symmetry. According to our current understanding, the chiral U(1) anomaly would provide the mass difference but it is very difficult to calculate such a mass difference practically since both mesons are bound states of quarks.

Lattice regularization is suitable for calculating such a nonperturbative effect. Wilson claimed[1] that the mass difference between the singlet and the nonsinglet are obtained in the strong-coupling limit by using lattice QCD with the Wilson fermion which gives the correct chiral anomaly in the continuum limit.[2]

In this paper we calculate the mass difference by using lattice QCD with the Wilson fermion of $r=1$ in the strong-coupling limit. The results are summarized as follows.

(1) From a tree-level analysis of the mesonic effective potential no mass difference can be obtained to all orders in the $1/N$ expansion if the parity is conserved ($\langle\bar{\psi}i\gamma_5\psi\rangle=0$).

(2) The parity-violating phase as well as the parity-conserving phase exist. In the parity-violating phase it is shown that the singlet meson is heavier than the nonsinglet meson.

This paper is organized as follows. In Sec. II we formulate the effective potential for mesons from QCD in the strong-coupling limit. A detailed calculation of the effective potential is given in Appendix A. In Sec. III we analyze the vacuum structure for the effective potential by the $1/N$ expansion and show that two phases exist. In Sec. IV we calculate the meson mass from the effective potential by the $1/N$ expansion and show that the above results (1) and (2) are true. In Sec. V we discuss the physical implications of our result. In Appendix B detailed calculations for the meson mass are given. In Appendix C the case that the Wilson parameter $r\neq 1$ is analyzed and it is shown that the result (1) is almost unchanged. In Appendix D we analyze QCD with the U(3) gauge and show that the results are the same.

II. EFFECTIVE POTENTIAL FOR MESONS

In this section using the $1/N$ expansion we define an effective potential for mesons in the strong-coupling limit ($1/g^2\to 0$). Detailed calculations are given in Appendix A.

The action of QCD on a d-dimensional lattice with the lattice spacing a is

$$S=S_F+S_G\ , \tag{2.1}$$

where

$$\begin{aligned}S_F=&\frac{1}{2a}a^4\sum_{n,\mu,f}(\bar{\psi}^f_n U_{n,\mu}\gamma_\mu\psi^f_{n+\hat{\mu}}-\bar{\psi}^f_{n+\hat{\mu}}U^\dagger_{n,\mu}\gamma_\mu\psi^f_n)\\&+a^4\sum_{n,f}M_f\bar{\psi}^f_n\psi^f_n\\&+\frac{r}{2a}a^4\sum_{n,\mu,f}(\bar{\psi}^f_n U_{n,\mu}\psi^f_{n+\hat{\mu}}+\bar{\psi}^f_{n+\hat{\mu}}U^\dagger_{n,\mu}\psi^f_n-2\bar{\psi}^f_n\psi^f_n)\ ,\end{aligned} \tag{2.2}$$

$$S_G=\frac{1}{g^2}\sum_{n,\mu>\nu}(\mathrm{Tr}U_{n,\mu}U_{n+\hat{\mu},\nu}U^\dagger_{n+\hat{\nu},\mu}U^\dagger_{n,\nu}+\mathrm{H.c.})\ .$$

Here we take $U_{n,\mu}$ as the element of U(N) rather than SU(N) since we want to treat only mesons without baryons. The $f(=1,2,\ldots,n_f)$ is a flavor index and M_f is a bare-quark mass of the flavor f. The second term in S_F is called the Wilson term which is necessary to remove a spectral doubling in the continuum limit. This term breaks the chiral symmetry explicitly; therefore, this action has no explicit chiral symmetry even if $M_f=0$.

The partition function Z with source $J^{\hat{\alpha}\hat{\beta}}(n)$ is given by

$$Z(J)=\int D\psi D\bar{\psi}D(U_{n,\mu})\times\exp\left[S_F+S_G+N\sum_n \mathrm{tr}J(n)\cdot M(n)\right]\ , \tag{2.3}$$

where

$$M^{\hat{\alpha}\hat{\beta}}(n)=\frac{a^3}{N}\sum_c(\bar{\psi}_n)^{\hat{\alpha}}_c(\psi_n)^{\hat{\beta}}_c \tag{2.4}$$

is the meson field, $\hat{\alpha},\hat{\beta}$ represent spinor-flavor indices $[\hat{\alpha}=(\alpha,f)]$, and c represents color index. Therefore $M(n)$ is a $C\times C$ matrix with $C=2^{[d/2]}n_f$. In four dimensions

$C=4n_f$. In the strong-coupling limit ($1/g^2 \to 0$) we can neglect the term S_G in (2.3). Correctly speaking, this limit means $1/g^2 N \to 0$.

After integration of the link variables $U_{n,\mu}$ we obtain[3]

$$Z(J)=\int DM(n)\exp\left[S_{\text{eff}}(M)+N\sum_n \operatorname{tr}J(n)\cdot M(n)\right], \tag{2.5}$$

where

$$S_{\text{eff}}(M)=N\sum_n[\operatorname{tr}\hat{M}M(n)-\operatorname{tr}\ln M(n)+W(M(n))] \tag{2.6}$$

is the effective potential for mesons. Here

$$\hat{M}_{\hat\alpha\hat\beta}=(M_f a+4r)\delta_{\alpha\beta}\delta_{ff'},$$

$$W(M(n))=\sum_\mu \sum_{k=0}^{\infty}\left[\frac{1}{N}\right]^k W_k(\Lambda_{n,\mu}),$$

$$\Lambda_{n,\mu}=M(n)(P_\mu^+)^T M(n+\hat\mu)(P_\mu^-)^T,$$

and

$$P_\mu^\pm=\frac{r\pm\gamma_\mu}{2}.$$

The form of $W_k(\Lambda_{n,\mu})$ is given in Appendix A.

III. VACUUM STRUCTURE WITH $r=1$

In this section we investigate the vacuum structure in the $1/N$ expansion with the Wilson parameter $r=1$ and show that two phases exist.

From the result in Sec. II and Appendix A we obtain

$$S_{\text{eff}}(M)=N\sum_n\left[\operatorname{tr}\hat{M}M(n)-\operatorname{tr}\ln M(n)+\sum_\mu\sum_{k=0}^{\infty}\left[\frac{1}{N}\right]^k W_k(\Lambda_{n,\mu})\right]. \tag{3.1}$$

If N becomes large we obtain

$$Z(J=0)=\int DM(n)\exp[S_{\text{eff}}(M)]\underset{N\to\infty}{\sim}\exp[S_{\text{eff}}(M_0)], \tag{3.2}$$

where $M_0(n)$ is the saddle point of $S_{\text{eff}}(M)$ (Ref. 4) and is interpreted as the physical vacuum.

To obtain the saddle point $M_0(n)$ we assume that the vacuum is translationally invariant and a flavor singlet; then the form of $M_0(n)$ is given by

$$M_0(n)^{\hat\alpha\hat\beta}=\sigma(e^{i\theta\gamma_5})_{\alpha\beta}\otimes 1_{ff'}, \tag{3.3}$$

where $\hat\alpha=(\alpha,f)$ and $\hat\beta=(\beta,f')$. Here we take into account the possibility that $\bar\psi_n\gamma_5\psi_n$ may develop a nonzero vacuum expectation value. Hereafter we set the Wilson parameter $r=1$. The case for $0<r<1$ will be considered in Appendix C.

We denote the vacuum expectation value of $f(M)$ which is an arbitrary function of $M(n)$ as

$$\langle f(M)\rangle=f(M_0). \tag{3.4}$$

Using this notation the saddle-point equation is written as

$$\left\langle\frac{\delta S_{\text{eff}}(M)}{\delta M(n)^{\hat\alpha\hat\beta}}\right\rangle=0. \tag{3.5}$$

If all the bare masses of quarks (M_f) are equal[5] Eq. (3.5) has solutions.

