Momentum projection and relativistic boost of solitons: Coherent states and projection

E. G. Lübeck, M. C. Birse, E. M. Henley, and L. Wilets
Institute for Nuclear Theory, Department of Physics, FM-15, University of Washington, Seattle, Washington 98195
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We present a method for calculating center-of-mass corrections to hadron properties in soliton models and we apply the method to the soliton bag model. A coherent state is used to provide a quantum wave function corresponding to the mean-field approximation. This state is projected onto a zero-momentum eigenstate. States of nonzero momentum can be constructed from this with a Lorentz boost operator. Hence center-of-mass corrections can be made in a properly relativistic way. The energy of the projected zero-momentum state is the hadron mass with spurious center-of-mass energy removed. We apply a variational principle to our projected state and use three "virial theorems" to test our approximate solution. We also study projection of general one-mode states. Projection reduces the nucleon energy by up to 25%. Variation after projection gives a further reduction of less than 20%. Somewhat larger reductions in the energy are found for meson states.

I. INTRODUCTION

Since their conception ten years ago, bag models[1,2] of hadron structure have suffered from problems due to the lack of translational invariance of their solutions. Various techniques have been proposed for the calculation of center-of-mass and recoil corrections to bag properties.[3–5] All of these are to some extent *ad hoc*. The basic problem is the bag boundary condition, which makes the treatment of translational motion (and other dynamical processes) virtually impossible.[6]

In contrast, soliton models[7–9] are based on field theories which provide a complete dynamical description. Hence, at least in principle, all dynamical processes can be calculated. The major obstacle to this is that these are strongly interacting field theories and so the standard perturbative methods are not applicable. However a variety of nonperturbative approaches have been[8,10–17] and are being developed.[18–22] These are, in many ways, analogous to techniques used in many-body physics. Soliton models can be used to study *NN* scattering,[23] meson emission and absorption,[24] and surface oscillations,[25] as well as the construction of momentum eigenstates which is the subject of this paper.[26]

The starting point for most of these treatments[27] is the semiclassical or mean-field approximation (MFA). In this, boson fields are replaced by their expectation values in the state of interest. Fluctuations of the fields about their mean values are neglected. In models with fermions, the mean boson fields play a role analogous to that of the Hartree-Fock potential in many-body systems. Like Hartree-Fock, the MFA can give solutions which are localized and hence not momentum eigenstates, even though the full theory is translationally invariant.

For such a localized solution, say $|\Psi\rangle$, we have

$$\langle\Psi|\mathbf{P}|\Psi\rangle=0\ ,\quad \text{but}\quad \langle\Psi|P^2|\Psi\rangle\neq 0\ .$$

Thus these localized states contain spurious center-of-mass energy and center-of-mass fluctuational motion. The underlying translational invariance shows up as spurious states in the excitation spectrum built on the Hartree-Fock or mean-field solution. In a field theory, these spurious states are known as "zero modes" and various approaches have been used to avoid the infrared divergences they produce.[11–15] In order to construct states of good momentum we need an explicitly quantum wave function, and not just expectation values of the fields (which are all we can get from a semiclassical treatment).

The techniques we will discuss here provide a wave function which embodies many of the features of the MFA. This wave function can be projected onto states of definite momentum and hence can be used to calculate center-of-mass corrections to soliton properties. The simplest such wave function is a *coherent state*,[28–30] constructed using a single mode of a quantum field. We have also studied more general single-mode states. This type of approach has been around since the early days of quantum field theory.[31] Recent applications can be found in the works of Bolsterli[18] and Parmentola.[32]

Momentum eigenstates are constructed from these states by a projection procedure[33] (often referred to as Peierls-Yoccoz projection). The mass of a hadron, corrected for spurious c.m. energy, is given by the expectation value of the Hamiltonian in the zero-momentum projected state. Variation of the expectation value of the Hamiltonian is applied after projection. We construct states of nonzero momentum by acting on the zero-momentum state with a Lorentz boost operator,[34,35] thus avoiding the problem that states projected onto different momenta need not have the same internal structure. At least to the level of approximation we use, this ensures that the center-of-mass motion is handled in a properly relativistic manner. This will permit calculation of magnetic moments, form factors, and other hadron properties. In the present paper we deal only with the construction of coherent states and their generalization and with the momentum projection. The boost and hadron properties will be described in a subsequent work.

Other works which use this approach (construction of a coherent state and projection) are those of Huang and Stump[16] and Fiebig and Hadjimichael.[22] However, some of the assumptions made in Ref. 22 are inconsistent with the structure of a quantum field theory. A path-integral version of the projection technique can be found in Ref. 15. It has also been applied to the construction of angular-momentum eigenstates in chiral models of baryons.[36]

As a specific example of these methods we apply them to the soliton bag model,[16,8] often known as the Freidberg-Lee model. The salient features of this model are briefly reviewed in Sec. II. The coherent state is introduced in Sec. III and momentum projection is applied to it in Sec. IV. The Lorentz boost is also described briefly. Then, in Sec. V, we introduce the general single-mode state. Various tests or virial theorems which we apply to our solutions are described in Sec. VI. Details of our numerical results are presented in Sec. VIII. Finally, a brief summary of our results is given in Sec. IX.

II. THE SOLITON BAG MODEL

The soliton bag model[8] (often known as the Friedberg-Lee model) is intended as a description of the low-momentum regime of quantum chromodynamics (QCD). It consists of quarks and gluons interacting with a phenomenological scalar field. Hadrons appear as solitons in this field with quarks trapped inside them. The model Hamiltonian is

$$H=\int d^3r\{\psi^\dagger(\mathbf{r})[-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}+g\beta\sigma(\mathbf{r})]\psi(\mathbf{r}) + \tfrac{1}{2}\pi(\mathbf{r})^2+\tfrac{1}{2}\,|\boldsymbol{\nabla}\sigma(\mathbf{r})|^2+U(\sigma)\}\ . \tag{2.1}$$

Here ψ is the quark-field operator with 4 (spinor) times 3 (color) times N_f (flavor) components. In the present work we include only up and down quarks and we neglect the current masses of these quarks. The scalar, isoscalar field is denoted by $\sigma(\mathbf{r})$ and its conjugate momentum by $\pi(\mathbf{r})$. The self-interaction of the σ field is described by the potential

$$U(\sigma)=\frac{a}{2}\sigma^2+\frac{b}{3!}\sigma^3+\frac{c}{4!}\sigma^4+B\ . \tag{2.2}$$

