Lattice gauge theory: Hamiltonian, Wilson fermions, and action

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We derive the gauge-theory Hamiltonian in the axial gauge directly from the path integral defined by the Wilson lattice action. We define the state space for the gauge field coupled to Wilson fermions and derive noncanonical equal-time anticommutation equations for Wilson fermions. We show that the Hamiltonian is nonlocal for fermions with canonical anticommutation. We derive the color charge operator and formulate Gauss's law for the system. We then evaluate the lattice action starting from a lattice fermionic Hamiltonian, and derive a boundary term in addition to the finite-time continuum action. Lastly we discuss our results.

I. INTRODUCTION

The Hamiltonian for the lattice gauge field, in particular the formulation given by Kogut and Susskind,[1] has been widely studied. The lattice fermions for the Hamiltonian, particularly those defined by Susskind[2] and by Drell, Weinstein, and Yankielowicz[3] are the two types of fermions that are most widely used. The earliest attempt to relate Wilson's lattice action[4] to the Hamiltonian was made by Creutz;[5] we will discuss this derivation in some detail later. Wilson[6] derived the matrix element of the operator $\exp(-aH)$, where H is the Hamiltonian for lattice quantum chromodynamics (QCD) and a the time lattice spacing; he also derived a noncanonical metric for the Hilbert space of the interacting theory. The Hamiltonian operator H for the free Dirac field with Wilson fermions has been derived in Ref. 7.

In this paper we derive the Hamiltonian operator H from the transfer matrix, defined as the operator $\exp(-aH)$. This will consist of using the metric on Hilbert space to obtain H as a differential operator with given noncanonical (anti)commutation relations for the field operators. We then reexpress the lattice Hamiltonian using fermions with canonical anticommutation, and this leads to nonlocal fermion and gauge-field interactions. We derive the quark color charge operator, which also becomes nonlocal in terms of canonical fermions. And lastly, we obtain the lattice action starting from a lattice Hamiltonian. The paper is organized as follows. In Sec. II we define the state space for the interacting theory and briefly discuss Wilson's derivation of $\exp(-aH)$ and the metric. In Sec. III we derive the inner product and anticommutation equation for the field operators. In Sec. IV we derive the Hamiltonian from an asymmetric Wilson action using an asymmetric space-time lattice with the time lattice spacing going to zero; we then transform to canonical fermions and solve for the free fermion sector. In Sec. V we derive certain properties of the chromoelectric field operator and the quark color charge operator, and use these to obtain Gauss's law for the interacting theory. In Sec. VI we derive the lattice action starting from a lattice QCD Hamiltonian which could have SLAC or Susskind fermions. We take the time continuum limit of the lattice action and obtain a boundary term in addition to the finite-time continuum action. In Sec. VII we briefly discuss our results.

II. THE TRANSFER MATRIX

Consider a d-dimensional Euclidean space-time-symmetric lattice with lattice spacing a and let n denote a lattice site. Let the $(d-1)$-dimensional "spatial" lattice be an infinite lattice and let the time lattice be open and of finite size M. Let $A_\mu^\alpha(x)$ be the continuum SU(N) non-Abelian gauge field and $\bar\psi(x)$, $\psi(x)$ the continuum SU(N) quark field considered as anticommuting Grassmann variables. The lattice degrees of freedom are dimensionless and are defined by (α and j are color indices)

$$B_{n\mu}^a = agA_\mu^\alpha(x),\ \ \mu=0,1,\ldots,(d-1)\ , \tag{2.1a}$$

$$\psi_{nj} = \left[\frac{a^{d-1}}{2K}\right]^{1/2}\psi_j(x)\ , \tag{2.1b}$$

$$\bar\psi_{nj} = \left[\frac{a^{d-1}}{2K}\right]^{1/2}\bar\psi_j(x)\ , \tag{2.1c}$$

$$2K=\frac{1}{d+m_0a},\ \ g=sg_0a^{(d-4)/2},\ \ x=na\ . \tag{2.1d}$$

The lattice quantities K and g are dimensionless and the dimensional continuum quantities sg_0 and m_0 are the bare coupling constant and bare quark mass, respectively; s is given in Eq. (2.4b). The quarks are in the fundamental representation and the gauge field in the adjoint representation of SU(N). Choose the $2^{d/2}\times 2^{d/2}$ Euclidean γ matrices (σ_i are the analogs of the Pauli matrices) for $d=$even in block form as

$$\gamma_0=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\ \ \gamma_i=i\begin{bmatrix}0 & -\sigma_i\\ \sigma_i & 0\end{bmatrix},\qquad i=1,2,\ldots,(d-1)\ . \tag{2.2}$$

We define the Dirac upper and lower component spinors, denoted by u and l, respectively, by

$$\psi_n = \begin{bmatrix} \psi_{nu} \\ \psi_{nl} \end{bmatrix} = \tfrac{1}{2}(1+\gamma_0)\psi_n + \tfrac{1}{2}(1-\gamma_0)\psi_n \ , \quad (2.3a)$$

$$\bar{\psi}_n = (\bar{\psi}_{nu}\ \bar{\psi}_{nl}) = \tfrac{1}{2}\bar{\psi}_n(1+\gamma_0) + \tfrac{1}{2}\bar{\psi}_n(1-\gamma_0) \ . \quad (2.3b)$$

Define the link variable connecting n to $n+\hat{\mu}$ ($\hat{\mu}$ is the unit lattice vector in the μ direction) by

$$U_{n\mu} = \exp(iB^{\alpha}_{n\mu}X_\alpha) \quad (2.4a)$$

with

$$[X_\alpha, X_\beta] = iC_{\alpha\beta\gamma}X_\gamma, \quad \mathrm{Tr}(X_\alpha X_\beta) = \frac{1}{s^2}\delta_{\alpha\beta} \ , \quad (2.4b)$$

where $U_{n\mu}$ is an element of SU(N). The spatial components are given by U_{ni}, $i=1,2,\ldots,(d-1)$; we also have $n=(n_0,\mathbf{n})$ where $\mathbf{n}$ is a $(d-1)$-dimensional spatial lattice point, and n_0 the time coordinate.

We will first evaluate the transfer matrix, i.e., the matrix elements of $\exp(-aH)$, between arbitrary initial and final field configurations. Field configurations are defined on the $(d-1)$-dimensional spatial lattice; the coordinate eigenstates for the field are given by[6–8]

$$|\bar{\psi}_l, \psi_u, U\rangle \equiv |\{\bar{\psi}_{\mathbf{n}l}, \psi_{\mathbf{n}u}, U_{\mathbf{n}i}\}\rangle \quad (2.5a)$$

such that, as expected,

$$(\hat{\bar{\psi}}_{\mathbf{n}l}, \hat{\psi}_{\mathbf{n}u}, \hat{U}_{\mathbf{n}i})\,|\bar{\psi}_l, \psi_u, U\rangle = (\bar{\psi}_{\mathbf{n}l}, \psi_{\mathbf{n}u}, U_{\mathbf{n}i})\,|\bar{\psi}_l, \psi_u, U\rangle \ . \quad (2.5b)$$

On the left-hand side, we have Schrödinger operators denoted by a caret acting on the state, and on the right-hand side the eigenvalues; note $\bar{\psi}_{\mathbf{n}l}, \psi_{\mathbf{n}u}$ are anticommuting Grassmannian eigenvalues.

The conjugate eigenstate is

$$\langle \bar{\psi}_u, \psi_l, U\,| \equiv \langle \{\bar{\psi}_{\mathbf{n}u}, \psi_{\mathbf{n}l}, U_{\mathbf{n}i}\}\,| \quad (2.6)$$

and satisfies equations similar to (2.5b). The coordinate and conjugate eigenstates are not artifacts of the lattice and it is shown in Ref. 7 how they arise from the continuum Dirac equation; in Ref. 8 we derive the eigenfunctionals and propagators of free continuum Dirac field starting from field coordinates given by (2.5) and (2.6). We further discuss the fermion calculus in Appendix A.

The Hilbert space for the interacting theory has a nontrivial metric $T(\bar{\psi},\psi,U)$; the metric determines the completeness equation on Hilbert space given by

$$\mathbb{1} = \prod_{\mathbf{n}i} \int d\bar{\zeta}_{\mathbf{n}} d\zeta_{\mathbf{n}} dV_{\mathbf{n}i}\, |\bar{\zeta}_l, \zeta_u, V\rangle T(\bar{\zeta},\zeta,V) \langle \bar{\zeta}_u, \zeta_l, V\,| \ , \quad (2.7)$$

where $d\bar{\zeta}$ and $d\zeta$ are fermion integrations and dV_{ni} are invariant SU(N) integrations.

The lattice Hamiltonian H is related to the lattice action by the Feynman path integral given by

$$\langle \bar{\psi}_u, \psi_l, U'\,|\exp(-aMH)|\,\bar{\psi}_l, \psi_u, U\rangle = \prod_{n_0=1}^{M-1} \prod_{\mathbf{n},\mu} \int d\bar{\psi}_n d\psi_n dU_{n\mu} \exp(A_M) \quad (2.8)$$

with boundary conditions

$$n_0=0:\ \bar{\psi}_{(0,\mathbf{n})l} = \bar{\psi}_{\mathbf{n}l},\ \ \psi_{(0,\mathbf{n})u} = \psi_{\mathbf{n}u},\ \ U_{(0,\mathbf{n})i} = U_{\mathbf{n}i} \ , \quad (2.9a)$$

$$n_0=M:\ \bar{\psi}_{(M,\mathbf{n})u} = \bar{\psi}_{\mathbf{n}u},\ \ \psi_{(M,\mathbf{n})l} = \psi_{\mathbf{n}l},\ \ U_{(M,\mathbf{n})i} = U'_{\mathbf{n}i} \ . \quad (2.9b)$$

The Wilson action for finite time with boundary conditions given by (2.9) is

$$\begin{aligned} A_M = {} & \frac{1}{g^2}\sum_{n_0=0}^{M-1}\sum_{\mathbf{n}i} \mathrm{Tr}(U_{ni}U_{n+\hat{i}0}U^\dagger_{n+\hat{0}i}U^\dagger_{n0} + U_{n0}U_{n+\hat{0}_i}U^\dagger_{n+\hat{i}0}U^\dagger_{ni}) \\ & + \frac{1}{g^2}\sum_{n_0=1}^{M-1}\sum_{\mathbf{n}ij}\mathrm{Tr}(W_{nij}) - \sum_{n_0=1}^{M-1}\sum_{\mathbf{n}}\left[\bar{\psi}_n\psi_n - K\sum_i[\bar{\psi}_n(1-\gamma_i)U_{ni}\psi_{n+\hat{1}} + \bar{\psi}_{n+\hat{1}}(1+\gamma_i)U^\dagger_{ni}\psi_n]\right] \\ & + K\sum_{n_0=0}^{M-1}\sum_{\mathbf{n}}[\bar{\psi}_n(1-\gamma_0)U_{n0}\psi_{n+\hat{0}} + \bar{\psi}_{n+\hat{0}}(1+\gamma_0 U^\dagger_{n0}\psi_n] \ , \end{aligned} \quad (2.10a)$$

where

$$W_{nij} = U_{ni}U_{n+\hat{1}j}U^\dagger_{n+\hat{j}i}U^\dagger_{nj} \ . \quad (2.10b)$$

Note only the first and the last terms in the action (2.10a) have $n_0=0$ in the time summation, and the boundary values given by (2.9) enter the action only through these terms coupling nearest neighbors in the time direction. Note also that at $n_0=0$ only $\bar{\psi}_l, \psi_u$ couple to later time and similarly at $n_0=M$ only $\bar{\psi}_u$, ψ_l couple to earlier time as is required by the boundary condition (2.9).