In the large-N limit the solution is given as[7]

$$\langle\bar\psi i\gamma_5\psi\rangle\frac{a^3}{4N}=\sigma\sin\theta
=\begin{cases}0 & \text{for } M_0{}^2\ge 4,\\ \dfrac{2[3(4-M_0{}^2)]^{1/2}}{16-M_0{}^2} & \text{for } M_0{}^2\le 4,\end{cases} \tag{3.6}$$

$$\langle\bar\psi\psi\rangle\frac{a^3}{4N}=\sigma\cos\theta
=\begin{cases}1/M_0 & \text{for } M_0{}^2\ge 4,\\ \dfrac{3M_0}{16-M_0{}^2} & \text{for } M_0{}^2\le 4,\end{cases} \tag{3.7}$$

where $M_0=\operatorname{tr}\hat{M}/n_f$ ($=M_f a+4r$ if all M_f are equal). Technical detail is given in Appendix B.

These results suggest that at the value that $M_0{}^2=4$ the phase transition occurs and its order parameter is $\langle\bar\psi i\gamma_5\psi\rangle$ rather than $\langle\bar\psi\psi\rangle$. In other words two phases exist: one is the parity-conserving phase ($\langle\bar\psi i\gamma_5\psi\rangle=0$) and the other is the parity-violating phase. It is easy to understand this phase transition from a statistical mechanical point of view. Now we consider the correlation function of a composite operator $\bar\psi i\gamma_5\psi$, which behaves as

$$\langle\bar\psi_n i\gamma_5\psi_n\bar\psi_0 i\gamma_5\psi_0\rangle\sim\exp(-m_\pi a|n|),\quad |n|\to\infty,$$

where m_π is identified with the lowest mass of the pseudoscalar meson (π meson). We assume the pseudoscalar meson becomes massless at some value of the parameters, for example, at $M_0{}^2=4$. Since this means that the correlation length diverges as $\xi\sim 1/m_\pi a$ the phase transition occurs and there is another phase where $\langle\bar\psi i\gamma_5\psi\rangle\ne 0$ if we vary the parameter. In this phase the mass of the pseudoscalar meson is defined by

$$\langle\bar\psi_n i\gamma_5\psi_n\bar\psi_0 i\gamma_5\psi_0\rangle-\langle\bar\psi_n i\gamma_5\psi_n\rangle\langle\bar\psi_0 i\gamma_5\psi_0\rangle\sim\exp(-m_\pi a|n|),\quad |n|\to\infty.$$

The phase transition which occurs at $M_0{}^2=4$ can be understood by this scenario and in the next section we show that the π meson becomes massless at $M_0{}^2=4$.

Next we consider the $1/N$ corrections to the above result. If we assume $\theta=0$ the solution is

$$\sigma=1/M_0 \tag{3.8}$$

to all orders in the $1/N$ expansion and this solution is the same as the result in the large-N limit [see (3.6) and (3.7)]. If we consider the solution with $\theta\ne 0$ the gap equation becomes

$$M_0\sigma=\cos\theta\ ,$$
$$\frac{\sin\theta}{\sigma}\left[-1+\frac{8\sigma^2}{1+(1-4\sigma^2\sin^2\theta)^{1/2}}+O(\sin^2\theta)\right]=0\ . \tag{3.9}$$

We take the limit as $\theta\to 0^+$ in Eq. (3.9) and obtain

$$\sigma^2=\tfrac{1}{4}\ \text{ and }\ M_0{}^2=4\ . \tag{3.10}$$

From (3.8) and (3.10) we can conclude that "the parameter value of M_0 where the phase transition of the parity violation ($\theta\neq 0$) occurs is the same as the value in the large-N limit ($M_0{}^2=4$) to all orders in the $1/N$ expansion." This result is different from the naive expectation.[8]

Finally we give the explicit form of the solution to the next leading order in the $1/N$ expansion:

$$\langle\bar\psi i\gamma^5\psi\rangle a^3/4N=\sigma\sin\theta=\begin{cases}0 \quad \text{for } M_0{}^2\geq 4\ ,\\ \dfrac{[12(4-M_0{}^2)]^{1/2}}{16-M_0{}^2}+\dfrac{1}{N}\dfrac{7M_0{}^2-6}{2(4-M_0{}^2)^{1/2}(16-M_0{}^2)^{1/2}}\sigma_1 \quad \text{for } M_0{}^2\leq 4\ ,\end{cases} \tag{3.11}$$

$$\langle\bar\psi\psi\rangle a^3/4N=\sigma\cos\theta=\begin{cases}1/M_0 \quad \text{for } M_0{}^2\geq 4\ ,\\ \dfrac{3M_0{}^2}{16-M_0{}^2}-\dfrac{1}{N}\dfrac{2\sqrt{3}M_0}{(16-M_0{}^2)^{1/2}}\sigma_1 \quad \text{for } M_0{}^2\leq 4\ ,\end{cases} \tag{3.12}$$

where

$$\sigma_1=6n_f\frac{8+M_0{}^2}{11M_0{}^2+16}\frac{(16-M_0{}^2)^{1/2}}{\sqrt{3}}\sum_{q_1,q_2}\left[\frac{48}{(16-M_0{}^2)^2}\right]^{q_1+q_2}(4-M_0{}^2)^{q_1+q_2-1}$$
$$\times\left[\sum_{n=0}^{q_2-1}+\sum_{n=0}^{q_1-1}\right]\left[\frac{(2n-1)!!}{(2n)!!}\frac{[2(q_1+q_2-n)-3]!!}{[2(q_1+q_2-n)]!!}\right]\ .$$

We plotted Eqs. (3.11) and (3.12) in the case that $N\to\infty$ and $N=3$ with two flavors in Fig. 1. It is worthwhile noting that the summation in $\sum_{q_1,q_2}$ is restricted in the case of $N=3$, so that

$$q_1+q_2\leq Nn_fC=3\times 2\times 4=24\ ,$$

because

$$(\psi\bar\psi)^{Nn_fC+1}=0$$

by the property of the Grassmann numbers $\bar\psi$ and ψ.

IV. THE MASS SPECTRUM FOR MESONS

In this section we calculate the meson masses by using the effective potential S_{eff}. Since it is very difficult to calculate the meson mass exactly, we calculate it approximately. First we expand S_{eff} around the vacuum $M_0(n)$:

$$S_{\text{eff}}(M(n))=S_{\text{eff}}(M_0(n))+\sum_{k=2}^{\infty}\sum_{n_1,\ldots,n_k}S_{\text{eff}}^{(k)}(n_1,\ldots,n_k)\times\Pi(n_1)\cdots\Pi(n_k), \tag{4.1}$$

where $\Pi(n)=M(n)-M_0(n)$ and

$$S_{\text{eff}}^{(k)}(n_1,n_2,\ldots,n_k)=\frac{1}{k!}\left\langle\frac{\partial^k S_{\text{eff}}}{\partial M(n_1)\partial M(n_2)\cdots\partial M(n_k)}\right\rangle\ .$$

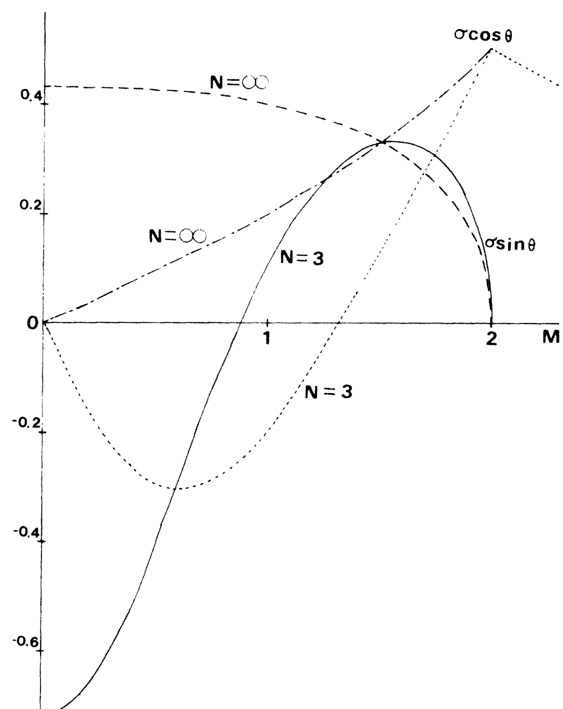

FIG. 1. Dependence of $\sigma\sin\theta$ and $\sigma\cos\theta$ on M solid line (——) represents $\sigma\sin\theta$ and dotted line (. . . .) represents $\sigma\cos\theta$ in the case that $N=3$. Dashed line (— — —) represents $\sigma\sin\theta$ and dash-dotted line (—·—·—·) represents $\sigma\cos\theta$ in the large-N limit.