This terminates in fourth order to ensure renormalizability, even though we are dealing with an effective theory. The parameters of the potential are chosen to give a stable minimum at $\sigma=\sigma_V$ (corresponding to the physical vacuum) and a local minimum (or in some cases an inflection point) at $\sigma=0$. The quantity $B=U(0)$ is identified with the bag constant, or volume energy density of a cavity. The full Hamiltonian of the model also contains gluonic terms[8] which have not been displayed here. They are discussed in detail elsewhere.[37]

The vacuum expectation value of the scalar field, σ_V, represents the nonperturbative features of the QCD vacuum. It may be interpreted as a gluon condensate, and it leads to quark and gluon confinement. The excitation quanta of this field may be regarded at 0^{++} glueballs. These quanta have a mass, m_σ, given by

$$m_\sigma{}^2=\left.\frac{d^2U}{d\sigma^2}\right|_{\sigma=\sigma_V}\ . \tag{2.3}$$

This quantity also provides a measure of the sharpness of the bag surface.

We work in the Schrödinger picture and so operators are time independent. The scalar field σ and its conjugate momentum satisfy the usual canonical commutation relations:

$$\begin{aligned}&[\pi(\mathbf{r}),\sigma(\mathbf{r}')]=-i\delta^3(\mathbf{r}-\mathbf{r}')\ ,\\&[\sigma(\mathbf{r}),\sigma(\mathbf{r}')]=0\ ,\\&[\pi(\mathbf{r}),\pi(\mathbf{r}')]=0\ .\end{aligned} \tag{2.4}$$

The total momentum operator, expressed in terms of these fields, is

$$\mathbf{P}=\int d^3r(\psi^\dagger(\mathbf{r})(-i\boldsymbol{\nabla})\psi(\mathbf{r}) -\tfrac{1}{2}\{\pi(\mathbf{r})\boldsymbol{\nabla}\sigma(\mathbf{r})+[\boldsymbol{\nabla}\sigma(\mathbf{r})]\pi(\mathbf{r})\})\ . \tag{2.5}$$

Note that in a static approximation $\pi(\mathbf{r})$ vanishes and only quarks contribute to the total momentum of the system.

III. THE COHERENT STATE

To obtain the approximation which we use here, it is convenient to work in the usual Fock-space representation of the field theory. The scalar field operators are expanded in terms of annihilation and creation operators:

$$\sigma(\mathbf{r})=\sigma_V+(2\pi)^{-3/2}\int d^3k\frac{1}{\sqrt{2\omega_k}}(e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}+e^{-i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}^\dagger)\ , \tag{3.1}$$

$$\pi(\mathbf{r})=-i(2\pi)^{-3/2}\int d^3k\left[\frac{\omega_k}{2}\right]^{1/2}(e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}+e^{-i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}}^\dagger)\ , \tag{3.2}$$

The creation and annihilation operators satisfy the usual commutation rules,

$$[a_{\mathbf{k}},a_{\mathbf{k}'}^\dagger]=\delta^3(\mathbf{k}-\mathbf{k}')\ , \tag{3.3}$$

which follow from (2.4). The satisfaction of the commutation relations (2.4) is, of course, independent of choice of ω_k.

In a weakly coupled field theory, it is convenient to choose

$$\omega_k=(m_\sigma{}^2+k^2)^{1/2}\ , \tag{3.4}$$

where m_σ is given by (2.3), since this choice diagonalizes the Hamiltonian for small-amplitude oscillations about the physical vacuum.

We note that the expansion basis (3.1)—(3.4) *is not a unique choice;* if one could solve the theory exactly, the results should be independent of the expansion basis. In practice, since we are going to make (fairly drastic) approximations, and work with coherent states built on the "vacuum" state defined by

$$a_{\mathbf{k}} | 0 \rangle = 0 \quad \text{for all } \mathbf{k} \ . \tag{3.5}$$

The quantum fluctuations in the state $|0\rangle$, and the coherent states built upon it, do depend on the choice of ω_k. In the present we stick closely to the MFA, and introduce quantum effects in order to treat translational invariance; hence our results will be ω_k dependent. The use of a plane-wave basis is convenient since it means that the state $|0\rangle$ is translationally invariant.

It is, of course, possible to improve on our approximations. For example, one can allow for distortion of the fluctuation modes by their interactions with the average field of the soliton. The plane waves in (3.1), (3.2) are then replaced by distorted waves [with a spectrum still given by (3.4)] and bound orbitals. This can be used as the starting point for a calculation to one-loop order.[38]

One can also treat the expansion functions and their corresponding frequencies as unrestricted variational parameters. Varying these to minimize the energy leads to the Hartree and related approximations.[39]

In the present paper, we do not attempt anything as ambitious as a one-loop or Hartree calculation. However, in view of the fact that the model contains large coupling constants, the basis defined by (3.4) need not be a good choice. In Sec. VII we describe calculations in which the frequencies are treated as variational parameters, although we stick to the plane-wave expansion. The choice of another set of frequencies, Ω_k, can be regarded as a Bogoliubov transformation on the creation and annihilation operators:

$$A_{\mathbf{k}} = \frac{1}{2}\left[\left[\frac{\Omega_k}{\omega_k}\right]^{1/2} + \left[\frac{\omega_k}{\Omega_k}\right]^{1/2}\right] a_{\mathbf{k}} + \frac{1}{2}\left[\left[\frac{\Omega_k}{\omega_k}\right]^{1/2} - \left[\frac{\omega_k}{\Omega_k}\right]^{1/2}\right] a^{\dagger}_{-\mathbf{k}} \ . \tag{3.6}$$

Such a transformation is well defined as long as no Ω_k vanishes.

We now turn to the construction of the coherent state.[28–30] This is done using a single mode of the σ field. We define the creation operator $A^{\dagger}$ by

$$\lambda A^{\dagger} = \int d^3k \left[\frac{\omega_k}{2}\right]^{1/2} f_{\mathbf{k}} a^{\dagger}_{\mathbf{k}} \ , \tag{3.7}$$

where

$$\lambda^2 = \int d^3k \frac{\omega_k}{2} |f_{\mathbf{k}}|^2 \ , \tag{3.8}$$

and hence $[A, A^{\dagger}] = 1$.

The coherent state is given by

$$|\sigma_0\rangle = e^{\lambda A^{\dagger}} |0\rangle \ . \tag{3.9}$$

We have labeled it by the expectation value of the σ field, σ_0. We will calculate this shortly.