Consider Eq. (2.8) for $M=2$. We have

$$\langle \bar{\psi}_u, \psi_l, U'\,|\exp(-2aH)|\,\bar{\psi}_l, \psi_u, U\rangle = \prod_{\mathbf{n},\mu}\int d\bar{\zeta}_{\mathbf{n}} d\zeta_{\mathbf{n}} dV_{\mathbf{n}\mu} \times \exp[A_2(\bar{\psi},\psi;U,U';\bar{\zeta},\zeta,V)] \ . \quad (2.11)$$

However, using the completeness equation (2.7) we also have

$$\langle \bar{\psi}_u,\psi_l,U' \,|\, \exp(-2aH) \,|\, \bar{\psi}_l,\psi_u,U\rangle = \prod_{\mathbf{n},i}\int d\bar{\xi}_{\mathbf{n}} d\xi_{\mathbf{n}} dV_{\mathbf{n}i} \langle \bar{\psi}_u,\psi_l,U' \,|\, e^{-aH} \,|\, \bar{\xi}_l,\xi_u,V\rangle \times T(\bar{\xi},\xi,V)\langle \bar{\xi}_u,\xi_l,V \,|\, e^{-aH} \,|\, \bar{\psi}_l,\psi_u,U\rangle \ . \tag{2.12}$$

By comparing (2.11) and (2.12), it can be shown[6] that, if we require a Hermitian Hamiltonian in the sense of (A4), we have a unique choice for the metric given by

$$T(\bar{\psi},\psi,U)=\exp\left[-\sum_{\mathbf{n}}\bar{\psi}_{\mathbf{n}}\psi_{\mathbf{n}} + K\sum_{\mathbf{n},i}(\bar{\psi}_{\mathbf{n}}U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{i}}+\bar{\psi}_{\mathbf{n}+\hat{i}}U^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}})\right] . \tag{2.13}$$

Note that the Wilson metric (2.13) depends on the Lagrangian through the parameter K. The norm on the Hilbert space is given by Eq. (A3). For positive norm, i.e., $\langle\Phi|\Phi\rangle\geq 0$ for all $|\Phi\rangle$, we must have $2K<1/(d-1)$; for asymptotically free theories $2K<1/d$ so that the Wilson metric gives a positive-definite norm for QCD (Ref. 6). The canonical fermion metric is obtained by setting K in (2.13) to zero.[5,7,8]

Using the notation $\phi_{\mathbf{n}}\equiv V_{\mathbf{n}0}$, we also have[6]

$$\begin{aligned}\langle \bar{\psi}_u,\psi_l,U' \,|\, \exp(-aH) \,|\, \bar{\psi}_l,\psi_u,U\rangle = \prod_{\mathbf{n}}\int d\phi_{\mathbf{n}}\exp\Bigg[&2K\sum_{\mathbf{n}}(\bar{\psi}_{\mathbf{n}l}\phi_{\mathbf{n}}\psi_{\mathbf{n}l}+\bar{\psi}_{\mathbf{n}u}\phi^{\dagger}_{\mathbf{n}}\psi_{\mathbf{n}u})\\ &+iK\sum_{\mathbf{n},i}(\bar{\psi}_{\mathbf{n}u}\sigma_i U'_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{1}l}+\bar{\psi}_{\mathbf{n}l}\sigma_i U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{1}u}-\bar{\psi}_{\mathbf{n}+\hat{1}u}\sigma_i U'^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}l}-\bar{\psi}_{\mathbf{n}+\hat{1}l}\sigma_i U^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}u})\\ &+\frac{1}{g^2}\sum_{\mathbf{n},i}\mathrm{Tr}(U_{\mathbf{n}i}\phi_{\mathbf{n}+\hat{1}}U'^{\dagger}_{\mathbf{n}i}\phi^{\dagger}_{\mathbf{n}}+\phi_{\mathbf{n}}U'_{\mathbf{n}i}\phi^{\dagger}_{\mathbf{n}+\hat{1}}U^{\dagger}_{\mathbf{n}i})+\frac{1}{2g^2}\sum_{\mathbf{n},ij}\mathrm{Tr}(W_{\mathbf{n}ij}+W'_{\mathbf{n}ij})\Bigg] . \end{aligned} \tag{2.14}$$

The metric $T(\bar{\psi},\psi,U)$ plays a crucial role in ensuring that $\exp(-aH)$ is Hermitian.[6] Note no gauge was chosen to arrive at $\exp(-aH)$. The $\phi_{\mathbf{n}}$ integrations ensure that $\exp(-aH)$ is invariant under time-independent gauge transformations separately for the coordinate eigenstates and the conjugate eigenstate; the metric is also gauge invariant.

Consider the operator $\exp(-aH)$ acting on the Hilbert space, i.e.,

$$\langle \bar{\psi}_u,\psi_l,U \,|\, \exp(-aH) \,|\, \tilde{\Phi}\rangle = \prod_{n,i}\int d\xi_{\mathbf{n}} d\xi_{\mathbf{n}} dV_{\mathbf{n}i} \langle \bar{\psi}_u,\psi_l,U \,|\, e^{-aH} \,|\, \bar{\xi}_l,\xi_u,V\rangle \times T(\bar{\xi},\xi,V)\langle \bar{\xi}_u,\xi_l,V \,|\, \tilde{\Phi}\rangle , \tag{2.15}$$

where $\tilde{\Phi}$ is an element of the Hilbert space. By performing the lattice gauge transformation in (2.15), namely,

$$\begin{aligned}&\xi_{\mathbf{n}l}\to\phi^{\dagger}_{\mathbf{n}}\xi_{\mathbf{n}l} ,\\ &\bar{\xi}_{\mathbf{n}u}\to\xi_{\mathbf{n}u}\phi_{\mathbf{n}} ,\\ &V_{\mathbf{n}i}\to\phi^{\dagger}_{\mathbf{n}}V_{\mathbf{n}i}\phi_{\mathbf{n}+\hat{1}}\equiv V_{\mathbf{n}i}(\phi) ,\end{aligned} \tag{2.16}$$

we perform a change of integration variables and shift the $\phi_{\mathbf{n}}$ integrations given in (2.14) from $\exp(-aH)$ to the wave functional $\tilde{\Phi}$; the $\phi_{\mathbf{n}}$ integrations sum $\tilde{\Phi}$ over all possible gauge transformations and in effect project out the gauge-invariant subspace of the full Hilbert space spanned by the coordinate eigenstates (2.5).

Define this gauge-invariant subspace by a collection of all Φ such that

$$\Phi(\bar{\psi}_u,\psi_l,U)=\prod_{\mathbf{n}}\int d\phi_{\mathbf{n}}\tilde{\Phi}(\bar{\psi}_u\phi^{\dagger},\phi\psi_l,U(\phi)) . \tag{2.17}$$

On this gauge-invariant subspace, the Hamiltonian is that obtained by setting $\phi_{\mathbf{n}}\equiv\mathbb{1}$ in (2.14) and in effect is the *axial* gauge. Hence, we have in the axial gauge

$$\begin{aligned}\langle \bar{\psi}_u,\psi_l,U' \,|\, \exp(-aH) \,|\, \bar{\psi}_l,\psi_u,U\rangle = \exp\Bigg[&2K\sum_{\mathbf{n}}(\bar{\psi}_{\mathbf{n}u}\psi_{\mathbf{n}u}+\bar{\psi}_{\mathbf{n}l}\psi_{\mathbf{n}l})\\ &+iK\sum_{\mathbf{n},i}(\bar{\psi}_{\mathbf{n}u}\sigma_i U'_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{1}l}+\bar{\psi}_{\mathbf{n}l}\sigma_i U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{1}u}-\bar{\psi}_{\mathbf{n}+\hat{1}u}\sigma_i U'^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}l}-\bar{\psi}_{\mathbf{n}+\hat{1}l}\sigma_i U^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}u})\\ &+\frac{1}{g^2}\sum_{\mathbf{n},i}\mathrm{Tr}(U_{\mathbf{n}i}U'^{\dagger}_{\mathbf{n}i}+U'_{\mathbf{n}i}U^{\dagger}_{\mathbf{n}i})+\frac{1}{2g^2}\sum_{\mathbf{n},ij}\mathrm{Tr}(W_{\mathbf{n}ij}+W'_{\mathbf{n}ij})\Bigg]\end{aligned} \tag{2.18}$$

acting on gauge-invariant wave functionals such that, from (2.17),

$$\Phi(\bar\psi_u,\psi_l,U)=\Phi(\bar\psi_u\phi^\dagger,\phi\psi_l,U(\phi)) \ . \tag{2.19}$$

Note that the gauge-invariant subspace is a result of the gauge-invariant definition of $\exp(-aH)$ stemming from the gauge-invariant action.

To extract H from the transfer matrix $\exp(-aH)$, we essentially have to divide out by the inner product

$$\langle \bar\psi_u,\psi_l,U' \mid \bar\psi_l,\psi_u,U\rangle \ .$$

We now examine this problem.

III. THE METRIC AND NONCANONICAL FERMION EQUAL-TIME ANTICOMMUTATION EQUATIONS

The metric $T(\bar\psi,\psi,U)$ determines the inner product

$$\langle \bar\psi_u,\psi_l,U' \mid \bar\psi_l,\psi_n,U\rangle$$

as well as the fermion anticommutation equation. In the usual scheme of Hamiltonian quantization, canonical anticommutation equations are simply postulated. In path-integral quantization, however, the commutation equation of the degree of freedom with its conjugate is determined by the action, and this was shown by Feynman for x and p in his original paper on path integration. The metric and anticommutation equations of the free Dirac field for the continuum and for the lattice have been discussed in Ref. 7.