$S^{(1)}_{\rm eff}(n)$ vanishes because of the saddle-point equation (3.5). We use only $S^{(2)}_{\rm eff}(n,m)$ in order to calculate the meson mass. In other words, we consider the tree diagrams for mesons and neglect the contribution of the meson loops. Rigorously speaking, the meson loops must be considered at the same time for a consistent large-N expansion and the contribution of the loops may be essential for the mass difference. Therefore we will discuss this point in Sec. V.

After some calculations (given in Appendix B) we obtain

$$\sum_{n,m} S^{(2)}_{\rm eff}(n,m)\Pi(n)\Pi(m)$$
$$= n_f \sum_{a=0}^{n_f^2-1} \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \Big[A\,{\rm Tr}\Pi^a(-p)e^{-i\theta\gamma_5}\Pi^a(p)e^{-i\theta\gamma_5} - B\sum_\mu {\rm Tr}[\Pi^a(-p)\gamma_5\Pi^a(p)\gamma_5 + \Pi^a(-p)\gamma_5\gamma_\mu^T\Pi^a(p)\gamma_5\gamma_\mu^T]$$
$$- C\sum_\mu {\rm Tr}\Pi^a(-p)(1+\gamma_\mu^T)\Pi^a(p)(1-\gamma_\mu^T)e^{ip_\mu a}\Big]$$
$$+ n_f \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} D \sum_\mu [{\rm Tr}\Pi^0(-p)i\gamma_5{\rm Tr}\Pi^0(p)i\gamma_5 + {\rm Tr}\Pi^0(-p)i\gamma_5\gamma_\mu^T{\rm Tr}\Pi^0(p)i\gamma_5\gamma_\mu^T$$
$$+ {\rm Tr}\Pi^0(-p)i\gamma_5(1-\gamma_\mu^T){\rm Tr}\Pi^0(p)i\gamma_5(1+\gamma_\mu^T)e^{ip_\mu a}] , \quad (4.2)$$

where

$$\Pi^a(p)_{\hat\alpha\hat\beta} = \sum_n e^{-ip\cdot a\cdot n} \sum_{a=0}^{n_f^2-1} \Pi^a(n)_{\alpha\beta} \times \tau^a_{ff'} ,$$

$\tau^0=1$, $\tau^a(a=1,\ldots,n_f^2-1)$ is the generator of SU(n_f) and ${\rm tr}\tau^a\tau^b=\delta_{ab}n_f$. Tr means the trace over the spinor index. A, B, C, and D are given by

$$A = \frac{1}{2\sigma^2} ,$$
$$B = \frac{\sigma^2\sin^2\theta}{[1+(1-4\sigma^2\sin^2\theta)^{1/2}]^2}\frac{1}{(1-4\sigma^2\sin^2\theta)^{1/2}} - B_1(\sigma^2\sin^2\theta) ,$$
$$C = \frac{1}{4(1-4\sigma^2\sin^2\theta)^{1/2}} - C_1(\sigma^2\sin^2\theta) ,$$
$$D = D_1(\sigma^2\sin^2\theta) ,$$

where B_1, C_1, and D_1 are $O(1/N)$ and satisfy $B_1(0)=C_1(0)=D_1(0)=0$. Their detailed forms are given in Appendix B. Here σ and θ satisfy the saddle-point equation (3.5).

The mass difference between the flavor-singlet meson and the nonsinglet mesons arises from the last term in (4.2). Now we obtain the important result of this paper.

"If $\theta=0$, which is the case for $M_0{}^2\geq 4$, $D(\theta=0)=0$. Then at the tree level of the mesons there is no mass difference between the singlet and the nonsinglet for $\theta=0$, which means that the parity is conserved, to all orders in the $1/N$ expansion." Furthermore, since $B_1(0)=C_1(0)=0$ there is no $1/N$ correction to meson masses for $M_0{}^2\geq 4$ ($\theta=0$).

From now on we look at the case for $M_0{}^2\leq 4$ ($\theta\neq 0$). We expand $\Pi^a(p)$ as

$$\Pi^a(p) = \sum_A \Pi^a_A(p)\Gamma^A , \quad (4.3)$$

where $\{\Gamma^A\}$ is a basis of 4×4 matrices that is given by

$$\Gamma^S = \frac{1}{2}1, \quad \Gamma^P = \frac{1}{2}\gamma_5, \quad \Gamma^{A(\rho)} = \frac{i}{2}\gamma_\rho^T\gamma_5, \quad \Gamma^{V(\rho)} = \frac{1}{2}\gamma_\rho^T ,$$

and

$$\Gamma^{T(\rho_\sigma)} = \frac{1}{2\sqrt{2}i}[\gamma_\rho^T,\gamma_\sigma^T] ,$$

then we obtain

$$\sum_{n,m} S^{(2)}_{\rm eff}(n,m)\Pi(n)\Pi(m) = n_f \sum_{a=0}^{n_f^2-1} \int \frac{d^4p}{(2\pi)^4} \sum_{A,B} \Pi^a_A(-p)D^a_{AB}(p)\Pi^a_B(p) , \quad (4.4)$$

where

$$D^a_{AB}(P)=\begin{array}{c} \\ S\\P\\A\\V\\T\end{array}\begin{array}{c}\begin{array}{cc} SPA & VT\end{array}\\ \left[\begin{array}{cc} D^a_{S-P-A}(P) & 0 \\ & \\ 0 & D^a_{V-T}(P)\end{array}\right]\end{array}, \tag{4.5}$$

$D^a_{V-T}(p)=$

$$\begin{array}{c} \\ V(\alpha) \\ T(\beta\gamma)\end{array}\left[\begin{array}{cc} V(\rho) & T(\mu\nu) \\ \delta_{\alpha\rho}\left[A+6B-2C\sum_{\mu\neq\alpha}\cos p_\mu a\right] & \sqrt{2}C(\delta_{\alpha\mu}\sin p_\nu a-\delta_{\alpha\nu}\sin p_\mu a) \\ -\sqrt{2}C(\delta_{\rho\beta}\sin p_\gamma a-\delta_{\rho\gamma}\sin p_\beta a) & \frac{1}{2}(\delta_{\beta\mu}\delta_{\gamma\nu}-\delta_{\beta\nu}\delta_{\alpha\mu})[A\cos2\theta-4B-2C(\cos p_\beta a+\cos p_\gamma a)]-\frac{ia}{2}e^{\beta\gamma\mu\nu}\sin2\theta\end{array}\right] \tag{4.6}$$

$(a=0,1,\ldots,n_f^2-1)$ for the vector-tensor (VT) sector,[9]

$$D^a_{S-P-A}(p)=\begin{array}{c} \\ S\\P\\A(\rho)\end{array}\left[\begin{array}{ccc} S & P & A(\nu) \\ A\cos2\theta & -iA\sin2\theta & 0 \\ -iA\sin2\theta & A\cos2\theta-8B^a-2C^a\Sigma_\mu\cos p_\mu a & 2C^a\sin p_\nu a \\ 0 & -2C^a\sin p_\rho a & \delta_{\rho\nu}(A+2B^a-2C^a\cos p_\nu a)\end{array}\right],$$

$$B^a=B \text{ and } C^a=C \text{ for } a=1,\ldots,n_f^2-1 \text{ (nonsinglet)}, \tag{4.7}$$

$$B^0=B+2D \text{ and } C^0=C+2D \text{ (singlet)}$$

for the scalar—pseudoscalar—axial-vector (SPA) sector.