First, we note that the state can also be written in the form

$$|\sigma_0\rangle = \prod_{\mathbf{k}} \exp\left[\left[\frac{\omega_k}{2}\right]^{1/2} f_{\mathbf{k}} a^{\dagger}_{\mathbf{k}}\right] |0\rangle \ . \tag{3.10}$$

From this we see that any state of the apparently two-mode form

$$e^{\lambda_1 A_1^{\dagger}} e^{\lambda_2 A_2^{\dagger}} |0\rangle \ ,$$

is equivalent to a one-mode coherent state. We also note that the coherent state is an eigenstate of all annihilation operators:

$$a_{\mathbf{k}} |\sigma_0\rangle = \left[\frac{\omega_k}{2}\right]^{1/2} f_{\mathbf{k}} |\sigma_0\rangle \ . \tag{3.11}$$

Using (3.1) and (3.10), we find that the expectation value of σ is

$$\frac{\langle \sigma_0 | \sigma(\mathbf{r}) | \sigma_0 \rangle}{\langle \sigma_0 | \sigma_0 \rangle} = \sigma_V + (2\pi)^{-3/2} \int d^3k \, f_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \equiv \sigma_0(\mathbf{r}) \ . \tag{3.12}$$

Expectation values in this state of normal-ordered products of field operators are given by

$$\frac{\langle \sigma_0 | :\sigma(\mathbf{r})^n: | \sigma_0 \rangle}{\langle \sigma_0 | \sigma_0 \rangle} = \sigma_0(\mathbf{r})^n \ , \tag{3.13a}$$

$$\langle \sigma_0 | :\pi(\mathbf{r})^n: | \sigma_0 \rangle = 0 \ . \tag{3.13b}$$

These expectation values have exactly the properties of the static MFA. By normal ordering the products of field operators, we have eliminated the (divergent) contributions to the expectation values from the fluctuations contained in the coherent state. In calculating the energy, we will take the expectation value of the normal-ordered Hamiltonian. This corresponds to subtracting off the (unobservable) energy of the vacuum state $|0\rangle$, as well as renormalization of certain of the model parameters.

The normalization of this state can be written as

$$\langle \sigma_0 | \sigma_0 \rangle = \exp([\lambda A, \lambda A^{\dagger}]) = \exp(\lambda^2) \tag{3.14}$$

since the commutator is a c number, and is equal to the average number of σ quanta in the coherent state

$$\langle n_0 \rangle = \left\langle \int d^3k \, a^{\dagger}_{\mathbf{k}} a_{\mathbf{k}} \right\rangle = \lambda^2 \ . \tag{3.15}$$

We also give the extensions of these results to overlaps and matrix elements between different coherent states. These will be needed when we come to the momentum projection. Consider two coherent states, $|\sigma_1\rangle$ and $|\sigma_2\rangle$, defined by

$$|\sigma_1\rangle = \exp\left[\lambda_1 \int d^3k \left[\frac{\omega_k}{2}\right]^{1/2} f_{\mathbf{k}} a^{\dagger}_{\mathbf{k}}\right] |0\rangle \ , \tag{3.16}$$

$$|\sigma_2\rangle = \exp\left[\lambda_2 \int d^3k \left[\frac{\omega_k}{2}\right]^{1/2} g_{\mathbf{k}} a^{\dagger}_{\mathbf{k}}\right] |0\rangle \ . \tag{3.17}$$

The scalar product of these states is

$$\langle \sigma_1 | \sigma_2 \rangle = \exp\left[\lambda_1 \lambda_2 \int d^3k \frac{\omega_k}{2} f^*_{\mathbf{k}} g_{\mathbf{k}}\right] \ . \tag{3.18}$$

Matrix elements of products of field operators can be found from

$$\langle \sigma_1 | :\sigma(\mathbf{r})^n: | \sigma_2 \rangle = \bar{\sigma}(\mathbf{r})^n \langle \sigma_1 | \sigma_2 \rangle , \quad (3.19a)$$

$$\langle \sigma_1 | :\pi(\mathbf{r})^n: | \sigma_2 \rangle = \bar{\pi}(\mathbf{r})^n \langle \sigma_1 | \sigma_2 \rangle , \quad (3.19b)$$

where we have defined

$$\bar{\sigma}(\mathbf{r}) = \frac{\sigma_1(\mathbf{r}) + \sigma_2(\mathbf{r})}{2} , \quad (3.20a)$$

$$\bar{\pi}(\mathbf{r}) = -i(2\pi)^{-3/2} \int d^3k \frac{\omega_k}{2} (g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} - f_{\mathbf{k}}^* e^{-i\mathbf{k}\cdot\mathbf{r}}) . \quad (3.20b)$$

The quark-field operator is expanded in the form

$$\psi(\mathbf{r}) = \sum_n b_n q_n(\mathbf{r}) + \sum_{\bar{n}} d_{\bar{n}}^\dagger q_{\bar{n}}^\dagger(\mathbf{r}) , \quad (3.21)$$

where the q's denote a complete set of spinor functions and the indices n and $\bar{n}$ refer to quark and antiquark states, respectively.

In our approximation, the model nucleon state is the direct product of a coherent σ state and a three-quark state:

$$| N \rangle = e^{\lambda A^\dagger} b_1^\dagger b_2^\dagger b_3^\dagger | 0 \rangle . \quad (3.22)$$

For the lowest-energy baryons (N and Δ) all quarks are put in the same spatial state. Gluonic interactions[37] have been ignored in the present calculation; hence we do not need to make explicit the color-spin-isospin structure of the wave function. In this paper we will give expressions only for the nucleon—the extensions to other baryons and mesons are straightforward.

With the help of (3.12) we can evaluate the expectation value of the normal-ordered Hamiltonian (2.1) in the state (3.22). This gives

$$\frac{\langle N | :H: | N \rangle}{\langle N | N \rangle} = \int d^3r \{ 3q_0^\dagger(\mathbf{r})[-i\boldsymbol{\alpha}\cdot\nabla + g\beta\sigma_0(\mathbf{r})] q_0(\mathbf{r}) + \tfrac{1}{2} | \nabla\sigma_0(\mathbf{r}) |^2 + U(\sigma_0) \} , \quad (3.23)$$

where $q_0(\mathbf{r})$ is the quark wave function corresponding to the $b_i^\dagger$ (all the quarks are put in the same spatial state) and $\sigma_0(\mathbf{r})$ is the function used to define $A^\dagger$.