Using the completeness equation (2.7) we have an integral equation for the inner product given by

$$\langle \bar\psi_u,\psi_l,U' \mid \bar\psi_l,\psi_u,U\rangle=\prod_{\mathrm{n},i}\int d\bar\xi_{\mathrm{n}}d\xi_{\mathrm{n}}dV_{\mathrm{n}i}\langle \bar\psi_u,\psi_l,U' \mid \bar\xi_l,\xi_u,V\rangle T(\bar\xi,\xi,V)\langle \bar\xi_u,\xi_l,V \mid \bar\psi_l,\psi_u,U\rangle \ . \tag{3.1}$$

We use the following notation for the Wilson metric:

$$T(\bar\xi,\xi,V)=\exp\left[-\sum_{\mathrm{nm},jk}\bar\xi_{\mathrm{n}j}M_{\mathrm{nm},jk}\xi_{\mathrm{m}k}\right] , \tag{3.2}$$

where, from (2.13), using j and k for non-Abelian indices

$$M_{\mathrm{nm},jk}=\delta_{\mathrm{mn}}\delta^{jk}-K\sum_i(\delta_{\mathrm{n}+\hat{1}\mathrm{m}}V^{jk}_{\mathrm{n}i}+\delta_{\mathrm{nm}+\hat{1}}V^{\dagger jk}_{\mathrm{m}i}) \tag{3.3a}$$

$$=M_{\mathrm{nm},jk}[V] \ . \tag{3.3b}$$

To solve Eq. (3.1) note that the matrix M does not couple upper to lower Dirac components. Using this property of M, it can be shown that

$$\langle \bar\psi_u,\psi_l,U' \mid \bar\psi_l,\psi_u,U\rangle=\frac{\exp\left[\sum_{\mathrm{nm},jk}\bar\psi_{\mathrm{n}j}M_{\mathrm{nm},jk}\psi_{\mathrm{m}k}\right]}{(\det M)^{2^{d/2}}}\times\prod_{\mathrm{n},i}\delta(U_{\mathrm{n}i}-U'_{\mathrm{n}i}) \ . \tag{3.4}$$

To derive fermion anticommutation equations, consider the α Dirac component and j color component of the operator $\bar\psi_{\mathrm{n}l}$ acting on Φ, i.e.,

$$\begin{aligned}\bar\psi^{\alpha}_{\mathrm{n}jl}\Phi(\bar\psi_u,\psi_l,U)&=\langle \bar\psi_u,\psi_l,U \mid \bar\psi^{\alpha}_{\mathrm{n}jl}\mid\Phi\rangle\\&=\prod_{\mathrm{m},i}\int d\bar\xi_{\mathrm{m}}d\xi_{\mathrm{m}}dV_{\mathrm{m}i}\bar\xi^{\alpha}_{\mathrm{n}jl}\\&\quad\times\langle \bar\psi_u,\psi_l,U \mid \bar\xi_l,\xi_u,V\rangle\\&\quad\times T(\bar\xi,\xi,V)\langle \bar\xi_u,\xi_l,V \mid \Phi\rangle \ ,\end{aligned} \tag{3.5}$$

where we have used (2.7) and (2.5) to derive (3.5). Note, however, from (3.4), using anticommuting fermion derivatives, we have

$$\bar\xi^{\alpha}_{\mathrm{n}jl}\langle \bar\psi_u,\psi_l,U \mid \bar\xi_l,\xi_u,V\rangle=-\sum_{\mathrm{m},k}M^{-1}{}_{\mathrm{mn},kj}[U]\frac{\delta}{\delta\psi^{\alpha}_{\mathrm{m}kl}}\langle \bar\psi_u,\psi_l,U \mid \bar\xi_l,\xi_u,V\rangle \ . \tag{3.6}$$

Hence,

$$\bar\psi^{\alpha}_{\mathrm{n}jl}\Phi(\bar\psi_u,\psi_l,U)=-\sum_{\mathrm{m},k}M^{-1}{}_{\mathrm{mn},kj}\frac{\delta}{\delta\psi^{\alpha}_{\mathrm{m}kl}}\Phi(\bar\psi_u,\psi_l,U) \ . \tag{3.7}$$

Similarly,

$$\psi^{\alpha}_{\mathrm{n}ju}\Phi(\bar\psi_u,\psi_l,U)=\sum_{\mathrm{m},k}M^{-1}{}_{\mathrm{nm},jk}\frac{\delta}{\delta\bar\psi^{\alpha}_{\mathrm{m}ku}}\Phi(\bar\psi_u,\psi_l,U) \ . \tag{3.8}$$

Using the anticommuting property of fermion variables, it follows from (3.7) and (3.8) that the Wilson fermion equal-time anticommutator is[9]

$$\{\bar\psi^{\alpha}_{\mathrm{n}j},\psi^{\beta}_{\mathrm{m}k}\}=M^{-1}{}_{\mathrm{mn},kj}[U]\gamma_0^{\alpha\beta} \ . \tag{3.9}$$

The noncanonical result obtained in (3.9) shows that the fermion equal-time anticommutator is nonlocal and depends on the gauge field $U_{\mathrm{n}i}$. Our result is analogous to anomalies in current commutators, the so-called Schwinger terms, which are well known from current algebra. The anticommutator (3.9) has two possible perturbative expansions: one is as a series in powers of K; the other is as a power series in the gauge-field variable B^{α}_{ni} and being appropriate for $g\to 0$. We discuss these expansions for $\det M$ in Appendix B and show to one loop that $\det M$ is free from ultraviolet mass divergences. Note for $K=0$, we recover the canonical anticommutation equation from (3.9) and it is a reflection of the canonical metric. To obtain the Hamiltonian operator from the transfer matrix $\exp(-aH)$ we use Eq. (3.4) to subtract out the inner product from Eq. (2.18).

IV. HAMILTONIAN

If we use a symmetric space-time lattice, we have to take the logarithm of the transfer matrix $\exp(-aH)$ to obtain the Hamiltonian; the result cannot be obtained in closed form and is not very useful. We instead start with an asymmetric lattice with the time lattice spacing being ϵ and the spatial lattice spacing being a, and take the limit of $\epsilon \to 0$ (Refs. 5 and 10); in this limit we will obtain a relatively simple Hamiltonian. The relations given in Eqs. (2.1) are all valid for the asymmetric lattice except that

$$B^{a}_{n0}=\epsilon g A^{a}_{0}(x), \quad x=(n_0\epsilon,\mathbf{n}a) . \tag{4.1}$$

We write an asymmetric lattice Lagrangian such that (i) we recover the expected classical continuum limit, (ii) we recover the Wilson Lagrangian for $\epsilon=a$ symmetric lattice, and (iii) we incorporate the metric and the inner product given by (2.13) and (3.4), respectively, into the Lagrangian so as to obtain a Hermitian Hamiltonian. Hence, we have the asymmetric lattice Lagrangian density given by

$$\begin{aligned}
\mathcal{L}=&\frac{1}{2}\left[\frac{\epsilon}{a}(2K-1)+1\right][\bar\psi_{n+\hat 0}(1+\gamma_0)U^\dagger_{n0}\psi_n+\bar\psi_n(1-\gamma_0)U_{n0}\psi_{n+\hat 0}]\\
&+\tfrac{1}{2}K\left[\frac{\epsilon}{a}-1\right]\sum_i[\bar\psi_{n+\hat 0}(1+\gamma_0)U_{n+\hat 0 i}U^\dagger_{n+\hat 1 0}\psi_{n+\hat 1}+\bar\psi_{n+\hat 1}U_{n+\hat 1 0}U^\dagger_{n+\hat 0 i}(1-\gamma_0)\psi_{n+\hat 0}\\
&\qquad+\bar\psi_n(1-\gamma_0)U_{ni}U_{n+\hat 1 0}\psi_{n+\hat 1+\hat 0}+\bar\psi_{n+\hat 1+\hat 0}(1+\gamma_0)U^\dagger_{n+\hat 1 0}U^\dagger_{ni}\psi_n]\\
&-\frac{\epsilon}{a}K\sum_i(\bar\psi_n\gamma_i U_{ni}\psi_{n+\hat 1}-\bar\psi_{n+\hat 1}\gamma_i U^\dagger_{ni}\psi_n)-\bar\psi_n\psi_n+K\sum_i(\bar\psi_n U_{ni}\psi_{n+\hat 1}+\bar\psi_{n+\hat 1}U^\dagger_{ni}\psi_n)\\
&+\frac{a}{\epsilon}\frac{1}{g^2}\sum_i \mathrm{Tr}(U_{ni}U_{n+\hat 1 0}U^\dagger_{n+\hat 0 i}U^\dagger_{n0}+U_{n0}U_{n+\hat 0 i}U^\dagger_{n+\hat 1 0}U^\dagger_{ni})\\
&+\frac{\epsilon}{a}\frac{1}{g^2}\sum_{ij}\mathrm{Tr}(W_{nij})+2^{d/2}\left[\frac{\epsilon}{a}-1\right]\ln\det M[U] .
\end{aligned} \tag{4.2}$$

We verify in Appendix C that this Lagrangian, because of some nontrivial cancellations, has the expected classical continuum limit. This Lagrangian is much more complicated than the asymmetric action used by Creutz[5] and is due to the more complicated Wilson metric. Note for $\epsilon=a$ we recover the symmetric Wilson action. There are two new types of term for the asymmetric lattice: namely, the second term and the last term in (4.2). The second term arises from the inner product given by (3.4) taken between states which are nearest neighbors in time. The last term also arises from the inner product, essentially the determinant of the matrix, and it is discussed in Appendix B; this term is local in time but nonlocal in space, and is purely a quantum effect which vanishes in the classical limit as $\hbar$.