The mass difference arises from only the SPA sector. The mixing between scalar and pseudoscalar in (4.7) is the consequence of the parity violation ($\theta\neq0$). Since there is no complete Lorentz symmetry on a lattice the mixing makes it difficult to decide the quantum number of the mesons. Following the ordinary method we identify the SPA sector as the π meson (nonsinglet) or η meson (singlet) and the VT sector as the ρ meson. Then we put $p_0=im_\pi(im_\eta)$, $p_k=0$ for the nonsinglet (singlet) SPA sector and $p_0=im_\rho$, $p_k=0$ for the VT sector into the equation that $\det D^a_{AB}(p)=0$.

In the large-N limit we obtain

$$\cosh m_\pi a=\cosh m_\eta a=\begin{cases} 1+\dfrac{2(4-M_0^2)(16-M_0^2)(8+M_0^2)}{15M_0^4-64M_0^2+256} & \text{for } M_0^2\le4, \\ 1+\dfrac{(M_0^2-4)(M_0^2-1)}{2M_0^2-3} & \text{for } M_0^2\ge4, \end{cases} \tag{4.8}$$

$$\cosh m_\rho a=\begin{cases} 1+\dfrac{(13M_0^4+112M_0^2-512)^2+192M_0^2(4-M_0^2)(8+M_0^2)^2}{6[(25M_0^4+208M_0^2-512)(13M_0^4+112M_0^2-512)+384M_0^2(4-M_0^2)(8+M_0^2)^2]} & \text{for } M_0^2\le4, \\ 1+\dfrac{(M_0^2-3)(M_0^2-2)}{2M_0^2-3} & \text{for } M_0^2\ge4. \end{cases} \tag{4.9}$$

Equation (4.8) shows that the π meson (and η meson) becomes massless at the value of $M_0^2=4$ where the phase transition occurs. The π meson is the massless mode associated with the phase transition of the spontaneous parity violation. As Witten mentioned[10] there is no mass difference in the large-N limit.

Up to the next leading order in the $1/N$ expansion we obtain

$$\cosh m_\pi a=\frac{(A+2B)A+2A\cos2\theta[2C-(A+2B)(4B+3C)]}{4AC[A\cos^2\theta-3\cos2\theta(B+C)]}, \tag{4.10}$$

$$\cosh m_\eta a=\frac{(A+2B')A+2A\cos2\theta[2C'(A+2B')(4B'+3C')]}{4AC'[A\cos^2\theta-3\cos2\theta(B'+C')]}, \tag{4.11}$$

$$\cosh m_\rho a=\frac{(A\cos2\theta-4B-4C)[(A+6B-4C)(A\cos2\theta-4B-2C)+4C^2]+A^2\sin^22\theta(A+6B-4C)}{2C[(A\cos2\theta+A+2B-6C)(A\cos2\theta-4B-4C)+A^2\sin^22\theta]}, \tag{4.12}$$

where

$$A=1/2\sigma^2 ,$$

$$B=\frac{\sigma^2\sin^2\theta}{[1+(1-4\sigma^2\sin^2\theta)^{1/2}]^2}\frac{1}{(1-4\sigma^2\sin^2\theta)^{1/2}}-\frac{n_f}{N}\sum_{q_1q_2}C^{(1)}_{q_1q_2}(-\sigma^2\sin^2\theta)^{q_1+q_2-1}[q_1(q_1-1)+q_2(q_2-1)] ,$$

$$C=\frac{1}{4(1-4\sigma^2\sin^2\theta)^{1/2}}-\frac{n_f}{N}\sum_{q_1q_2}C^{(1)}_{q_1q_2}(-\sigma^2\sin^2\theta)^{q_1+q_2-1}(q_1{}^2+q_2{}^2) ,$$

$$D=\frac{n_f}{N}\sum_{q_1q_2}C^{(1)}_{q_1q_2}(-\sigma^2\sin^2\theta)^{q_1+q_2-1}q_1q_2/2 ,$$

$$B'=B+2D,\quad C'=C+2D ,$$

and

$$C^{(1)}_{q_1q_2}=\frac{-(-4)^{q_1+q_2-1}}{(q_1+q_2)}\left[\sum_{n=0}^{q_1-1}+\sum_{n=0}^{q_2-1}\right]\left[\frac{(2n-1)!!}{(2n)!!}\frac{[2(q_1+q_2-n)-3]!!}{[2(q_1+q_2-n)]!!}\right] .$$

Here σ and θ are given in (3.11) and (3.12).

We plotted m_π, m_η, and m_ρ both in the large-N limit [(4.8) and (4.9)] and in the case of $N=3$ with two flavors [(4.10)–(4.12)] in Fig. 2. In our calculation for 4.10–(4.12) we dropped terms of the order $(1/N)^2$ or more. The oscillatory behavior of the $N=3$ curves in Fig. 2, in particular that m_ρ dips below zero, show that the higher order of $1/N$ is important for $N=3$ in this region of M_0. From Fig. 2 the domain of validity of the $N=3$ case is roughly estimated as

$$M_0{}^2>2.4 .$$

Our main interest is the difference between m_π and m_η, then we calculate it analytically up to the next leading order. The result is

$$\Delta m(\eta-\pi)=m_\eta a-m_\pi a=\frac{1}{N}\frac{d_1 f(M_0{}^2)}{\sinh m_\pi(N=\infty)a} , \tag{4.13}$$

where

$$d_1=-n_f\sum_{q_1q_2}q_1q_2C^{(1)}_{q_1q^2}\left[\frac{12(M_0{}^2-4)}{(16-M_0{}^2)^2}\right]^{q_1+q_2-1}\geq 0 ,$$

$$f(M_0{}^2)=\frac{16(8+M_0{}^2)}{(16-M_0{}^2)(15M_0{}^4-64M_0{}^2+256)}(15M_0{}^8-112M_0{}^6+3488M_0{}^4-16\,384M_0{}^2+24\,576)\geq 0 ,$$

and

$$\sinh m_\pi(N=\infty)a=\frac{[2(4-M_0{}^2)(16-M_0{}^2)(8+M_0{}^2)]^{1/2}}{15M_0{}^4-64M_0{}^2+256}(M_0{}^6+3M_0{}^4-160M_0{}^2+768)^{1/2} .$$

It is easy to check

$$\Delta m(\eta-\pi)>0\quad\text{for } M_0{}^2<4$$

and

$$\lim_{M_0{}^2\to 4}\Delta m(\eta-\pi)=0 .$$

In Fig. 3 we plotted $\Delta m(\eta-\pi)$ in the case of $N=3$ with two flavors.

Before ending this section we mention the case for $0<r^2<1$. The result in Appendix C shows that for $0<r^2<1$ a mass difference exists between the singlet and the nonsinglet for only the scalar meson if $\theta=0$.

V. CONCLUSIONS AND DISCUSSIONS

In this section we summarize results obtained in this paper and discuss their physical implications.

There are three main results in the strong-coupling limit.

(1) A pseudoscalar meson (π meson) is identified as the massless mode associated with the parity-violating phase transition rather than as the Nambu-Goldstone boson of the chiral-symmetry breaking.

(2) At the tree level of the mesons there is no mass difference between the flavor-nonsinglet mesons (π meson) and the flavor-singlet meson (η meson) to all orders in the $1/N$ expansion if the parity is conserved.

(3) In the parity-violating phase, contrary to the result (2), such a mass difference exists and

$$\Delta m=m_\eta a-m_\pi a>0 .$$

In other words, results (2) and (3) mean that the U(1) problem is solved only in the parity-violating phase in the strong-coupling limit.