If we require $\langle :H: \rangle$ to be stationary with respect to variations of $q_0(\mathbf{r})$ and $\sigma_0(\mathbf{r})$, then we obtain equations which are identical to those of the mean-field approximation:

$$[-i\boldsymbol{\alpha}\cdot\nabla + g\beta\sigma_0(\mathbf{r})] q_0(\mathbf{r}) = \epsilon_0 q_0(\mathbf{r}) , \quad (3.24a)$$

$$-\nabla^2\sigma_0(\mathbf{r}) + U'(\sigma_0) + 3g\bar{q}_0(\mathbf{r}) q_0(\mathbf{r}) = 0 . \quad (3.24b)$$

These equations have been solved numerically by Goldflam and Wilets[19] and other groups.[40] With a suitable choice of parameters, the solution can have a MIT-bag-like form. In fact, there is a fairly wide variety of choices of the parameters (a, b, c, and g) which lead to reasonable values for nucleon properties.[41] The present calculations place some restrictions on the acceptable values for these parameters. Other constraints on them can be found in Ref. 37.

IV. MOMENTUM PROJECTION AND BOOST

The soliton coherent state described in the previous section is localized and so has no definite momentum. To construct a momentum eigenstate, we use Peierls-Yoccoz projection.[33] The zero-momentum projected state is

$$| \mathbf{P}=0 \rangle = \int d^3X | \mathbf{X} \rangle , \quad (4.1)$$

where $| \mathbf{X} \rangle$ is a nucleon state localized at the point $\mathbf{X}$ and has the form

$$| \mathbf{X} \rangle = e^{\lambda A^\dagger(\mathbf{X})} b_1^\dagger(\mathbf{X}) b_2^\dagger(\mathbf{X}) b_3^\dagger(\mathbf{X}) | 0 \rangle . \quad (4.2)$$

The creation operators are for a σ mode and quark states centered on $\mathbf{X}$. The operator $A^\dagger(\mathbf{X})$ can be written

$$\lambda A^\dagger(\mathbf{X}) = \int d^3k \left[\frac{\omega_k}{2} \right]^{1/2} f_{\mathbf{k}}(\mathbf{X}) a_{\mathbf{k}}^\dagger , \quad (4.3)$$

where λ is still given by (3.8) and $f_{\mathbf{k}}(\mathbf{X})$ is the Fourier transform of the σ field distribution centered at $\mathbf{X}$:

$$f_{\mathbf{k}}(\mathbf{X}) = e^{-i\mathbf{k}\cdot\mathbf{X}} f_{\mathbf{k}}(0) . \quad (4.4)$$

The expectation value of σ in this shifted state is $\sigma_0(\mathbf{r}-\mathbf{X})$ and the quark wave functions are $q_0(\mathbf{r}-\mathbf{X})$.

Since the σ field is expanded in terms of plane waves, the σ vacuum state defined by (3.5) is translationally invariant. The quark basis we have used, (3.21), is not translationally invariant, *but we neglect differences from unity of the overlaps of quark vacua centered on different points*. This procedure cannot be exact, but no obvious problems have arisen as yet. If it does cause trouble, one can always go to a plane-wave basis and work with a Dirac Hamiltonian projected onto positive-energy plane waves. The methods of Huang and Stump[16] avoided this problem by assuming that m_σ was very large. Then the overlap of two coherent states centered on different points was essentially a δ function and there was no need to calculate overlaps between different vacua.

From (4.1) the expectation value of an operator O in the projected state is

$$\langle O \rangle = \frac{\langle \mathbf{P}=0 | O | \mathbf{P}=0 \rangle}{\langle \mathbf{P}=0 | \mathbf{P}=0 \rangle} = \frac{\int d^3X\, d^3Y \langle \mathbf{X} | O | \mathbf{Y} \rangle}{\int d^3X\, d^3Y \langle \mathbf{X} | \mathbf{Y} \rangle} . \quad (4.5)$$

Provided O is translationally invariant, the integrals over $\frac{1}{2}(\mathbf{X}+\mathbf{Y})$ give trivial volume factors which cancel in (4.5). Hence we can write

$$\langle O \rangle = \frac{\int d^3Z \langle -\frac{1}{2}\mathbf{Z} | O | \frac{1}{2}\mathbf{Z} \rangle}{\int d^3Z \langle -\frac{1}{2}\mathbf{Z} | \frac{1}{2}\mathbf{Z} \rangle} . \quad (4.6)$$

The integrand in the normalization is a product of σ and quark factors:

$$\langle -\tfrac{1}{2}\mathbf{Z} | \tfrac{1}{2}\mathbf{Z} \rangle = N_\sigma(\mathbf{Z}) N_q(\mathbf{Z})^3 , \quad (4.7)$$

where

$$N_\sigma(\mathbf{Z}) = \exp \left[\int d^3k \frac{\omega_k}{2} f_{\mathbf{k}}^*(-\tfrac{1}{2}\mathbf{Z}) f_{\mathbf{k}}(\tfrac{1}{2}\mathbf{Z}) \right] \quad (4.8)$$

[obtained from (3.17) and (3.18)] and

$$N_q(\mathbf{Z})=\int d^3r\, q_0^\dagger(\mathbf{r}+\tfrac{1}{2}\mathbf{Z})q_0(\mathbf{r}-\tfrac{1}{2}\mathbf{Z}) \ . \tag{4.9}$$

The integrands for the expectation values of normal-ordered products of field operators are, from (3.19),

$$\langle -\tfrac{1}{2}\mathbf{Z}\,|\,:\sigma(\mathbf{r})^n:\,|\,\tfrac{1}{2}\mathbf{Z}\rangle=\bar\sigma(\mathbf{r};\mathbf{Z})^n N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^3 \ , \tag{4.10a}$$

$$\langle -\tfrac{1}{2}\mathbf{Z}\,|\,:\pi(\mathbf{r})^n:\,|\,\tfrac{1}{2}\mathbf{Z}\rangle=\bar\pi(\mathbf{r};\mathbf{Z})^n N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^3 \ , \tag{4.10b}$$

where

$$\bar\sigma(\mathbf{r};\mathbf{Z})=\tfrac{1}{2}[\sigma_0(\mathbf{r}-\tfrac{1}{2}\mathbf{Z})+\sigma_0(\mathbf{r}+\tfrac{1}{2}\mathbf{Z})] \ , \tag{4.11a}$$

$$\bar\pi(\mathbf{r};\mathbf{Z})=-i(2\pi)^{3/2}\int d^3k\frac{\omega_k}{2}[f_{\mathbf{k}}(\tfrac{1}{2}\mathbf{Z})e^{i\mathbf{k}\cdot\mathbf{r}} - f_{\mathbf{k}}^*(-\tfrac{1}{2}\mathbf{Z})e^{-i\mathbf{k}\cdot\mathbf{r}}] \ . \tag{4.11b}$$