From the Lagrangian given by (4.2), we construct the action and obtain, repeating Eqs. (2.11) to (2.14) in the axial gauge as in (2.18),

$$\begin{aligned}
\langle\bar\psi_u,\psi_l,U'\,|\exp(-\epsilon H)|\,\bar\psi_l,\psi_u,U\rangle=\exp\Bigg[&\left[\frac{\epsilon}{a}(2K-1)+1\right]\sum_{\mathbf{n}}(\bar\psi_{\mathbf{n}u}\psi_{\mathbf{n}u}+\bar\psi_{\mathbf{n}l}\psi_{\mathbf{n}l})\\
&+K\left[\frac{\epsilon}{a}-1\right]\sum_{\mathbf{n},i}(\bar\psi_{\mathbf{n}u}U'_{\mathbf{n}i}\psi_{\mathbf{n}+\hat 1 u}+\bar\psi_{\mathbf{n}+\hat 1 l}U'^\dagger_{\mathbf{n}i}\psi_{\mathbf{n}l}+\bar\psi_{\mathbf{n}l}U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat 1 l}+\bar\psi_{\mathbf{n}+\hat 1 u}U^\dagger_{\mathbf{n}i}\psi_{\mathbf{n}u})\\
&+i\frac{\epsilon}{a}K\sum_{\mathbf{n}i}(\bar\psi_{\mathbf{n}u}\sigma_i U'_{\mathbf{n}i}\psi_{\mathbf{n}+\hat 1 l}-\bar\psi_{n+\hat 1 u}\sigma_i U'^\dagger_{\mathbf{n}i}\psi_{\mathbf{n}l}+\bar\psi_{\mathbf{n}l}\sigma_i U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat 1 u}-\bar\psi_{\mathbf{n}+\hat 1 l}\sigma_i U^\dagger_{\mathbf{n}i}\psi_{\mathbf{n}u})\\
&+\frac{a}{\epsilon}\frac{1}{g^2}\sum_{\mathbf{n},i}\mathrm{Tr}(U_{\mathbf{n}i}U'^\dagger_{\mathbf{n}i}+U'_{\mathbf{n}i}U^\dagger_{\mathbf{n}i})+\frac{\epsilon}{a}\frac{1}{2g^2}\sum_{\mathbf{n},ij}\mathrm{Tr}(W_{\mathbf{n}ij}+W'_{\mathbf{n}ij})\\
&+\frac{2^{d/2}}{2}\left[\frac{\epsilon}{a}-1\right](\ln\det M[U]+\ln\det M[U'])\Bigg] .
\end{aligned} \tag{4.3}$$

Note the terms independent of ϵ in the Lagrangian yield the metric and the inner product. For $\epsilon \to 0$, we have

$$\exp\left[+\frac{g^2\epsilon}{2a}\sum_{\mathbf{n},i}\nabla^2(U_{\mathbf{n}i})\right]\langle\bar\psi_u,\psi_l,U'\,|\,\bar\psi_l,\psi_u,U\rangle \cong \frac{\exp\left[\sum_{\mathbf{nm},jk}\bar\psi_{\mathbf{n}j}M_{\mathbf{nm},jk}[U]\psi_{\mathbf{m}k}\right]}{[\det\mathbf{M}]^{2^{d/2}}}\exp\left[\frac{a}{\epsilon}\frac{1}{g^2}\sum_{\mathbf{n}i}\mathrm{Tr}(U_{\mathbf{n}i}U'^{\dagger}_{\mathbf{n}i}+U'_{\mathbf{n}i}U^{\dagger}_{\mathbf{n}i})\right], \tag{4.4}$$

where ∇^2 is the Laplace-Beltrami operator for SU(N).

We also have from Eq. (A4a), to leading order in ϵ, the Hamiltonian differential operator H given by

$$\langle\bar\psi_u,\psi_l,U'\,|\exp(-\epsilon H)\,|\,\bar\psi_l,\psi_u,U\rangle$$

$$\simeq e^{-\epsilon\hat H(\bar\psi,\psi,U,U')}\langle\bar\psi_u,\psi_l,U'\,|\,\bar\psi_l,\psi_u,U\rangle \tag{4.5}$$

$$=\exp\left[-\epsilon\hat H(\bar\psi,\psi,U)+\sum_{\mathbf{nm},jk}\bar\psi_{\mathbf{n}jk}M_{\mathbf{nm},jk}[U]\psi_{\mathbf{m}k}-2^{d/2}\ln\det M[U]\right]\prod_{\mathbf{n}i}\delta(U_{\mathbf{n}i}-U'_{\mathbf{n}i}), \tag{4.6}$$

where we have used Eq. (3.4) to obtain (4.6), and that, due to (4.4), $U_{ni}\cong U'_{ni}$ to $O(\epsilon)$ in Eq. (4.3). Hence, from (4.3), (4.4), and (4.6) we have, taking ϵ to zero,

$$\hat H=-\frac{g^2}{2a}\sum_{\mathbf{n},i}\nabla^2(U_{\mathbf{n}i})-\frac{1}{ag^2}\sum_{\mathbf{n},ij}\mathrm{Tr}(W_{\mathbf{n}ij})-\frac{1}{a}(2K-1)\sum_{\mathbf{n}}\bar\psi_{\mathbf{n}}\psi_{\mathbf{n}}$$
$$-\frac{K}{a}\sum_{\mathbf{n},i}[\bar\psi_{\mathbf{n}}(1-\gamma_i)U_{\mathbf{n}i}\psi_{\mathbf{n}+\hat{1}}+\bar\psi_{\mathbf{n}+\hat{1}}(1+\gamma_i)U^{\dagger}_{\mathbf{n}i}\psi_{\mathbf{n}}]-\frac{1}{a}2^{d/2}\ln\det M[U]. \tag{4.7}$$

This is a Hamiltonian for the lattice gauge theory with Wilson fermions. Recall from (3.9) the fermion equal-time anticommutation equation is

$$\{\bar\psi^{\alpha}_{\mathbf{n}j},\psi^{\beta}_{\mathbf{m}k}\}=M^{-1}{}_{\mathbf{mn},kj}[U]\gamma_0^{\alpha\beta}. \tag{4.8}$$

Equations (4.7) and (4.8) provide a complete description for the operator formulation of the lattice theory.

The pure gauge field part of H is the well-known Hamiltonian derived by Kogut and Susskind.[1] The fermion part of (4.7) is similar to Creutz's[5] result, except that he has canonical anticommutation equations for the fermions instead of (4.8). Also, Creutz uses coherent fermion states for the initial and final field configurations and these have a generalization, as will be shown, which can yield the Wilson metric. The last term in the Hamiltonian given by (4.7) has not been previously derived and is a direct reflection of the Wilson metric.

The noncanonical anticommutation equation (4.8) as well as the metric (2.13) can be reduced to the canonical form by the following transformation:

$$\psi_{\mathbf{n}j}=\sum_{\mathbf{m},k}M^{-1/2}{}_{\mathbf{nm},jk}\chi_{\mathbf{m}k}, \tag{4.9a}$$

$$\bar\psi_{\mathbf{m}k}=\sum_{\mathbf{n},j}\bar\chi_{\mathbf{n}j}M^{-1/2}{}_{\mathbf{nm},jk}. \tag{4.9b}$$

Note only $\bar\psi_{\mathbf{n}u}$ and $\psi_{\mathbf{n}l}$ are independent variables and the transformation of $\bar\psi_{\mathbf{n}l}$ and $\psi_{\mathbf{n}u}$ are fixed by (3.7) and (3.8), respectively; Eq. (4.9) is to be understood in this sense. This yields, from (4.8) and (4.9), the canonical anticommutator

$$\{\bar\chi^{\alpha}_{\mathbf{n}j},\chi^{\beta}_{\mathbf{m}k}\}=\delta_{\mathbf{nm}}\delta_{jk}\gamma_0^{\alpha\beta}. \tag{4.10}$$

For the fermion sector of the Hamiltonian, from (4.7)

$$-aH_F=-\sum_{\mathbf{n}}\bar\chi_{\mathbf{n}}\chi_{\mathbf{n}}+2K\sum_{\mathbf{nm}}\bar\chi_{\mathbf{n}}M^{-1}{}_{\mathbf{nm}}\chi_{\mathbf{m}}$$
$$-K\sum_{\mathbf{pnm},i}\bar\chi_{\mathbf{n}}M^{-1/2}{}_{\mathbf{nm}}\gamma_i U_{\mathbf{m}i}M^{-1/2}{}_{\mathbf{m}+\hat{1}l}\chi_{\mathbf{p}}$$
$$+K\sum_{\mathbf{pnm},i}\bar\chi_{\mathbf{n}}M^{-1/2}{}_{\mathbf{nm}+\hat{1}}\gamma_i U^{\dagger}_{\mathbf{m}i}M^{-1/2}{}_{\mathbf{m}l}\chi_{\mathbf{p}}. \tag{4.11}$$

And finally, from (2.13) and (4.9), we obtain the canonical metric:

$$T(\bar\chi,\chi)=\exp\left[-\sum_{\mathbf{n}}\bar\chi_{\mathbf{n}}\chi_{\mathbf{n}}\right]. \tag{4.12}$$

The canonical form for H involving $\bar\chi$ and χ is more suitable for perturbation theory since we can use the Fock basis for the $\bar\chi,\chi$ fields. Note that the canonical form for H_F in (4.11) makes H_F nonlocal; noncanonical anticommutation leads to new interactions in H involving the matrix $M^{-1/2}$.

We can expand $M^{-1/2}{}_{\mathbf{nm}}$ as a power series in K, and obtain strings of gauge-field links of length L of the type K^L $(\prod_{\mathbf{p}}U_{\mathbf{p}})$ running from $\mathbf{n}$ to $\mathbf{m}$ (Fig. 1). H_F has three-site nonlocal interactions between a quark at point $\mathbf{n}$, a gauge-field link between $\mathbf{m}$ and $\mathbf{m}+\hat{1}$, and an antiquark at l (Fig. 2). These nonlocal interactions also are present in the quark-color charge operator given in (5.12).

We briefly discuss the relation of the coordinate eigenstates $|\bar\psi_l,\psi_u,U\rangle$ with the fermion coherent state formalism used by Creutz.[5] Using the canonical fermion operators $\bar\chi_n,\chi_n$ given by (4.10), we define the "bare vacuum" state $|0\rangle$ by[5,7]

$$\chi_{nu}|0\rangle=\bar\chi_{nl}|0\rangle=0, \tag{4.13a}$$

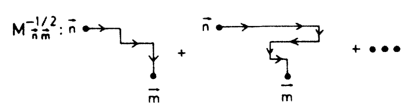

FIG. 1. Expansion of $M^{-1/2}{}_{nm}$ in terms of gauge-field string operators.

$$\langle 0 | 0 \rangle = 1 \ . \tag{4.13b}$$

The coordinate eigenstates satisfying the completeness equation (2.7) with the Wilson metric (2.13) can be defined (suppressing summation on all lattice and internal indices, and denoting the gauge-field coordinate eigenstate by $| U \rangle\rangle$) as (c is a function of $\det M[U]$)

$$| \bar{\psi}_l, \psi_u, U \rangle = c \exp(\bar{\psi}_l M^{1/2} \chi_l + \bar{\chi}_u M^{1/2} \psi_u) | 0 \rangle \, | U \rangle \tag{4.14a}$$

and, using the rules of conjugation given in (A2), we have

$$\langle \bar{\psi}_u, \psi_l, U | = c \langle U | \langle 0 | \exp(\bar{\chi}_l M^{1/2} \psi_l + \bar{\psi}_u M^{1/2} \chi_l) \ , \tag{4.14b}$$

where we have used the property that M is Hermitian to obtain the above equation.

Using the canonical anticommutation of $\bar{\chi}$ and χ and (4.13) we obtain (fixing c appropriately)

$$\langle \bar{\psi}_u, \psi_l, U' | \bar{\psi}_l, \psi_u, U \rangle = \frac{\exp(\bar{\psi} M \psi)}{(\det M)^{2^{d/2}}} \prod_{n,i} \delta(U'_{ni} - U_{ni}) \tag{4.14c}$$

which is simply Eq. (3.4). We see that Eq. (4.14) is a nontrivial generalization of fermion coherent states as the gauge field is directly involved in its construction.