Now let us discuss these results, unexpected by the author prior to calculations, and consider their physical im-

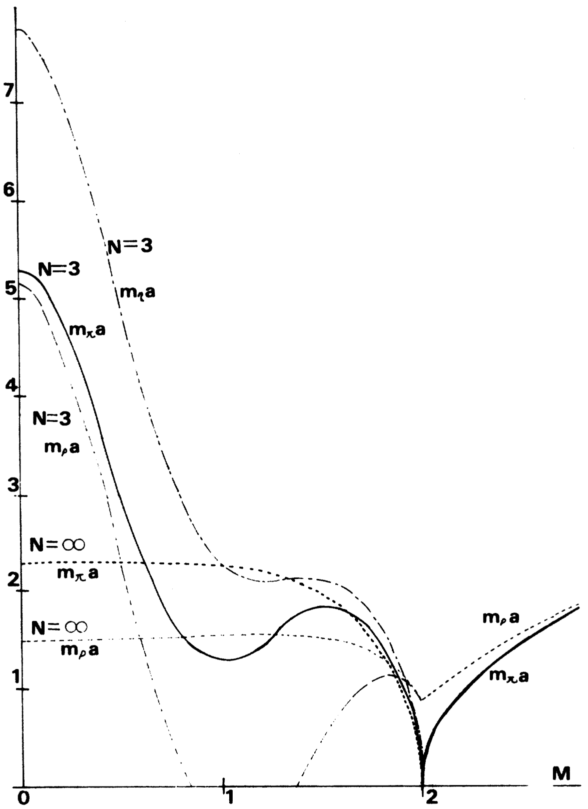

FIG. 2. Dependence of $m_\pi a$, $m_\eta a$, and $m_\rho a$ on M. Solid line (——) represents $m_\pi a$, dashed-dotted line (—·—·—·) represents $m_\eta a$, and dashed line (— — —) represents $m_\rho a$ in the case that $N=3$. Two dotted lines (· · · ·) represent $m_\pi a$ $(=m_\eta a)$ and $m_\rho a$ in the large-N limit.

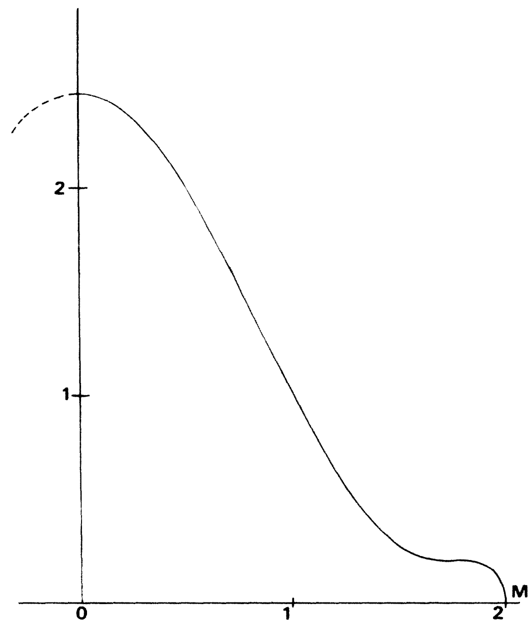

FIG. 3. Dependence of $\Delta m = m_\eta a - m_\pi a$ on M in the case that $N=3$.

plications.

(a) The reason why the mass difference cannot be obtained in the parity-conserving phase will be considered.

(i) The $1/N$ expansion is not correct and for the physical value of $N(=3)$ the $1/N$ expansion may be divergent. In Appendix D we will treat the case of $N=3$ directly without the $1/N$ expansion and get the same results as those of the $1/N$ expansion. Therefore the $1/N$ expansion is not responsible for the result (2).

(ii) The approximation in order to calculate meson masses from the effective potential $S_{\text{eff}}(M)$ is wrong. We calculate meson masses from the quadratic part of S_{eff}; $S_{\text{eff}}^{(2)}(n,m)\Pi(n)\Pi(m)$ (see Sec. IV). Correctly speaking, meson masses are defined by the behavior of the two-point function:

$$\int \prod_s dM(s) M(n)\cdot M(0) \exp[S_{\text{eff}}(M)] \sim \exp(-man) + \langle M \rangle^2, \quad n \to \infty , \quad (5.1)$$

where m is the lowest mass corresponding to the field $M(n)$ and

$$\langle M(n) \rangle = \int \prod_s dM(s) M(n) \exp[S_{\text{eff}}(M)] .$$

Our approximation is

$$\int \prod_s dM(s) M(n)\cdot M(0) \exp[S_{\text{eff}}(M)] - \langle M \rangle^2 = \int \prod_s dM(s) \Pi(n)\cdot\Pi(0) \exp\left[\sum_{n,m} S_{\text{eff}}^{(2)}(n,m)\Pi(n)\Pi(m) + O(\Pi^3)\right] \approx \text{tr}[S_{\text{eff}}^{(2)}(n,0)]^{-1} . \quad (5.2)$$

In other words we neglect the interaction among mesons which represents $O(\Pi^3)$ in (5.2). If we include this interaction and calculate the loop diagrams of the mesons,[11] for example, the self-energy diagram of the meson propagator, it may be possible that the mass difference can be obtained. But unfortunately it is very difficult to calculate the loop diagrams since the meson propagator [the inverse of $D^a{}_{AB}(p)$] is very complicated on a lattice. Loop corrections will indeed give the different behavior of the two-point function as Wilson pointed out,[1] but we cannot show that this difference assures the correct mass relation $(m_\eta > m_\pi)$. In further investigations we must include the loop diagrams by using another method (for example, the Monte Carlo simulations).

(iii) The introduction of the meson field $M(n)$ and description of the theory by $S_{\text{eff}}(M)$ are wrong. $S_{\text{eff}}(M)$ has no stable vacuum since $-\text{tr}\ln(M)$ is unbounded above. Indeed in the large-N limit the vacuum stays at the saddle point where $\langle M \rangle = \sigma \neq 0$. This shortcoming of $S_{\text{eff}}(M)$ may be overcome by introducing baryon fields.

The effective action for baryon and meson has no $-\mathrm{tr}\ln M$ term.[12] We must change gauge group from U(N) to SU(N) and investigate such a theory in the future.

(iv) The calculation in the strong-coupling limit is not sufficient to detect the mass difference and we must make the strong-coupling expansion.[13] Now we have finished the calculations and will publish results in the next paper.

(b) Although parity violation is very weak in nature, we discuss the property of the parity violation in the lattice QCD hereafter. This property may become useful for further development of the investigation, for example, the model building or the Monte Carlo simulation.

(i) Probably result (1) may hold in the continuum limit. Indeed the Gross-Neveu model on a lattice is such an example.[7,14] If the bare coupling g^2 of QCD becomes small enough, the phase structure on a M_0-1/g^2N becomes complicated so that spectral doublings of the lattice fermion are separated from each other.[7] Furthermore, we may construct the parity-violating QCD if we take the continuum limit from the parity-violating phase. Indeed we can do so for the Gross-Neveu model and obtain the continuum limit:[7,14]

$$g^2[\langle\bar{\psi}\psi\rangle^2+\langle\bar{\psi}i\gamma_5\psi\rangle^2]^{1/2}=\frac{4}{a}\exp(-\pi/g^2N)\ .$$

(ii) The existence of parity-violating QCD seems to be inconsistent with the result of Vafa and Witten.[15] They assume that the fermion part of the action is $\Sigma\bar{\psi}(\not{D}+M)\psi$, but the Wilson fermion does not satisfy this assumption. Therefore the parity violation is possible in our case.