With these results we can now evaluate the nucleon mass in this approximation:

$$M=\langle :H:\rangle=\frac{\langle \mathbf{P}=0\,|\,:H:\,|\,\mathbf{P}=0\rangle}{\langle \mathbf{P}=0\,|\,\mathbf{P}=0\rangle} \ , \tag{4.12}$$

where H is the Hamiltonian (2.1). For the pieces of H involving the quarks we have

$$\langle :H_q+H_{q\sigma}:\rangle=3\int d^3Z\, N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^2\frac{\int d^3r\, q_0^\dagger(\mathbf{r}+\frac{1}{2}\mathbf{Z})[-i\boldsymbol{\alpha}\cdot\nabla+g\beta\bar\sigma(\mathbf{r};\mathbf{Z})]q_0(\mathbf{r}-\frac{1}{2}\mathbf{Z})}{\int d^3Z\, N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^3} \ , \tag{4.13}$$

while the pieces involving only σ give a contribution,

$$\langle :H_\sigma:\rangle=\frac{\int d^3Z\, \mathscr{E}_\sigma(\mathbf{Z})N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^3}{\int d^3Z\, N_\sigma(\mathbf{Z})N_q(\mathbf{Z})^3} \ , \tag{4.14}$$

where

$$\mathscr{E}_\sigma(\mathbf{Z})=\int d^3r[\tfrac{1}{2}\bar\pi(\mathbf{r};\mathbf{Z})^2+\tfrac{1}{2}\,|\,\nabla\bar\sigma(\mathbf{r};\mathbf{Z})\,|^2+U(\bar\sigma)] \ . \tag{4.15}$$

To calculate other nucleon properties (e.g., magnetic moments) we need states with nonzero momentum. These could be constructed using finite-momentum projection. However, such a procedure has well-known difficulties: namely, the various states of good momentum are not related to each other by the appropriate Lorentz transformations. Instead we operate on the zero-momentum state with a Lorentz boost operator to produce an approximate four-momentum eigenstate. The boosted state is defined by[34,35]

$$|\,\mathbf{y}\rangle=e^{i\mathbf{y}\cdot\mathbf{K}}\,|\,\mathbf{P}=0\rangle \ , \tag{4.16}$$

where

$$\mathbf{K}=\int d^3r\, \mathbf{r}\mathscr{H}(\mathbf{r}) \ , \tag{4.17}$$

and $\mathscr{H}(\mathbf{r})$ is the Hamiltonian density. The quantity $\mathbf{y}$ is the rapidity in the direction of the velocity:

$$\mathbf{y}=\hat{\mathbf{v}}\tfrac{1}{2}\ln\left[\frac{1+v}{1-v}\right] . \tag{4.18}$$

Unless the state $|\,\mathbf{P}=0\rangle$ is an exact energy eigenstate, the state $|\,\mathbf{y}\rangle$ is not an exact momentum eigenstate. Even so, it leads to expectation values of energy and momentum with the correct Lorentz transformation properties. This can be seen as from the following.

The operator $\mathbf{K}$ obeys the commutation rules:

$$\begin{aligned}&[\mathbf{K},H]=i\mathbf{P} \ ,\\ &[K_i,P_j]=i\delta_{ij}H \ .\end{aligned} \tag{4.19}$$

If we define

$$\begin{aligned}&E(y)=\langle \mathbf{y}\,|\,H\,|\,\mathbf{y}\rangle \ ,\\ &P_{\parallel}(y)=\langle \mathbf{y}\,|\,\hat{\mathbf{v}}\cdot\mathbf{P}\,|\,\mathbf{y}\rangle \ ,\end{aligned} \tag{4.20}$$

then from (4.19) we have

$$\begin{aligned}&\frac{dE}{dy}=P_{\parallel}(y) \ ,\\ &\frac{dP_{\parallel}}{dy}=E(y) \ .\end{aligned} \tag{4.21}$$

These differential equations have solutions [for the boundary condition $E(0)=M$, $P_{\parallel}(0)=0$]:

$$\begin{aligned}&E(y)=M\cosh(y) \ ,\\ &P_{\parallel}(y)=M\sinh(y) \ ,\end{aligned} \tag{4.22}$$

where M is the expectation value of H in the $\mathbf{P}=0$ state. The boosted state is more conveniently labeled by the expectation value of the momentum, $\mathbf{P}=\hat{\mathbf{v}}M\sinh(y)$. The use of this boost to calculate magnetic moments and form factors will be described elsewhere.

V. THE GENERAL SINGLE-MODE STATE

The coherent state introduced by Eq. (3.9) is a special case of the general single-mode state[31] which can be written in the form

$$|\,\sigma_0;F\rangle=F(A^\dagger)\,|\,0\rangle \ , \tag{5.1}$$

where $F(A^\dagger)$ is an arbitrary function of the chosen mode creation operator (3.7) and σ_0 characterizes the mode through the Fourier coefficients f_k.

The coherent state can be regarded as a Gaussian wave functional in the infinite-dimensional space corresponding to a field theory.[39,26,29] The state (5.1) allows for a more general functional form and so includes some of the dynamical effects from fluctuations in the chosen mode.