We examine the limiting case of free Wilson fermions, as this is the first step in any weak-coupling calculation. We set the gauge field to zero, obtaining from (4.7), (4.8), and (3.3), up to a constant,

$$H = -\frac{1}{a}(2K-1) \sum_{\mathbf{n}} \bar{\psi}_{\mathbf{n}} \psi_{\mathbf{n}} - \frac{1}{a} K \sum_{\mathbf{n},i} [\bar{\psi}_{\mathbf{n}} (1-\gamma_i) \psi_{\mathbf{n}+\hat{1}} + \bar{\psi}_{\mathbf{n}+\hat{1}} (1+\gamma_i) \psi_{\mathbf{n}}] \tag{4.15a}$$

and

$$\{ \bar{\psi}^{\alpha}_{\mathbf{n}j}, \psi^{\beta}_{\mathbf{m}k} \} = \delta_{jk} \widetilde{M}^{-1}{}_{\mathbf{mn}} \gamma_0^{\alpha\beta} \ , \tag{4.15b}$$

$$\widetilde{M}_{\mathbf{nm}} = \delta_{\mathbf{n},\mathbf{m}} - K \sum_i (\delta_{\mathbf{n}+\hat{1},\mathbf{m}} + \delta_{\mathbf{n},\mathbf{m}+\hat{1}}) \ . \tag{4.15c}$$

For the infinite spatial lattice, define Fourier transforms for the canonical field

$$\chi_{\mathbf{n}} = \int_p e^{i\mathbf{p}\cdot\mathbf{n}} \chi_{\mathbf{p}}, \quad \bar{\chi}_{\mathbf{n}} = \int_p e^{-i\mathbf{p}\cdot\mathbf{n}} \bar{\chi}_{\mathbf{p}} \tag{4.16a}$$

with

$$\int_p \equiv \int_{-\pi}^{+\pi} \frac{d^{d-1}p}{(2\pi)^{d-1}} \ . \tag{4.16b}$$

We then have from (4.9) and (4.11)

$$H = \int_p \bar{\chi}_{\mathbf{p}} \frac{1}{\lambda_p} (\alpha + i\boldsymbol{\gamma}\cdot\boldsymbol{\beta}) \chi_{\mathbf{p}} \tag{4.17}$$

and from (4.10)

$$\{ \bar{\chi}_{\mathbf{p}'}, \chi_{\mathbf{p}} \} = \gamma_0 \delta_{\mathbf{p}',\mathbf{p}} \ , \tag{4.18}$$

where

$$\alpha = \frac{1}{a} \left[(1-2K) - 2K \sum_i \cos p_i \right] , \tag{4.19a}$$

$$\beta_i = \frac{1}{a} 2K \sin p_i \ , \tag{4.19b}$$

$$\lambda_{\mathbf{p}} = 1 - 2K \sum_i \cos p_i \ . \tag{4.19c}$$

We obtain the energy eigenspectrum of equally spaced energy levels in (4.17), where the energy of a single particle or antiparticle excitation is given by[7,11]

$$E_{\mathbf{p}} = \frac{1}{\lambda_p} (\alpha^2 + \boldsymbol{\beta}^2)^{1/2} \tag{4.20a}$$

$$= \frac{1}{a} \frac{\left[\left[1 - 2K - 2K \sum_i \cos p_i \right]^2 + 4K^2 \sum_i \sin^2 p_i \right]^{1/2}}{\left[1 - 2K \sum_i \cos p_i \right]} \ . \tag{4.20b}$$

We make the following observations. First, energy E_p is a monotonic function of p; this is due to the cosine term in the numerator of (4.20b) coming from the Wilson projectors $1 \pm \gamma_i$. If this cosine term were absent, then the energy of a zero-momentum quark and that of a quark of momentum π for some p_i would be equal, since $\sin p_i = 0$ for both of these cases. In the continuum limit, taking all possibilities into account, this would result in 2^{d-1} quark species[11] and would be the reflection of the overcounting of quark states in the lattice Hamiltonian formulation. Second, the metric yields the factor of $1/\lambda p$ in the expression for energy. In the case of coupling to the gauge field, the noncanonical anticommutation equation (4.8) yields more complicated contributions to the energy given in (4.11). Third, taking the continuum limit of $a \to 0$ for (4.20) using (2.1) and setting $p = ka$ ($k \in [-\infty, +\infty]$ is the continuum momentum) we have $E_k = (k^2 + m_0{}^2)^{1/2}$ as expected. Fourth, energy E_p can be directly obtained from the Lagrangian (4.2) by locating the pole of the propagator, setting $p_0 = -i\epsilon E_p$, and taking $\epsilon \to 0$ limit.

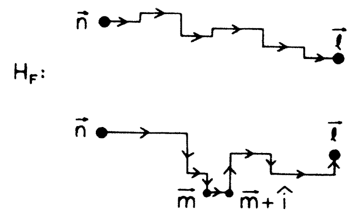

FIG. 2. Nonlocal interactions in the Hamiltonian for the fermion sector.

V. COLOR CHARGE OPERATOR AND GAUSS'S LAW

In the Lagrangian approach, the charge operator and Gauss's law are obtained by exploiting symmetries of the Lagrangian. In the Hamiltonian formulation, these result from transformation properties and symmetries of wave functionals.

Consider first the gauge field for SU(N): right and left group multiplication for group elements are given by Hermitian generators E_a^L and E_a^R, respectively, where[12]

$$[E_a^L, E_b^L] = iC_{abc}E_c^L \ , \tag{5.1a}$$

$$[E_a^R, E_b^R] = -iC_{abc}E_c^R \ , \tag{5.1b}$$

$$[E_a^L, E_b^R] = 0 \ . \tag{5.1c}$$

Note the minus sign in (5.1b). E_a^R and E_a^L are first-order Hermitian differential operators on $L_2(\mathrm{SU}(N))$. For the group element U we have the operator equation

$$E_a^L(U) = R_{ba}(U)E_b^R(U) \ , \tag{5.2}$$

where $R_{ab}(U)$ is the adjoint representation of U. We hence have for an arbitrary function of U, the SU(N) Taylor theorem[12]

$$f(e^{i\phi^a X_a} U e^{i\sigma^a X_a}) = e^{i\phi^a E_a^R(U)} e^{i\sigma^a E_a^L(U)} f(U) \tag{5.3a}$$

$$= e^{i(\phi^a E_a^R + \sigma^a E_a^L)} f(U) \ , \tag{5.3b}$$

where we have used (5.1c) to obtain (5.3b).

It can be shown that for SU(N)

$$-\nabla^2(U) = \sum_a E_a^R(U)E_a^R(U) = \sum_a E_a^L(U)E_a^L(U) \ . \tag{5.3c}$$

Form Eq. (5.3c), we see that either E_a^R or E_a^L can be identified as the chromoelectric field operator for the lattice gauge field. The analogue of Eq. (5.3b) for the fermions is more complicated. We discuss this in Appendix A and have, from Eq. (A12),

$$h(\bar\psi_u e^{-i\phi^a X_a}, e^{i\sigma^a X_a}\psi_l) = \exp\left[-i\phi^a X_a^{jk} \bar\psi_{ju}^{\alpha} \frac{\delta}{\delta \bar\psi_{ku}^{\alpha}} + i\sigma^a X_a^{jk} \psi_{kl}^{\alpha} \frac{\delta}{\delta \psi_{jl}^{\alpha}} \right] \times h(\bar\psi_u, \psi_l) \ . \tag{5.4}$$

We define the charge operator as the generator of gauge transformations in the following manner. Consider the (time-independent) gauge transformation

$$U_{\mathbf{n}i} \to \phi_{\mathbf{n}} U_{\mathbf{n}i} \phi^{\dagger}_{\mathbf{n}+\hat{i}}, \quad \phi_{\mathbf{n}} = e^{i\phi_{\mathbf{n}}^a X_a} \ , \tag{5.5a}$$

$$\psi_{\mathbf{n}} \to \phi_{\mathbf{n}} \psi_{\mathbf{n}}, \quad \bar\psi_{\mathbf{n}} \to \bar\psi_{\mathbf{n}} \phi_{\mathbf{n}}^{\dagger} \ . \tag{5.5b}$$

Hence, from (5.3) and (5.4), we have, for the wave functional,

$$\Phi(\bar\psi_u, \psi_l, U) \to \Phi(\bar\psi_u \phi^{\dagger}, \phi\psi_l, U(\phi)) \tag{5.6a}$$

$$= \prod_{\mathbf{n},i} e^{i\phi_{\mathbf{n}}^a E_a^R(U_{\mathbf{n}i})} e^{-i\phi^a_{\mathbf{n}+\hat{i}} E_a^L(U_{\mathbf{n}i})} \prod_{\mathbf{n}} \exp\left[i\phi_{\mathbf{n}}^a X_a^{jk} \left[\psi_{\mathbf{n}kl}^{\alpha} \frac{\delta}{\delta\psi_{\mathbf{n}jl}^{\alpha}} - \bar\psi_{\mathbf{n}ju}^{\alpha} \frac{\delta}{\delta\bar\psi_{\mathbf{n}ku}^{\alpha}} \right] \right] \Phi(\bar\psi_u, \psi_l, U) \tag{5.6b}$$

$$= \exp\left[i \sum_{\mathbf{n}} \phi_{\mathbf{n}}^a \left[\sum_i [E_a^R(U_{\mathbf{n}i}) - E_a^L(U_{\mathbf{n}-\hat{i},i})] - \rho_{\mathbf{n}a}(\bar\psi, \psi) \right] \right] \Phi(\bar\psi_u, \psi_l, U) \ , \tag{5.6c}$$

where $\rho_{\mathbf{n}a}$ is the quark color charge operator, and we have ignored topologically significant surface terms in combining the chromoelectric field operators.

For the wave functionals to be gauge invariant as required by Eq. (2.19), they have to be independent of $\phi_{\mathbf{n}}^a$. Since $\phi_{\mathbf{n}}^a$ is arbitrary in (5.6c) we must have for gauge invariance

$$\sum_i [E_a^R(U_{\mathbf{n}i}) - E_a^L(U_{\mathbf{n}-\hat{i},i})] - \rho_{\mathbf{n}a} = 0 \ . \tag{5.7}$$

Using Eq. (5.2), we have from (5.7)

$$\sum_i [E_a^R(U_{\mathbf{n}i}) - R_{ab}(U^{\dagger}_{\mathbf{n}-\hat{i},i}) E_b^R(U_{\mathbf{n}-\hat{i},i})] = \rho_{\mathbf{n}a} \ . \tag{5.8}$$

If we choose to identify E_a^R as the chromoelectric field operator, then (5.8) is Gauss's law using only E_a^R.