(iii) We mention the connection between the parity violation and the vacuum of QCD. In our case if we change

$$\psi_n\to\psi'_n=e^{-i\theta\gamma_5/2}\psi_n\ ,$$
$$\bar{\psi}\to\bar{\psi}'_n=\bar{\psi}_n e^{-i\theta\gamma_5/2}$$

then we can set

$$\langle\bar{\psi}'\psi'\rangle=\sigma \text{ and } \langle\psi' i\gamma_5\psi'\rangle=0\ .$$

Under the above transformation the fermion action becomes

$$S_F=\sum_{n,\mu}(\bar{\psi}_n\gamma_\mu\nabla_\mu\psi_n+ar\bar{\psi}_n e^{i\theta\gamma_5}\Box\psi_n+M\bar{\psi}_n e^{i\theta\gamma_5}\psi_n)\ . \tag{5.3}$$

This action is equal to the action in Ref. 16 if we change $M\to Me^{i\theta\gamma_5}$. It is noted that θ depends on M in our case. In Ref. 16 the θ angle of the Wilson term is proven to be equal to the θ angle of the vacuum. If this is the case in the action (5.3) the θ angle of the parity violation due to the QCD interaction is interpreted as the θ angle of QCD vacuum. If so, by experiments and the effective Lagrangian method,[6] we get the upper bound

$$\theta\lesssim 3\times10^{-3}\ .$$

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APPENDIX A: CALCULATION FOR THE EFFECTIVE POTENTIAL IN THE $1/N$ EXPANSION

In this appendix we present the detailed calculations for the effective potential in the $1/N$ expansion. We use the method of Ref. 17. The partition function (2.3) in the strong-coupling limit is written as

$$Z(J)=\int D\psi D\bar{\psi}\exp\left[N\sum_n \mathrm{tr}(\hat{M}+J)M(n)\right]\prod_{n,\mu}Z_{n,\mu}\ , \tag{A1}$$

where $\hat{M}$,J and $M(n)$ are defined in the text. The one-link integral $Z_{n,\mu}$ is defined as

$$Z_{n,\mu}=\int d\,U_{n,\mu}\exp[a^3\mathrm{Tr}(U_{n,\mu}D^\dagger_{n,\mu}+D_{n,\mu}U^\dagger_{n,\mu})] \tag{A2}$$

where

$$(D_{n,\mu})_{ab}=-\Sigma_f(\bar{\psi}^f_{n+\hat{\mu}})_b P^+_\mu(\psi^f_n)_a\ ,$$
$$(D^\dagger_{n,\mu})_{ab}=-\Sigma_f(\bar{\psi}^f_n)_b P^-_\mu(\psi^f_{n+\hat{\mu}})_a\ ,$$

and

$$\mathrm{Tr}U_{n,\mu}D^\dagger_{n,\mu}=(U_{n,\mu})_{ab}(D^\dagger_{n,\mu})_{ba},\ \text{etc}\ .$$

For simplicity we drop the suffixes (n,μ) hereafter. Using $U^\dagger U=1$ we derive the Schwinger-Dyson equation:

$$\frac{\partial^2 Z}{\partial D_{ab}\partial D^\dagger_{bc}}=\delta_{ac}\ . \tag{A3}$$

Since Z is gauge invariant, Z depends on only gauge-invariant traces λ_q that are defined by (Ref. 18)

$$\lambda_q=\mathrm{Tr}\Lambda^q,\ \ q=1,2,\ldots,$$
$$\Lambda_{ab}=\frac{a^5}{N^2}(D^\dagger D)_{ab}\ .$$

Setting $Z\equiv\exp(NW)$ we write (A3) in terms of λ_q:

$$\Lambda=\sum_{q=1}^{\infty}\frac{\partial W}{\partial\lambda_q}q\left[\Lambda^q+\frac{1}{N}\sum_{p=1}^{q-1}\Lambda^{q-p}\lambda_p\right]$$
$$+\sum_{q,t=1}^{\infty}qt\left[\frac{1}{N}\frac{\partial^2 W}{\partial\lambda_q\partial\lambda_t}+\frac{\partial W}{\partial\lambda_q}\frac{\partial W}{\partial\lambda_t}\right]\Lambda^{q+t}. \tag{A4}$$

Now we solve Eq. (A4) with the boundary condition $W(\Lambda=0)=0$. In the large-N limit the solution was obtained in Ref. 17:

$$W_0(\Lambda)=\mathrm{Tr}\left[(1+4\Lambda)^{1/2}-1-\ln\left[\frac{1+(1+4\Lambda)^{1/2}}{2}\right]\right] . \tag{A5}$$

Now we expand W so that

$$W(\Lambda)=\sum_{k=0}^{\infty}\left[\frac{1}{N}\right]^k W_k(\Lambda)$$

and put this form into (A4). Then we obtain the recursion equation for $k\geq 1$:

$$\sum_{q=1}^{\infty} q\Lambda^q\frac{\partial W_k}{\partial\lambda_q}=-(1+4\Lambda)^{-1/2}\left[\sum_{q=1}^{\infty}\sum_{p=1}^{q-1} q\lambda_q\Lambda^{q-p}\frac{\partial W_{k-1}}{\partial\lambda_q}+\sum_{q,t=1}^{\infty} qt\left[\frac{\partial^2 W_{k-1}}{\partial\lambda_q\partial\lambda_t}+\sum_{l=1}^{k-1}\frac{\partial W_{k-1}}{\partial\lambda_t}\frac{\partial W_l}{\partial\lambda_q}\right]\Lambda^{q+t}\right] . \tag{A6}$$

Since we know $W_0(\Lambda)$ we can calculate W_k for $k\geq 1$, in principle, from (A6).

From Eq. (A5) it is easy to prove two statements by using the mathematical induction in the order of k.

(1) For $k=2l$ (even) W_{2l} has the form

$$W_{2l}=\sum_{s=0}^{l} C^{(2l+1)}_{q_1\cdots q_{2s+2}}\lambda_{q_1}\cdots\lambda_{q_{2s+1}}$$

and for $k=2l+1$ (odd) W_{2l+1} has the form

$$W_{2l+1}=\sum_{s=0}^{l} C^{(2l+1)}_{q_1\cdots q_{2s+2}}\lambda_{q_1}\cdots\lambda_{q_{2s+2}} . \tag{A7}$$

(2) For $k\geq 1$ $C_1^{(k)}$ vanishes, i.e., $C_1^{(k)}=0$, where $C_q^{(k)}$ is defined in (A7).

The above two statements are important in analyzing the vacuum structure and calculating the meson masses.

Now we calculate the explicit form of W_1. From (A6) and (A7) for $k=1$ we obtain

$$W_1=\sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty} C^{(1)}_{q_1q_2}\lambda_{q_1}\lambda_{q_2} , \tag{A8}$$

where

$$C^{(1)}_{q_1q_2}=-\frac{(-4)^{q_1+q_2-1}}{q_1+q_2}\left[\sum_{n=0}^{q_1-1}+\sum_{n=0}^{q_2-1}\right]\times\left[\frac{(2n-1)!!}{(2n)!!}\frac{[2(q_1+q_2-n)-3]!!}{[2(q_1+q_2-n)]!!}\right] . \tag{A9}$$

Unfortunately Eqs. (A8) and (A9) cannot be written in a closed form.

Now we return to the $Z(J)$. From (A2) we obtain

$$Z(J)=\int D\psi D\bar\psi \exp N\left[\sum_n \mathrm{tr}(\hat M+J)M(n)+\sum_{n,\mu} W(\Lambda_{n,\mu})\right] . \tag{A10}$$

After changing integration variables in (A10) from ψ_n and $\bar\psi_n$ to $M(n)$ (Ref. 3) we obtain the final result (2.5) and (2.6) in the text. In order to write (A4), (A7), and (A8) in terms of $M(n)$ we use the formula

$$\lambda_q=N^{-2q}\mathrm{Tr}(D^\dagger_{n,\mu}D_{n,\mu})^q$$
$$=-\mathrm{tr}[M(n)(P^+_\mu)^T M(n+\hat\mu)(P^-_\mu)^T]^q , \tag{A11}$$

where tr means the trace over the spinor-flavor indices. We must be careful about the sign factor.