It is convenient to expand $F(A^\dagger)$ as a power series in $A^\dagger$ and to write the model nucleon state in the form

$$|N\rangle=\sum_{n=0}^{\infty}\frac{F_n}{n!}(A^\dagger)^n|3q\rangle\ , \tag{5.2}$$

where $|3q\rangle$ denotes the bare three-quark state [compare (3.22)]. It follows from the commutation relation $[A,A^\dagger]=1$ that

$$\langle 3q|A^m(A^\dagger)^n|3q\rangle=\delta_{mn}n!\ , \tag{5.3}$$

which shows that this expansion corresponds to an orthogonal basis—in fact it is just the usual Fock-space representation. Similar use of the commutation rules yields

$$\langle 3q|A^m a_{\mathbf{k}}(A^\dagger)^n|3q\rangle=\left[\frac{\omega_k}{2}\right]^{1/2} f_{\mathbf{k}}\delta_{m,n-1}n!\ . \tag{5.4}$$

With this we can evaluate the expectation value of $\sigma(\mathbf{r})$ in (5.2),

$$\frac{\langle N|\sigma(\mathbf{r})|N\rangle}{\langle N|N\rangle}=\sigma_V+\frac{\sum_n(1/n!)F_nF_{n-1}}{\sum_n(1/n!)F_n{}^2}(2\pi)^{-3/2}\times\int d^3k\, f_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}\equiv\sigma_0(\mathbf{r})\ . \tag{5.5}$$

From similar straightforward, but tedious, use of the commutation rules, we can get expressions for the expectation values of normal-ordered products of field operators and the Hamiltonian. Variation of the energy with respect to the coefficients F_n leads to a set of linear equations of the form

$$\sum_m H_{nm}F_m=EF_n\ , \tag{5.6}$$

where

$$H_{nm}=\frac{\langle 3q|A^n{:}H{:}(A^\dagger)^m|3q\rangle}{m!}\ . \tag{5.7}$$

The solution to this set of equations with the lowest eigenvalue is the nucleon ground state in this approximation. The other eigenvectors are artifacts of the approximation.

We find that 12 terms in the expansion (5.2) are more than adequate for convergence of the numerical results. For the unprojected state (and also the projected one) the results are fairly similar to those of the coherent state; the leading coefficients in (5.2) have a behavior close to that of a coherent state: the ratio F_{n+1}/F_n is approximately constant (see Table I). The energy is also close to that of the coherent state (about 10% lower).

We have also investigated a representation of the general single-mode state in terms of a superposition of coherent states:

$$|\sigma_0;F\rangle=\int d\alpha\, G(\alpha)e^{\alpha A^\dagger}|0\rangle\ . \tag{5.8}$$

This considerably simplifies the algebra involved in taking expectation values, but it means working in a basis of states which are not orthogonal and, more seriously, not linearly independent. The integral equation obtained by varying the energy with respect to $G(\alpha)$ does not have well-defined solutions. The function $G(\alpha)$ is found to oscillate wildly, and it does not converge as the number of points used in discretizing the integral is increased. This is characteristic of generator coordinate solutions. However, the results for the energy do converge and are in good agreement with the (stable) power-series expansion.

TABLE I. Ratios of successive coefficients in the Fock-space expansion of the general single-mode states with and without projection. The model parameters used are those of set I (see Table II).

	F_{n+1}/F_n	
n	Unprojected	Projected
1	0.577	1.093
2	1.035	1.352
3	1.217	1.469
4	1.092	1.312
5	1.009	1.142
6	0.857	0.904
7	0.345	0.220

We can project (5.2) onto a zero-momentum eigenstate in the same way as we do for the coherent state. Variation of the projected energy leads to a set of equations similar to (5.6):

$$\sum_m \tilde{H}_{nm}F_m=E\tilde{N}_nF_n\ , \tag{5.9}$$

where

$$\tilde{H}_{nm}=\frac{1}{m!}\int d^3Z\times\langle 3q;-\tfrac{1}{2}\mathbf{Z}|A(-\tfrac{1}{2}\mathbf{Z})^n{:}H{:}A^\dagger(\tfrac{1}{2}\mathbf{Z})^m|3q;\tfrac{1}{2}\mathbf{Z}\rangle\ , \tag{5.10}$$

$$\tilde{N}_n=\int d^3Z\left[\int d^3k\frac{\omega_k}{2}f_{\mathbf{k}}^*(-\tfrac{1}{2}\mathbf{Z})f_{\mathbf{k}}(\tfrac{1}{2}\mathbf{Z})\right]^n N_q(\mathbf{Z})^3\ . \tag{5.11}$$

The matrix $\tilde{H}_{nm}$ can be evaluated with a lot of commutation algebra. We will not present its explicit form here.

VI. VIRIAL THEOREMS

The approximations we use in this work, along with the variational method, do not give us an exact solution of even the model Hamiltonian (2.1). Hence we need some way to test how well we are approximating a true eigenstate. We obtain these tests by noting that the time derivative of the expectation value of any operator should vanish in an exact eigenstate:

$$\frac{d}{dt}\langle O\rangle=i\langle[H,O]\rangle=0\ . \tag{6.1}$$

By taking expectation values of the commutators of various operators with the (normal-ordered) Hamiltonian, we obtain a set of virial theorems. These are analogous to the virial theorem used in nonrelativistic quantum mechanics. Of course, they are only necessary, not sufficient, conditions for a good solution to the field theory.

We have considered the following operators:

$$\int d^3r\, \psi^\dagger(\mathbf{r})(-i\mathbf{r}\cdot\nabla)\psi(\mathbf{r})\ ,\quad \int d^3r\, \pi(\mathbf{r})\ ,$$
$$\int d^3r :\sigma(\mathbf{r})\pi(\mathbf{r}):\ .$$

Computation of these operators with $:H:$ leads to the corresponding virial theorems. These require the vanishing of

$$V_1 \equiv \int d^3r \langle \psi^\dagger(\mathbf{r})[-i\boldsymbol{\alpha}\cdot\nabla - g\beta\mathbf{r}\cdot\nabla\sigma(\mathbf{r})]\psi(\mathbf{r})\rangle(\langle :H:\rangle)^{-1}\ , \tag{6.2a}$$

$$V_2 \equiv \int d^3r \langle :[-\nabla^2\sigma(\mathbf{r}) + U'(\sigma) + 3g\bar{\psi}(\mathbf{r})\psi(\mathbf{r})]:\rangle\ , \tag{6.2b}$$

$$V_3 \equiv \int d^3r \left\langle : \left[-\pi(\mathbf{r})^2 + \sigma(\mathbf{r})\frac{\partial H}{\partial\sigma(\mathbf{r})} \right] : \right\rangle \Big/ \langle :H:\rangle\ . \tag{6.2c}$$

The first is a Dirac version of the familiar virial theorem of nonrelativistic quantum mechanics. In that case, it is equivalent to energy minimization with respect to the length scale of the wave function. The second is the volume integral of the field equation for σ. It corresponds to the second time derivative of the σ field; the first derivative vanishes for both unprojected and projected coherent states. The third is a field-theoretic analogue of the first. We have included factors of $\langle :H:\rangle^{-1}$ in (6.2a) and (6.2c) so that all of these quantities are dimensionless.

In the first two cases, commutation does not destroy normal ordering, while in the last case it does. The commutator (6.2c) should then be written as a normal-ordered piece plus divergent terms. With ω_k given by (3.4), the divergent terms cancel, leaving only the finite part.