Equation (5.7) is Gauss's law for the lattice gauge theory; it is understood that this is not an operator equation and that the operator on the left-hand side of (5.7) is acting on wave functionals. From our derivation, it is clear that Gauss's law is a differential statement of the wave functionals being gauge invariant and expresses local conservation of color charge at the lattice site $\mathbf{n}$.

To completely define the operator $\rho_{\mathbf{n}a}$, note from its definition

$$\rho_{\mathbf{n}a} = X_a^{jk} \left[\bar\psi_{\mathbf{n}ju}^{\alpha} \frac{\delta}{\delta\bar\psi_{\mathbf{n}ku}^{\alpha}} - \psi_{\mathbf{n}kl}^{\alpha} \frac{\delta}{\delta\psi_{\mathbf{n}jl}^{\alpha}} \right] \ . \tag{5.9a}$$

It follows from (5.9a) that

$$[\rho_{\mathbf{n}a}, \rho_{\mathbf{m}b}] = iC_{abc}\rho_{\mathbf{n}c}\delta_{\mathbf{n}\mathbf{m}} \tag{5.9b}$$

and hence $\rho_{\mathbf{n}a}$ are generators of SU(N) local gauge transformations. Inverting Eqs. (3.7) and (3.8) we have

$$\frac{\delta}{\delta\bar\psi_{\mathbf{n}ku}^{\alpha}} = \sum_{\mathbf{m},j} M_{\mathbf{n}\mathbf{m},kj}[U]\psi_{\mathbf{m}ju}^{\alpha} \tag{5.10a}$$

and

$$\frac{\delta}{\delta\psi_{\mathbf{n}jl}^{\alpha}} = -\sum_{\mathbf{m},k} M_{\mathbf{m}\mathbf{n},kj}[U]\bar\psi_{\mathbf{m}kl}^{\alpha} \ . \tag{5.10b}$$

Combining (5.9a) and (5.10) we have, using Eq. (3.3a) and the fact that X_a's are traceless,

$$\rho_{\mathrm{n}a}=\bar{\psi}_{\mathrm{n}}\gamma_0 X_a\psi_{\mathrm{n}}-K\sum_i(\bar{\psi}_{\mathrm{n}u}X_a U_{\mathrm{n}i}\psi_{\mathrm{n}+\hat{1}u}+\bar{\psi}_{\mathrm{n}u}X_a U^{\dagger}_{\mathrm{n}-\hat{1},i}\psi_{\mathrm{n}-\hat{1}u})+K\sum_i(\bar{\psi}_{\mathrm{n}-\hat{1}l}U_{\mathrm{n}-\hat{1}i}X_a\psi_{\mathrm{n}l}+\bar{\psi}_{\mathrm{n}+\hat{1}l}U^{\dagger}_{\mathrm{n}i}X_a\psi_{\mathrm{n}l}) . \tag{5.11}$$

Note that the regulated quark charge operator $\rho_{\mathrm{n}a}$ involves the gauge field due to the Wilson metric; for $K=0$ we obtain the expected canonical result.

Equation (5.11) together with (5.7) and (4.8) gives a complete definition of Gauss's law.

In terms of canonical fermions $\bar{\chi}$ and χ given by (4.9) and (4.10), we have the charge-density operator

$$\rho_{\mathrm{n}a}=\sum_{\mathrm{mp}}(\bar{\chi}_{\mathrm{m}u}M^{-1/2}{}_{\mathrm{mn}}X_a M^{1/2}{}_{\mathrm{np}}\chi_{\mathrm{p}u}-\bar{\chi}_{\mathrm{m}l}M^{1/2}{}_{\mathrm{mn}}X_a M^{-1/2}{}_{\mathrm{np}}\chi_{\mathrm{p}l}) . \tag{5.12}$$

The definition of quark charge operator involves the matrices $M^{-1/2}$ and $M^{1/2}$. Note for the Abelian field the charge operator is in effect local, i.e., for total charge

$$Q^{\mathrm{Abelian}}=\sum_{\mathrm{n}}\rho_{\mathrm{n}}^{\mathrm{Abelian}}=\sum_{\mathrm{n}}\bar{\chi}_{\mathrm{n}}\gamma_0\chi_{\mathrm{n}} . \tag{5.13}$$

Hence, only for the non-Abelian case does M couple to the charge operator and render it truly nonlocal.

We illustrate the lattice result (5.8) by taking the $a\to 0$ classical continuum limit; in this limit, the continuum chromoelectric field operator is simply the differential operator,

$$E_{ai}^{\mathrm{cont}}(\mathbf{x})=\frac{\delta}{i\delta A_i^a(\mathbf{x})} \tag{5.14}$$

and for which, unlike (5.1), the different color components commute. Using

$$E_a^R(U_{\mathrm{n}i})=[\delta_{ab}-\tfrac{1}{2}C_{abc}B^c_{\mathrm{n}i}+O(B^2)]\frac{\partial}{i\partial B^b_{\mathrm{n}i}} , \tag{5.15a}$$

$$R_{ab}(U_{\mathrm{n}i})=\delta_{ab}-C_{abc}B^c_{\mathrm{n}i}+O(B^2) , \tag{5.15b}$$

and

$$\frac{\partial f(\mathbf{x})}{\partial x_i}=\lim_{a\to 0}\frac{1}{a}(f_{\mathrm{n}}-f_{\mathrm{n}-\hat{1}}) \tag{5.16}$$

we have from (2.1), (5.8), (5.11), and (5.15) the expected continuum Gauss's law

$$\sum_i\left[\frac{\partial}{\partial x_i}E_{ai}^{\mathrm{cont}}(\mathbf{x})+C_{abc}A_i^b(\mathbf{x})E_{ci}^{\mathrm{cont}}(\mathbf{x})\right]=\bar{\psi}(\mathbf{x})\gamma_0 X_a\psi(\mathbf{x}) . \tag{5.17}$$

Equation (5.17) is valid only classically since we have assumed the fields $A_\mu(x)$, $\psi(x)$, and $\bar{\psi}(x)$ are continuous and differentiable. For the quantum case, the new terms in (5.8) which arise due to the (lattice) cutoff all contribute to the renormalized quantum continuum limit.

VI. LATTICE ACTION FROM LATTICE HAMILTONIAN

Suppose the starting point for the theory is taken to be the Hamiltonian. We now derive the action from the Hamiltonian. Suppose we have in the axial gauge

$$\hat{H}(\bar{\psi},\psi)=m_0\sum_{\mathrm{n}}\bar{\psi}_{\mathrm{n}}\psi_{\mathrm{n}}+\sum_{\mathrm{nm}}\bar{\psi}_{\mathrm{n}}\begin{bmatrix}0 & h_{\mathrm{nm}}\\ h^{\dagger}_{\mathrm{nm}} & 0\end{bmatrix}\psi_{\mathrm{m}}+H_{\mathrm{GF}} \tag{6.1a}$$

with canonical anticommutation equation

$$\{\bar{\psi}^{\alpha}_{\mathrm{n}j},\psi^{\beta}_{\mathrm{m}k}\}=\gamma_0^{\alpha\beta}\delta_{\mathrm{nm}}\delta_{jk} . \tag{6.1b}$$

The off-diagonal coupling h_{nm} can be nonlocal, and involves the gauge field for the interacting theory. The Hamiltonian in (6.1) for the case of $m_0=0$ includes the chirally invariant SLAC lattice Hamiltonian[3] and the Susskind fermions.[2]

For the metric we have from (6.1b)

$$T(\bar{\psi},\psi)=\exp\left[-\sum_{\mathrm{n}}\bar{\psi}_{\mathrm{n}}\psi_{\mathrm{n}}\right] , \tag{6.2}$$

and from (3.1) we have

$$\langle\bar{\psi}_u,\psi_l,U'\,|\,\bar{\psi}_l,\psi_u,U\rangle=\exp\left[\sum_{\mathrm{n}}\bar{\psi}_{\mathrm{n}}\psi_{\mathrm{n}}\right]\prod_{\mathrm{n},i}\delta(U_{\mathrm{n}i}-U'_{\mathrm{n}i}) . \tag{6.3}$$

Hence, for the evolution kernel we have

$$K=\langle\bar{\psi}_u,\psi_l,U'\,|\,e^{-\epsilon MH}\,|\,\bar{\psi}_l,\psi_u,U\rangle \tag{6.4}$$

$$=\prod_{n_0=1}^{M-1}\prod_{n\mu}\int d\bar{\psi}_n d\psi_n dU_{n\mu}e^{A} , \tag{6.5}$$

where the boundary conditions for (6.5) are given in Eq. (2.9). Using the completeness equation (2.7) in (6.4) $M-1$ times and using (6.2) for the metric, we have

$$K=\prod_{n_0=1}^{M-1}\prod_{ni}\int d\bar{\psi}_n d\psi_n dU_{ni}e^{-\bar{\psi}_n\psi_n}\langle\bar{\psi}_{n+\hat{0}u},\psi_{n+\hat{0}l},U_{n+\hat{0}i}\,|\,e^{-\epsilon H}\,|\,\bar{\psi}_{nl},\psi_{nl},U_n\rangle . \tag{6.6}$$

And hence, from (6.3) and (A4a), for the fermion part we have to leading order in ϵ

$$K_F\cong\prod_n\int d\bar{\psi}_n d\psi_n dU_{ni}\exp\left[-\sum_n\bar{\psi}_n\psi_n+\sum_n(\bar{\psi}_{n+\hat{0}u}\psi_{nu}+\bar{\psi}_{nl}\psi_{n+\hat{0}l})-\epsilon\sum_{n_0}H_F(\bar{\psi}_{n+\hat{0}u},\bar{\psi}_{nl};\psi_{nu},\psi_{n+\hat{0}l})\right] . \tag{6.7}$$

From (6.1) we have

$$H_F(\bar{\psi}_{n+\hat{0}u},\bar{\psi}_{nl};\psi_{nu},\psi_{n+\hat{0}l})=+m_0\sum_{n}(\bar{\psi}_{n+\hat{0}u}\psi_{nu}+\bar{\psi}_{nl}\psi_{n+\hat{0}l})+\sum_{nm}(\bar{\psi}_{n+\hat{0}u}h_{nm}\psi_{m+\hat{0}l}+\bar{\psi}_{nl}h^{\dagger}_{nm}\psi_{mu}) . \tag{6.8}$$

From (6.7) and (6.8) we obtain the action

$$A_F=-\sum_{n}\bar{\psi}_n\psi_n+(1-m_0\epsilon)\sum_{n}(\bar{\psi}_{n+\hat{0}u}\psi_{nu}+\bar{\psi}_{nl}\psi_{n+\hat{0}l})-\epsilon\sum_{nm}\bar{\psi}_n\begin{bmatrix}0 & h_{nm}\\ h^{\dagger}_{nm} & 0\end{bmatrix}\psi_m\delta_{n_0,m_0} . \tag{6.9}$$

Transforming from the axial gauge back to a manifestly gauge-invariant expression we have the lattice action

$$A=-\sum_{n}\bar{\psi}_n\psi_n+\tfrac{1}{2}(1-m_0\epsilon)\sum_{n}[\bar{\psi}_{n+\hat{0}}(1+\gamma_0)U^{\dagger}_{n0}\psi_n+\bar{\psi}_n(1-\gamma_0)U_{n0}\psi_{n+\hat{0}}]-\epsilon\sum_{nm}\bar{\psi}_n\begin{bmatrix}0 & h_{nm}\\ h^{\dagger}_{nm} & 0\end{bmatrix}\psi_m\delta_{n_0,m_0}+A_{GF} . \tag{6.10}$$

The action has Wilson projectors $1\pm\gamma_0$ for the time coupling and is due to the fermionic coordinates of the Hilbert space. Note action A in (6.10) explicity breaks chiral symmetry, even if $m_0=0$ and the Hamiltonian is chiral invariant. The breaking of chiral symmetry is due to the structure of the fermionic Hilbert space, which has nonchiral invariant field coordinates, namely, $|\bar{\psi}_l,\psi_u\rangle$ and $\langle\bar{\psi}_u,\psi_l|$.