APPENDIX B: DETAILED CALCULATIONS FOR THE VACUUM STRUCTURE AND THE MESON MASS

In this appendix we give the details of the calculations used in Sec. III and Sec. IV. The useful formulas for calculations are listed below:

$$\left\langle\frac{\partial f(\Lambda_{n,\mu})}{\partial\Lambda^{\hat\alpha\hat\beta}_{n,\mu}}\right\rangle=[f''(-\sigma^2\sin^2\theta)A^T_\mu+f'(0)B^T_\mu]_{\hat\alpha\hat\beta}\otimes 1_{f'f} , \tag{B1}$$

$$\left\langle\frac{\partial^2 f(\Lambda_{n,\mu})}{\partial\Lambda^{\hat\alpha\hat\beta}_{n,\mu}\partial\Lambda^{\hat\gamma\hat\delta}_{n,\mu}}\right\rangle=\left[f''(-\sigma^2\sin^2\theta)A^T_\mu\otimes B^T_\mu+\frac{f'(-\sigma^2\sin^2\theta)}{(-\sigma^2\sin^2\theta)}(A^T_\mu\otimes B^T_\mu+B^T_\mu\otimes A^T_\mu)+f''(0)B^T_\mu\otimes B^T_\mu\right]_{\beta\gamma,\delta\alpha}\otimes(1\otimes 1)_{f'f,g'g} , \tag{B2}$$

where f is an arbitrary function of $\Lambda_{n,\mu}$, $f'(x)=df(x)/dx$, $f''(x)=d^2f(x)/dx^2$, $\hat\alpha=(\alpha,f)$, $\hat\beta=(\beta,f')$, $\hat\gamma=(\gamma,g)$, and $\hat\delta=(\delta,g')$. Here

$$\langle\Lambda_{n,\mu}\rangle=T_\mu\begin{bmatrix}-\sigma^2\sin^2\theta 1 & 0\\ 0 & 0\end{bmatrix}T_\mu{}^{-1}\otimes 1 ,$$

$$(A_\mu)_{\alpha\beta}=\sum_{\gamma=1}^{2}(T_\mu)_{\alpha\gamma}(T_\mu{}^{-1})_{\gamma\beta},\quad (B_\mu)_{\alpha\beta}=\sum_{\gamma=3}^{4}(T_\mu)_{\alpha\gamma}(T_\mu{}^{-1})_{\gamma\beta} .$$

It is easy to see that A_μ and B_μ satisfy

$$A_\mu+B_\mu=1,\ \ (P_\mu^-)^T A_\mu=(P_\mu^-)^T,\ \text{ and }\ (P_\mu^-)^T B_\mu=0\ .$$

Using Eq. (B1) the gap equation (3.5) becomes

$$\left[1\otimes\hat{M}-\sigma e^{-i\theta\gamma_5^T}\otimes 1+\sum_\mu \sigma\sin\theta\left[W_0'(-\sigma^2\sin^2\theta)-\sum_{k=1}^\infty N^{-2k}\sum_{l=0}^k (4n_f)^{2l}\hat{C}_{\{q\}}^{(2k)}\sum_{j=1}^{2l+1}4q_j\right.\right.$$
$$\left.\left.-\sum_{k=1}^\infty N^{-2k+1}\sum_{l=1}^k (4n_f)^{2l-1}\hat{C}_{\{q\}}^{(2k-1)}\sum_{j=1}^{2l}4q_j\right]i\gamma_5^T\otimes 1\right]_{\alpha\beta,f'f}=0\ , \tag{B3}$$

where

$$\hat{C}_{\{q\}}^{(s)}=C_{q_1}^{(s)}\cdots\times(-\sigma^2\sin^2\theta)^{\sum q-1}\ .$$

If the matrix $\hat{M}$ is equal to $M_0\cdot 1$, which means that all bare mass of quarks are equal, Eq. (B3) is equivalent to

$$M_0\cdot\sigma=\cos\theta\ , \tag{B4}$$

$$\sigma\sin\theta\left[-\sigma^{-2}+\frac{8}{1+(1-4\sigma^2\sin^2\theta)^{1/2}}\right.$$
$$\left.-\sum_{k=1}^\infty N^{-2k}\sum_{l=0}^k(4n_f)^{2l}\hat{C}_{\{q\}}^{(2k)}\sum_{j=1}^{2l+1}4q_j-\sum_{k=1}^\infty N^{-2k+1}\sum_{l=1}^k(4n_f)^{2l-1}\hat{C}_{\{q\}}^{(2k-1)}\sum_{j=1}^{2l}4q_j\right]=0\ . \tag{B5}$$

From (B4) and (B5) we can derive Eqs. (3.6) and (3.7) or Eqs. (3.11) and (3.12). Furthermore, from fact (2) in Appendix A we obtain Eq. (3.9).

Using Eq. (B2) it is easy to obtain Eqs. (4.2) and (4.3). B_1, C_1, and D_1 are given by

$$B_1(\sigma^2\sin^2\theta)=\sum_{k=1}^\infty\left[N^{-2k}\sum_{l=0}^k(4n_f)^{2l}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k)}\sum_{j=1}^{2l+1}q_j(q_j-1)/4+N^{-2k+1}\sum_{l=1}^k(4n_f)^{2l-1}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k-1)}\sum_{j=1}^{2l}q_j(q_j-4)/4\right]\ ,$$

$$C_1(\sigma^2\sin^2\theta)=\sum_{k=1}^\infty\left[N^{-2k}\sum_{l=0}^k(4n_f)^{2l}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k)}\sum_{j=1}^{2l+1}q_j{}^2/4+N^{-2k+1}\sum_{l=1}^k(4n_f)^{2l-1}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k-1)}\sum_{j=1}^{2l}q_j{}^2/4\right]\ ,$$

$$D_1(\sigma^2\sin^2\theta)=\sum_{k=1}^\infty\left[N^{-2k}\sum_{l=0}^k(4n_f)^{2l}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k)}\sum_{i\neq j}^{2l+1}q_i\cdot q_j/16+N^{-2k+1}\sum_{l=1}^k(4n_f)^{2l-1}\sum_{\{q\}}\hat{C}_{\{q\}}^{(2k-1)}\sum_{i\neq j}^{2l}q_i\cdot q_j/16\right]\ .$$

From fact (2) in Appendix A it is easy to check that $B_1(0)=C_1(0)=D_1(0)=0$.

APPENDIX C: $1/N$ CORRECTIONS WITH $r\neq 1$ AND $\theta=0$

In Sec. IV it was shown that the mass difference does not exist in the parity-conserving phase to all orders of the $1/N$ expansion. However, we might suspect that this conclusion is special for the case of $r=1$. Indeed for $r=1$, $P_\mu^\pm(r)$ satisfy

$$P_\mu^+(r)P_\mu^-(r)=0$$

and due to the above property of $P_\mu^\pm(r)$ there are no $1/N$ corrections to the meson mass and no mass difference for $r=0$. Therefore, in this appendix we analyze the case that $r\neq 1$ and $\theta=0$. We use $P_\mu^\pm(r)=(r\pm\gamma_\mu)/2$ instead of $P_\mu^\pm=(1\pm\gamma_\mu)/2$. The physical positivity demands that $0<r^2<1$.

For $\theta=0$ the gap equation becomes

$$4n_f\left\{M_0-\sigma^{-1}+\frac{4\sigma(1-r^2)}{1+[1-(1-r^2)\sigma^2]^{1/2}}\right.$$
$$+2\sigma(r^2-1)\sum_{k=1}^\infty\left[N^{-2k}\sum_{l=0}^k(4n_f)^{2l}\sum_{\{q\}}C_{\{q\}}^{(2k)}\left[\frac{\sigma^2(r^2-1)}{4}\right]^{\sum q-1}\sum q\right.$$
$$\left.\left.+N^{-2k+1}\sum_{l=1}^k(4n_f)^{2l-1}\sum_{\{q\}}C_{\{q\}}^{(2k-1)}\left[\frac{\sigma^2(r^2-1)}{4}\right]^{\sum q-1}\sum q\right]\right\}=0\ . \tag{C1}$$

If we expand σ as

$$\sigma=\sum_{k=0}^{\infty} N^{-k}\sigma_k \tag{C2}$$

we obtain

$$\sigma_0=\frac{-3M_0+4[M_0{}^2+7(1-r^2)]^{1/2}}{M_0{}^2+16(1-r^2)}\ , \tag{C3}$$

$$\sigma_1=\frac{-32n_f\sum_{q_1q_2} C^{(1)}_{q_1q_2}(q_1+q_2)[\sigma_0{}^2(r^2-1)/4]^{q_1+q_2}}{M_0+\dfrac{8(1-r^2)\sigma_0}{1+[1-(1-r^2)\sigma_0{}^2]^{1/2}}+\dfrac{(1-r^2)\sigma_0}{\{1+[1-(1-r^2)\sigma_0{}^2]^{1/2}\}^2[1-(1-r^2)\sigma_0{}^2]^{1/2}}}\ , \tag{C4}$$

etc.