Each of these quantities vanishes in the MFA, if one has obtained self-consistent solutions. Similarly, they vanish for the unprojected coherent state constructed from the MFA solutions. However, none of these quantities vanishes automatically in the projected state. Hence they can be used as tests of our solutions and approximations.

VII. VARIATION AFTER PROJECTION

It is well known in nuclear physics that energy variation before projection can be a dangerous procedure. In the present case, the expectation value of the energy in the projected coherent state depends upon the functions $u(r)$, $v(r)$, and $\sigma_0(r)$ where u and v are the upper and lower components of $q_0(\mathbf{r})$. A full variation with respect to all of these functions would lead to a set of coupled integral equations. Solving these exactly appears to be prohibitively complicated at present. Instead, we utilize the functional forms of the MFA solutions, now denoted with a tilde, and rescale them as follows:

$$\sigma_0(r) - \sigma_V = \xi[\tilde{\sigma}_0(r/\lambda) - \sigma_V]\ , \tag{7.1a}$$

$$u(r) = \tilde{u}(r/\delta)\ ,\quad v(r) = \gamma\tilde{v}(r/\delta)\ . \tag{7.1b}$$

The normalization of q_0 must, of course, be readjusted. We have also tried an alternative parametrization of σ_0, using a Fermi-function form,

$$\sigma_0(r) - \sigma_V = -\xi \frac{\sigma_v}{1 + \exp\left[\dfrac{r - r_0}{\mu}\right]}\ . \tag{7.2}$$

This allows independent variations of the surface thickness and radius of σ_0.

In the general single-mode state, we have the F_n, defined by (5.2), as variational parameters, as well as those just described.

As noted above, the commutation relations (2.4) are satisfied for any set of ω_k, and the choice (3.4) is not necessarily the best for the nucleon state. Another choice, say, Ω_k, defines operators $A_\mathbf{k}$ which are linear combinations of $a_\mathbf{k}$ and $a_\mathbf{k}^\dagger$, as in Eq. (3.6). The corresponding vacuum state $|\Omega\rangle$, defined by

$$A_\mathbf{k}|\Omega\rangle = 0 \quad \text{for all } \mathbf{k}\ , \tag{7.3}$$

is a Gaussian wave packet in functional space. Its principal axes are given by the plane-wave expansion functions, and its widths are $\Omega_\mathbf{k}^{-1/2}$. In general, it will be nothing like the physical vacuum. This is in contrast with the vacuum state defined using (3.4). That expansion basis diagonalizes the Hamiltonian for small oscillations about the mean σ field in the vacuum, and it minimizes the energy of the vacuum. Hence, at least for weak coupling, it can be regarded as an approximation to the physical vacuum.

The coherent state used to describe the nucleon is a displaced Gaussian wave packet of the same form as $|\Omega\rangle$, but centered on the classical field configuration $\sigma_0(r)$. The general single-mode state defined in Sec. V allows for multiplication of this Gaussian by a more general function of the chosen single mode.

A better starting point for the nucleon state would be to expand the σ field in distorted waves. However, we have committed ourselves to a plane-wave basis in order to facilitate projection. In choosing a coherent state with $\Omega_k \neq \omega_k$ to describe the nucleon, we have changed the vacuum far from the nucleon, and produced an infinite shift in the vacuum contribution to the energy. Our procedure is to calculate only the (finite) differences in energies, and other properties, between the nucleon and the (unobservable) vacuum state, $|\Omega\rangle$.

Varying the energies of the nucleon and vacuum states independently would lead, in our approximations, to the choice (3.4) for both states. This is because the nucleon state is an empty vacuum except for a localized distortion in the vicinity of the nucleon. To try to improve the description of quantum fluctuations in the vicinity of the nucleon, we invoke the variational principle for the difference in energy between the nucleon and the vacuum. Since the energy of each eigenstate is stationary with respect to arbitrary variations, so also is the energy difference, at least for independent variations of the parameters used to describe both states. Here we constrain certain parameters (the Ω_k) to be the same in both states—a somewhat dangerous procedure. It is quite possible that the energy difference is only weakly dependent on some parameters, and hence that it extremizes for values of these parameters which are far from those that would be obtained by varying either of the energies separately. We expect that the parameters for which this happens are those associated with the unobservable vacuum, and so should have little effect on the localized part of the nucleon state. To test whether this procedure is meaningful we use the virial theorems of the previous section. As shown below, optimization with respect to the Ω_k leads to significant improvement in the satisfaction of the virial theorems.

In normal ordering the Hamiltonian (2.1), we not only subtract off the energy of the vacuum state $|\Omega\rangle$, we also renormalize the coefficients of the linear and quadratic terms in $U(\sigma)$. (The renormalized coefficient of the linear term is set to zero.) To ensure that this procedure is well defined and independent of the state we are studying, we define our renormalized coupling constants a, b, and c by normal ordering with respect to the physical vacuum basis, defined by (3.4). When we normal order with respect to the basis defined by Ω_k, the resulting Hamiltonian density can be written

$$\mathscr{H}=\psi^\dagger(-i\alpha\cdot\nabla+g\beta\sigma)\psi+\tfrac{1}{2}\pi^2+\tfrac{1}{2}|\nabla\sigma|^2 +\bar{d}\sigma+\tfrac{1}{2}\bar{a}{:}\sigma^2{:}+\frac{1}{3!}b{:}\sigma^3{:}+\frac{1}{4!}c{:}\sigma^4{:}+\bar{K}\ , \tag{7.4}$$

where the barred quantities are related to the unbarred (vacuum) quantities by

$$\bar{a}=a+\tfrac{1}{4}c\Delta\ , \tag{7.5a}$$

$$\bar{d}=\tfrac{1}{4}b\Delta\ , \tag{7.5b}$$

with

$$\Delta=\frac{1}{2\pi^2}\int k^2dk\left[\frac{1}{\Omega_k}-\frac{1}{\omega_k}\right]. \tag{7.6}$$

The constant c number part of the field, $\bar{\sigma}_V$, satisfies the equation

$$\bar{d}+\bar{a}\bar{\sigma}_V+\tfrac{1}{2}b\bar{\sigma}_V{}^2+\tfrac{1}{6}c\bar{\sigma}_V{}^3=0\ , \tag{7.7}$$

and so is not the same as the physical vacuum. The Ω_k-dependent constant $\bar{K}$ drops out of the energy difference between the nucleon and vacuum states.