The lattice action obtained in (6.10) contains more information than finite time continuum action. To see this, take the limit of $M\to\infty$ and $\epsilon\to 0$ with $T=M\epsilon$ fixed, and for notational simplicity consider the continuum QCD Hamiltonian. We then have from (6.10)

$$A=\int_0^T dt\int d\mathbf{x}\,\mathscr{L}_{\rm YM}+A_{\rm boundary} \tag{6.11}$$

and

$$A_{\rm boundary}=\int d\mathbf{x}[\bar{\psi}_u(\mathbf{x})\psi_u(\mathbf{x},T)+\bar{\psi}_l(\mathbf{x})\psi_l(\mathbf{x},0)] , \tag{6.12}$$

where $\mathscr{L}_{\rm YM}$ is given in (C11). The first term in (6.11) is the finite-time action; the second term is the *boundary term* and is given by (6.12); $\bar{\psi}_u(\mathbf{x})$ and $\bar{\psi}_l(\mathbf{x})$ are part of the boundary conditions given in (2.9), whereas $\psi_u(\mathbf{x},T)$ and $\psi_l(\mathbf{x},0)$ are integration variables; note boundary values $\psi_u(\mathbf{x})$ and $\psi_l(\mathbf{x})$ are also coupled to the action. The importance of the boundary term can be seen in the case of the free Dirac field, where[7,8]

$$K(T)=\langle\bar{\psi}_u,\psi_l|\exp(-TH_{\rm Dirac})|\bar{\psi}_l,\psi_u\rangle \tag{6.13}$$

$$=C(T)\exp[A_{\rm boundary}(\bar{\psi},\psi,T)] \tag{6.14}$$

with $\psi(\mathbf{x},t)$ satisfying the classical field equations with boundary conditions given by Eq. (2.9), and $C(T)$ is a normalization function.

VII. CONCLUSIONS

We derived the lattice Hamiltonian using Wilson fermions. We found that the Hamiltonian was nonlocal due to the nontrivial Wilson metric; and, in fact, using canonical fermions made the Hamiltonian pick up new types of nonlocal interactions. We derived the gauge-field color charge operator as well as the quark color charge operator directly from properties of the wave functionals. The lattice quark charge operator for the non-Abelian case has anomalous pieces which depend upon the gauge field.

The Hamiltonian derived here is not the unique Hamiltonian which corresponds to the Wilson action, both for the gauge field as well as for the fermions. The reason for this is that to derive the Hamiltonian the time lattice spacing has to be taken much smaller than the spatial lattice spacing, and this extension to infinitesimal time is highly nonunique. Hence, Creutz's expressions reduce to the Wilson action for a symmetric space-time lattice, but give a very different result for the Hamiltonian compared to the one derived here. In particular, Creutz's Hamiltonian has a hopping parameter different from the value of K, and a different Hilbert space. Creutz's result is consistent, and we ascribe the different results to the different schemes for extending the Wilson action to infinitesimal time. Of course, some regularization schemes yield the physics more clearly than others. The results of this paper incorporate interactions in Hilbert space via the Wilson metric in a transparent way and may allow the study of anomalies using the nontrivial properties of the Hilbert space of the interacting theory.

We derived the lattice action from a given Hamiltonian, and showed that chiral symmetry is explicitly broken in the action by the coordinates of the fermionic Hilbert space. It should be possible to choose fermionic field coordinates such that a chiral-invariant lattice Hamiltonian leads to a chiral-invariant lattice action.

Using the results obtained, we can now perturbatively study the fermion Hamiltonian both in the strong- and the weak-coupling sectors. To study the QCD Hamiltonian in weak coupling, we also have to gauge fix the lattice gauge field degrees of freedom. We discuss this in a separate publication.[13]

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APPENDIX A: FERMION CALCULUS

Let $|\Phi\rangle$ be a wave functional of the interacting quark–gauge-field system. Its coordinate representation form (2.6) is

$$\Phi(\bar{\psi}_u,\psi_l,U)=\langle\bar{\psi}_u,\psi_l,U|\Phi\rangle . \tag{A1}$$

Conjugation is defined by[6,8] (1) reverse the order of the fermion variables and complex conjugate the coefficients, (2) $\psi\to\bar\psi\gamma_0$, $\bar\psi\to\gamma_0\psi$, (3) $U_{ni}\to U_{ni}^\dagger$. Hence we have for $\Phi^\dagger$, the conjugate of Φ,

$$\Phi^\dagger(\bar\psi_u,\psi_l,U)=\Phi^*(\psi_u,-\bar\psi_l,U^\dagger) \tag{A2a}$$

$$=\langle\Phi\mid\bar\psi_l,\psi_u,U\rangle \ . \tag{A2b}$$

The scalar product on the Hilbert space, using (A1), (A2), and the completeness equation (2.7), is given by

$$\langle f\mid h\rangle=\int d\bar\psi d\psi dU\langle f\mid\bar\psi_l,\psi_u,U\rangle \times T(\bar\psi,\psi,U)\langle\bar\psi_u,\psi_l,U\mid h\rangle \ . \tag{A3}$$

The matrix elements of an operator G are given by

$$\langle\bar\psi_u,\psi_l,U'\mid G\mid\bar\psi_l,\psi_u,U\rangle =G(\bar\psi,\psi;U,U')\langle\bar\psi_u,\psi_l,U'\mid\bar\psi_l,\psi_u,U\rangle \tag{A4a}$$

and Hermitian conjugation is defined as usual by

$$\langle h\mid G\mid f\rangle^*=\langle f\mid G^\dagger\mid h\rangle \ . \tag{A4b}$$

Note in (A4b), the metric via Eq. (2.7) has to be used to define the matrix element $\langle f\mid G\mid h\rangle$ and plays a central role in defining the Hermiticity of an operator. In particular, the transfer matrix $\exp(-aH)$ given in Eq. (2.14) is Hermitian only with the Wilson matrix given in (2.13).

The fermionic "Fourier transform" of $\Phi(\bar\psi_u,\psi_l,U)$ is defined using the metric $T(\bar\psi,\psi,U)$; for the Wilson metric (2.13) we have

$$\Phi(\bar\psi_u,\psi_l,U)=\int d\bar\psi_l d\psi_u T(\bar\psi,\psi,U)\Phi'(\bar\psi_l,\psi_u,U) \ , \tag{A5a}$$

where Φ' denotes the Fourier transform of Φ.

Inverting Eq. (A5a) we have, using (3.4),

$$\Phi'(\bar\psi_l,\psi_u,U)=\int d\bar\psi_u d\psi_l dV\langle\bar\psi_u,\psi_l,V\mid\bar\psi_l,\psi_u,U\rangle \times\Phi(\bar\psi_u,\psi_l,V) \ . \tag{A5b}$$

The additional bosonic integration dV in (A5b) is needed to compensate the δ function in the inner product in (3.4). To prove Eqs. (A5a) and (A5b) we need the identity

$$\delta(\eta-\epsilon)=\int d\bar\psi\exp[\bar\psi(\eta-\epsilon)] \tag{A6a}$$

and

$$\delta(\bar\eta-\bar\epsilon)=\int d\psi\exp[(\bar\eta-\bar\epsilon)\psi] \ , \tag{A6b}$$

where $\bar\eta,\eta,\bar\xi,\xi$ are fermionic variables and the left-hand side of (A6) are fermionic δ functions having the usual definition of

$$\int d\eta\,\delta(\eta-\epsilon)f(\eta)=f(\epsilon), \ \text{etc} \ . \tag{A6c}$$

Perform the gauge transformation

$$\psi_l'=e^{i\phi^aX_a}\psi_l \ , \tag{A7a}$$

$$\bar\psi_u'=\bar\psi_u e^{-i\phi^aX_a} \ . \tag{A7b}$$

Then, from (2.13) and (A5a) we have in abbreviated notation

$$\Phi(\bar\psi_u',\psi_l',U)=\int d\bar\psi_l d\psi_u\exp(-\bar\psi_u e^{-i\phi^aX_a}M\psi_u -\bar\psi_l M e^{i\phi^aX_a}\psi_l) \times\Phi'(\bar\psi_l,\psi_u,U) \ . \tag{A8}$$

Let $h_u=M\psi_u$ and $\bar h_l=\bar\psi_l M$. We have to prove

$$\exp(-\bar\psi_u e^{-i\phi^aX_a}h_u)=\exp\left[-i\phi^aX_a^{jk}\bar\psi_{ju}\frac{\delta}{\delta\bar\psi_{ku}}\right]e^{-\bar\psi_u h_u} \ . \tag{A9}$$

For infinitesimal ϕ^a we have for the right-hand side of (A9):

$$\left[1-i\phi^aX_a^{jk}\bar\psi_{ju}\frac{\delta}{\delta\bar\psi_{ku}}+O(\phi^2)\right]e^{-\bar\psi_u h_u} =(1+i\phi^a\bar\psi_u X_a h_u)e^{-\bar\psi_u h_u} \simeq\exp[-\bar\psi_u e^{-i\phi^aX_a}h_u+O(\phi^2)] \ . \tag{A10}$$

Iterating Eq. (A10), we obtain Eq. (A9) for finite ϕ^a. Similarly, we have

$$\exp(-\bar h_l e^{i\phi^aX_a}\psi_l)=\exp\left[i\phi^aX_a^{jk}\psi_{kl}\frac{\delta}{\delta\psi_{jl}}\right]e^{-\bar h_l\psi_l} \ . \tag{A11}$$

Hence, from (A8), (A9), and (A11) we have

$$\Phi(\bar\psi_u e^{-i\phi^aX_a},e^{i\phi^aX_a}\psi_l,U) =\exp\left[i\phi^aX_a^{jk}\left[-\bar\psi_{ju}\frac{\delta}{\delta\bar\psi_{ku}}+\psi_{kl}\frac{\delta}{\delta\psi_{jl}}\right]\right] \times\Phi(\bar\psi_u,\psi_l,U) \ , \tag{A12}$$

where we have obtained (A12) by using the commutativity of the exponents in (A12).