After little calculations we obtain

$$\sum_{n,m} S^{(2)}_{\rm eff}(n,m)\Pi(n)\Pi(m)=n_f\sum_{a=0}^{n_f{}^2-1}\int_p\left[A\,{\rm tr}\Pi^a(-p)\Pi^a(p)+\sum_\mu B\,{\rm tr}\Pi^a(-p)(P_\mu^+)^T\Pi^a(p)(P_\mu^-)^Te^{ip_\mu a}\right]$$
$$+n_f{}^2D\int_p\sum_\mu {\rm tr}\Pi^0(-p){\rm tr}\Pi^0(p)(1+e^{ip_\mu a})\ , \tag{C5}$$

where

$$A=(2\sigma^2)^{-1}+W_0''[\sigma^2(r^2-1)/4]\sigma^2(r^2-1)^2/2+A_1(\sigma^2)\ ,$$

$$B=W_0''[s\,2(r\,2-1)/4]\sigma^2(r^2-1)/4$$
$$+W_0'[\sigma^2(r^2-1)/4]+B_1(\sigma^2)\ ,$$

$$D=D_1(\sigma^2)\ .$$

Here $A_1(\sigma^2)$, $B_1(\sigma^2)$, and $D_1(\sigma^2)$ are $O(1/N)$. From the last term in (C5) the mass difference arises. If we write it as

$$n_f{}^2D\int_p\sum_\mu {\rm tr}\Pi^0_S(-p)\Pi^0_S(p)(1+e^{ip_\mu a}) \tag{C6}$$

it is easy to see that in the case of $r\neq 1$ there exists a mass difference between the singlet and the nonsinglet only scalar mesons rather than pseudoscalar mesons if the parity is conserved.

APPENDIX D: ANALYSIS FOR U(3) GAUGE

In order to show that our results obtained in Secs. III and IV in the text are not a special case for the $1/N$ expansion, we analyze the theory with the U(3) gauge in this appendix. The main result in Sec. III is that there is no mass difference between the singlet and the nonsinglet mesons to all orders in the $1/N$ expansion if the parity is conserved. Here we will show that this property holds for the case of the U(3) gauge.

First we calculate one link integral $Z_{n,\mu}$ given in (A2) for the U(3) group. Here we drop suffix (n,μ):

$$Z=\int_{{\rm U}(3)} d\,U\exp[{\rm Tr}(U\,D^\dagger+D\,U^\dagger)]\ . \tag{D1}$$

Schwinger-Dyson equations are given by

$$\sum_b\frac{\partial^2 Z}{\partial D^\dagger_{ab}\partial D_{bc}}=\delta_{ac}\ . \tag{D2}$$

In the case of SU(3), Z has been calculated by Hoek.[19] We use his method here. We define W so that

$$Z\equiv e^W\ , \tag{D3}$$

and expand it so that

$$W=\sum_{m,1,s\geq 0} C_{mls}\kappa^m\lambda^l\mu^s\ , \tag{D4}$$

where $\kappa={\rm Tr}(D^\dagger D)$, $\lambda={\rm Tr}(D^\dagger D)^2$, and $\mu={\rm Tr}(D^\dagger D)^3$ since W depends only on gauge-invariant traces ${\rm Tr}(D^\dagger D)^k$ $(k=1,2,\ldots)$ and ${\rm Tr}(D^\dagger D)^k$ $(k>3)$ are expressed in terms of κ, λ, and μ by the Caley-Hamilton theorem. For example,

$${\rm Tr}(D^\dagger D)^4=\lambda^2/2+4\mu\kappa/3-\lambda\kappa^2+\kappa^4/6\ ,$$

$${\rm Tr}(D^\dagger D)^5=\tfrac{1}{6}(\kappa^5-5\lambda\kappa^3+5\kappa^2\mu+5\lambda\mu)\ .$$

We insert (D3) and (D4) into (D2) and then we obtain the recursion equation for C_{mls}. By using this recursion equation we can calculate C_{mls} from the smaller value of $m+2l+3s$. For example, $C_{000}=0$ and $C_{100}=\frac{1}{3}$. But we will not use the definite value of C_{mls} in this appendix and will use only the fact that W has the form (D4). Especially if D and $D^\dagger$ are the Grassmann variables the summation of (D4) is restricted such that $m+2l+3s\leq 3\times n_f\times C$. Therefore the summation is finite and this expansion is well defined.

The effective potential in the strong-coupling limit becomes

$$S_{\rm eff}(M)=3\sum_n\left[{\rm tr}\widehat{M}M(n)-{\rm tr}\ln M(n)\right.$$
$$+\sum_\mu\sum_{m,l,s}C_{mls}(-1)^{m+1+s}$$
$$\left.\times 9^{m+2l+3s}\kappa^m_{n,\mu}\lambda^l_{n,\mu}\mu^s_{n,\mu}\tfrac{1}{3}\right]\ , \tag{D5}$$

where

$$\kappa_{n,\mu}=\mathrm{tr}M(n)(P_\mu^+)^T M(n+\hat{\mu})(P_\mu^-)^T ,$$

$$\lambda_{n,\mu}=\mathrm{tr}[M(n)(P_\mu^+)^T M(n+\hat{\mu})(P_\mu^-)^T]^2 ,$$

$$\mu_{n,\mu}=\mathrm{tr}[M(n)(P_\mu^+)^T M(n+\hat{\mu})(P_\mu^-)^T]^3 .$$

We assume the vacuum expectation of $M(n)$ has the form $\langle M(n)\rangle=\sigma\cdot e^{i\theta\gamma_5}$; then we obtain gap equations:

$$M_0\sigma=\cos\theta , \quad \text{(D6)}$$

$$\sin\theta\cos\theta[-1+f(\sigma,\theta)]=0 , \quad \text{(D7)}$$

where

$$f(\sigma,\theta)=12\sigma^2\sum_{m,l,s}C_{mls}(-1)^l(n_f C/2)^{m+l+s-1} \times(9\sigma^2\sin^2\theta)^{m+2l+3s-1} .$$

If the equation $f(\cos\theta/M_0,\theta)=1$ has the nonzero solution θ_0, the parity is broken in this vacuum. We, however, analyze only the case that $\theta=0$ hereafter. In this case

$$\sigma=1/M_0 . \quad \text{(D8)}$$

This solution coincides with the solution in Eq. (3.12) in the text. The value where $f(\cos\theta/M_0,\theta)=1$ has the $\theta=0$ solution is given by

$$4\sigma^2=1\rightarrow M_0{}^2=4 \quad \text{(D9)}$$

(note that $C_{100}=\frac{1}{3}$). This value also coincides with the value where the parity-violating phase transition occurs in the $1/N$ expansion.

We calculate meson masses with $\theta=0$

$$\sum_{m,n}S_{\rm eff}^{(2)}(n,m)\Pi(m)\Pi(n) =\sum_{a=0}^{n_f{}^2-1} n_f\int_p\left[\frac{1}{2\sigma^2}\mathrm{tr}\Pi^a(-p)\Pi^a(p) -\sum_\mu \mathrm{tr}\Pi^a(-p)(1+\gamma_\mu^T)\Pi^a(p) \times(1-\gamma_\mu^T)e^{ip_\mu a}/4\right] . \quad \text{(D10)}$$

Equation (D10) coincides with the result in the $1/N$ expansion. From (D10) we conclude that there is no mass difference for the U(3) gauge in the strong-coupling limit with $r=1$ if the parity is conserved. Furthermore, at the value $M_0{}^2=4$ the π meson becomes massless, and for $M_0{}^2<4$ the parity may be broken, and these properties are also the same as the $1/N$ expansion.

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