APPENDIX B: THE MATRIX M

We discuss the matrix M which appears in the Hamiltonian and the noncanonical commutation equations for the Wilson fermions. From (3.2) and (3.3), using $\bar\psi,\psi$ which carry only color charge since M carries no Dirac indices, let

$$A(\bar\psi,\psi,U)=-\sum_{\mathrm{nm},jk}\bar\psi_{\mathrm{n}j}M_{\mathrm{nm},jk}\psi_{\mathrm{m}k} \tag{B1}$$

and the matrix M is given by

$$M_{\mathrm{nm},jk}=\delta_{\mathrm{nm}}\delta^{jk}-K\sum_i(\delta_{\mathrm{n}+\hat{1},\mathrm{m}}U_{\mathrm{n}i}^{jk}+\delta_{\mathrm{n},\mathrm{m}+\hat{1}}U_{\mathrm{m}i}^{\dagger jk}) \ . \tag{B2}$$

There are two expansions for M, namely, as a power series in K and in $B_{\mathrm{n}i}^a$, where $U_{\mathrm{n}i}=\exp(iB_{\mathrm{n}i}^aX_a)$; the expansion in $B_{\mathrm{n}i}^a$ is in effect an expansion as a power series in the coupling constant g.

(a) Weak-coupling expansion. We have

$$U_{\mathrm{n}i}=1+iB_{\mathrm{n}i}^aX_a+\frac{i^2}{2}(B_{\mathrm{n}i}^aX_a)^2+O(B^3) \ . \tag{B3}$$

Defining

$$\lambda_p = 1 - 2K \sum_i \cos p_i \tag{B4a}$$

and

$$\sum_a X_a{}^2 = c_2 1 \tag{B4b}$$

we have from (B1) and (B3) (see Fig. 3)

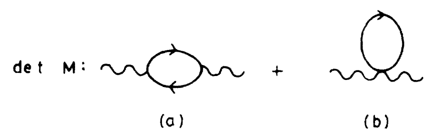

FIG. 3. Feynman diagrams for expansion of $\det M$ in powers of the gauge field.

$$\det M = \prod_{\mathbf{n}} \int d\bar{\psi}_{\mathbf{n}} d\psi_{\mathbf{n}} \exp[A(\bar{\psi},\psi,B)]$$

$$= \exp\left[\tfrac{1}{2}c_2 KJ \sum_{nia} (B^a_{\mathbf{n}i})^2 - \tfrac{1}{2}c_2 K^2 \sum_{\mathbf{nm},ija} B^a_{\mathbf{n}i} B^a_{\mathbf{m}j} \int_q e^{i\mathbf{q}(\mathbf{n}-\mathbf{m})} \Gamma_{ij}(\mathbf{q}) + O(B^3)\right], \tag{B5}$$

where [defining p,q integrations as in (4.16b)]

$$J = \int_p \left[\frac{e^{ip_i} + e^{-ip_i}}{\lambda_p}\right], \tag{B6a}$$

$$\Gamma_{ij}(\mathbf{q}) = \int_p \frac{(e^{i(p+q)_i} - e^{-ip_j})(e^{i(p+q)_j} - e^{-ip_i})}{\lambda_p \lambda_{p+q}}. \tag{B6b}$$

The first term in (B5) comes from Fig. 3(a) and the second term from Fig. 3(b).

Note for $d=4$, J is linearly divergent, and so is $\Gamma_{ij}(q)$, and would give a divergent mass term for the gauge field. However, these two divergences cancel exactly (due to lattice gauge invariance) since we can prove the numerical identity in d dimensions:[14]

$$\delta_{ij} J = K \Gamma_{ij}(\mathbf{q}=0). \tag{B7}$$

Hence we obtain

$$\det M = \exp\left[-\frac{K^2}{2} c_2 \sum_{\mathbf{nm}ija} B^a_{\mathbf{n}i} B^a_{\mathbf{m}j} \int_q e^{i\mathbf{q}\cdot(\mathbf{n}-\mathbf{m})} [\Gamma_{ij}(\mathbf{q}) - \Gamma_{ij}(0)] + O(B^3)\right]. \tag{B8}$$

The ultraviolet divergences of M have to be studied using the weak-coupling expansion.[14]

(b) Strong-coupling expansion. For the case of $g \gg 1$, the theory is expanded as a power series in K and $1/g^2$. To leading order in K we have, from (B1) and (B5),

$$\det M \simeq \exp\left[\frac{K^4}{4} \sum_{\mathbf{n},ij} \mathrm{Tr}(U_{\mathbf{n}i} U_{\mathbf{n}+\hat{i}\,j} U^\dagger_{\mathbf{n}+\hat{j}\,i} U^\dagger_{\mathbf{n}j} + U_{\mathbf{n}j} U_{\mathbf{n}+\hat{j}\,i} U^\dagger_{\mathbf{n}+\hat{i}\,j} U^\dagger_{\mathbf{n}i}) + O(K^6)\right]. \tag{B9}$$

APPENDIX C: CLASSICAL CONTINUUM LIMIT

By the classical continuum limit is meant the limit of $\hbar \to 0$ and $a \to 0$ for the lattice theory with the field values and their derivatives being continuous and differentiable. The classical continuum limit for the symmetric lattice has been derived by Wilson.[11] We essentially redo this calculation for the asymmetric lattice theory given by (4.2).

Recall from (2.1) and (4.1) we have for spatial lattice spacing a and time lattice spacing ϵ the following:

$$x = (n_0 \epsilon, \mathbf{n} a), \tag{C1a}$$

$$B^\alpha_{n0} = \epsilon g A^\alpha_0(x), \tag{C1b}$$

$$B^\alpha_{ni} = a g A^\alpha_i(x), \tag{C1c}$$

$$\psi_n = \left[\frac{a^{d-1}}{2K}\right]^{1/2} \psi(x), \tag{C1d}$$

$$\bar{\psi}_n = \left[\frac{a^{d-1}}{2K}\right]^{1/2} \bar{\psi}(x), \tag{C1e}$$

$$2K = \frac{1}{d + m_0 a}, \quad g = g_0 s a^{(d-4)/2}. \tag{C1f}$$

We will take the limit of $a \to 0$, $\epsilon \to 0$ for the action defined on an infinite space-time lattice, i.e.,

$$A = \sum_n \mathscr{L}, \tag{C2}$$

where $\mathscr{L}$ is given by (4.2). The $\ln \det M$ term in (4.2) goes as $\hbar$ in the action (C2) and vanishes in the classical limit;

hence it will be dropped.

In the classical limit, we can expand $A_\mu^a(na+\hat{\mu}a)$ in a Taylor's series about $A_\mu^a(na)$ and similarly for $\psi(na+\hat{\mu}a)$, etc.; in particular, for the pure gauge-field part of action, it can be readily shown, using (C1a) and (C1b) that[11]

$$A_{\rm GF}=\frac{1}{g^2}\frac{a}{\epsilon}\sum_{\mathbf{n},i}\mathrm{Tr}(U_{ni}U_{n+\hat{1}0}U^\dagger_{n+\hat{0}i}U^\dagger_{n0}+\mathrm{H.c.})+\frac{\epsilon}{a}\frac{1}{g^2}\sum_{\mathbf{n},ij}\mathrm{Tr}(W_{nij})\simeq-\tfrac{1}{4}\epsilon a^{d-1}\sum_n\left[\sum_{ia}[F^a_{0i}(x)]^2+\sum_{ija}[F^a_{ij}(x)]^2\right], \tag{C3}$$

where

$$F^a_{\mu\nu}(x)=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+sg_0C_{abc}A^b_\mu A^c_\nu \ . \tag{C4}$$

Using

$$\epsilon a^{d-1}\sum_n\to\int d^dx \tag{C5}$$

we obtain the continuum Yang-Mills action from (C3).

The space-time asymmetry is more nontrivial for the fermion sector. We Taylor expand the fields using (C1):

$$\psi_{n+\hat{0}}=\left[\frac{a^{d-1}}{2K}\right]^{1/2}[\psi(x)+\epsilon\partial_0\psi(x)+O(\epsilon^2)] \ , \tag{C6a}$$

$$\psi_{n+\hat{1}}=\left[\frac{a^{d-1}}{2K}\right]^{1/2}[\psi(x)+a\partial_i\psi(x)+O(a^2)] \ , \tag{C6b}$$

$$U_{n+\hat{0}i}\cong 1+iga[A_i^a(x)+\epsilon\partial_0A_i^a(x)+\cdots] \ , \tag{C6c}$$

$$U_{n+\hat{1}0}\simeq 1+ig\epsilon[A_0^a(x)+a\partial_iA_0^a(x)+\cdots] \ , \tag{C6d}$$

etc.

On carrying out the Taylor's expansion of the action, we keep terms only of $O(\epsilon a^{d-1})$ and discard all higher-order terms. All terms of lower order must cancel (as, in fact, they do) to have a finite classical limit. For the fermion part of the action, after considerable simplifications, using $A_\mu\equiv A_\mu^aX_a$ we have

$$A_F\simeq-\epsilon a^{d-1}\sum_n\left\{\frac{1}{a}\left[\frac{1}{2K}-d\right]\bar{\psi}(x)\psi(x)+\left[1+\left[\frac{\epsilon}{a}-1\right]\left[d-\frac{1}{2K}\right]\right]\bar{\psi}(x)\gamma_0[\partial_0+isg_0A_0(x)]\psi(x)+\bar{\psi}(x)\gamma_i[\partial_i+isg_0A_i(x)]\psi(x)\right\}+O(\epsilon^2a^{d-1},\epsilon a^d) \ . \tag{C7}$$

From (C1f) we have

$$\frac{1}{2K}-d=m_0a \ . \tag{C8}$$

Hence, the coefficient of the time-derivative term in (C7) becomes, from (C8),

$$1-m_0a\left[\frac{\epsilon}{a}-1\right]=1+O(a) \tag{C9}$$

and consequently the time and space asymmetry vanishes in the classical continuum limit. We finally have, from (C7), (C8), and (C3), (C5),

$$A=A_F+A_{\rm GF} \tag{C10}$$

$$=-\int d^dx\{m_0\bar{\psi}(x)\psi(x)+\bar{\psi}(x)\gamma_\mu[\partial_\mu+isg_0A_\mu(x)]\psi(x)\}-\tfrac{1}{4}\int d^dx[F^a_{\mu\nu}(x)]^2 \tag{C11}$$

which is the color gauge theory with bare coupling constant sg_0 and bare quark mass m_0.

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