We choose the Ω_k to be such that $\Omega_k\to\omega_k$ as $k\to\infty$ sufficiently rapidly so that Δ is finite. For example, the choice $\Omega_k{}^2=m^2+k^2$ $(m\neq m_\sigma)$ leads to a logarithmic divergence for Δ, so that a cutoff is needed. A possible parametrization would be $\Omega_k=m$ for $k<k_c$ and $\Omega_k=\omega_k$ for $k>k_c$. A more convenient two-parameter form is

$$\Omega_k{}^2=\omega_k{}^2+\delta m^2\exp(-k^2/k_c{}^2)\ . \tag{7.8}$$

We emphasize that in variation we seek only a stationary point, not necessarily a minimum, in the energy. Within our approximations, the Hamiltonian is not bounded from below. As already noted we are calculating the difference of two energies. Furthermore, the quark Dirac energies are unbounded from below, since we are including only the valence quarks. We do find that variation of most parameter results in minima, but there are cases where we find maxima.

VIII. NUMERICAL RESULTS

The Hamiltonian for the Friedberg-Lee model without gluons, (2.1), contains four adjustable parameters. Here we give detailed results for three representative parameters sets, shown in Table II. The sets were chosen[41] by requiring that the MFA energy (with the spurious c.m. energy of the quarks subtracted off, see below) fit the average of the observed N and Δ masses, and that the proton charge radius match the experimental value of 0.83 fm. These conditions fix two of the parameters. The remaining two parameters are not well determined, and a fairly wide range of choices can give reasonable agreement with other nucleon properties. Eventually we intend to refit all the parameters, including both center-of-mass corrections and gluonic interactions.[37] For the present, we give only an indication of the parameter dependence of the c.m. corrections to the energy, as well as more detailed results for the sets of Table II. In the detailed results we include calculations for mesons (two quark states) as well as for baryons (three quarks).

TABLE II. Parameter sets for the soliton bag model. Also shown are the corresponding glueball masses, m_σ, and vacuum expectation values of the σ field. All dimensioned quantities are in appropriate powers of fm (1 fm^{-1}=197 MeV).

	Set I	Set II	Set III
a	30.0	0.0	0.0
b	−610.0	−105.14	−58.52
c	4000.0	1000.0	500.0
g	10.0	9.04	9.16
m_σ	5.99	4.07	3.21
σ_V	0.314	0.315	0.321

In addition to the results of projection and variation, we present an estimate of the mass based on the inequality[5]

$$M^2\le\langle H^2\rangle-\langle P^2\rangle\ , \tag{8.1}$$

where the expectation values are taken in the localized MFA state. If we assume that the state is approximately an eigenstate of H, then we can replace $\langle H^2\rangle$ by $\langle H\rangle^2$ to obtain

$$M\lesssim(\langle H\rangle^2-\langle P^2\rangle)^{1/2}\ . \tag{8.2}$$

This should be a reasonable estimate provided that the c.m. effects are small and so the localized state can be regarded as an approximate energy eigenstate. The quark contribution to $\langle P^2\rangle$ for a localized bag state is[5]

$$\langle P_q{}^2\rangle=-3\int d^3r\, q_0^\dagger(\mathbf{r})\nabla^2 q(\mathbf{r})\ . \tag{8.3}$$

The σ contribution to $\langle P^2\rangle$ cannot be calculated in the usual MFA since $\mathbf{P}$ contains $\pi(\mathbf{r})$ [see Eq. (2.5)]. However, we noted in Sec. III that the unprojected coherent state is equivalent to the MFA. Hence the σ piece can be calculated in the coherent state approximation from

$$\langle P_\sigma{}^2\rangle\equiv\int d^3k\, k^2\frac{\omega_k}{2}f_{\mathbf{k}}{}^2\ , \tag{8.4}$$

where $f_{\mathbf{k}}$ is given in terms of the mean σ field by (3.12),

TABLE III. Baryon energies (in units of fm^{-1}) obtained by projection and variation, for parameter set I. Also listed are the percentage reduction from the MFA energy, and the three virial tests of (6.2). The results shown are for the unprojected MFA, the estimate (8.2) of the recoil-corrected mass, projection without variation, projection with variation of the parameters in (7.1), projection with variation only of δm^2 in (7.8), and projection with full variation of both (7.1) and (7.8). The coherent state results are labeled CSA; the general single-mode state, GSM.

Approximation		Energy	%	V_1	V_2	V_3
MFA		6.621		0.0	0.0	0.0
$(\langle H\rangle^2-\langle P^2\rangle)^{1/2}$		5.060	23.6			
Projection	CSA	4.758	28.0	−0.152	4.133	−0.237
	GSM	3.653	44.8	−0.302	−5.554	−0.651
Variation	CSA	4.115	37.8	−0.350	−12.154	−1.408
	GSM	3.690	44.3	−0.404	−4.274	−0.859
δm^2 only	CSA	4.076	38.4	0.005	9.564	0.167
	GSM	3.207	51.6	−0.174	0.749	−0.289
Full variation	CSA	3.025	54.3	−0.048	−13.640	−1.264
	GSM	2.983	54.9	−0.163	2.925	−0.781

and the frequencies of (3.4) are used. The parameter sets of Ref. 40 were chosen so that the mass (8.2), including only the quark piece of $\langle P^2\rangle$, fit the average of the N and Δ masses. However, the quark and σ terms are of similar size, and so both should be included.

For comparison, we show also the mass obtained, in the nonrelativistic limit, by projecting the soliton (described as a coherent state) onto a state of nonzero momentum. The projected state is[33]

$$|\mathbf{P}\rangle=\int d^3X\, e^{i\mathbf{P}\cdot\mathbf{X}}\,|\mathbf{X}\rangle\ . \tag{8.5}$$

Expanded in powers of P, the energy of this state is

$$\frac{\langle\mathbf{P}\,|{:}H{:}|\,\mathbf{P}\rangle}{\langle\mathbf{P}\,|\,\mathbf{P}\rangle}=M+\frac{P^2}{2M^*}+\cdots\ , \tag{8.6}$$

where M is the mass from Eq. (4.12), and M^* is given by

$$\frac{1}{M^*}=\frac{\frac{1}{3}\int d^3Z\,Z^2\langle-\frac{1}{2}\mathbf{Z}\,|\,({:}H{:}-M)\,|\,\frac{1}{2}\mathbf{Z}\rangle}{\int d^3Z\langle-\frac{1}{2}\mathbf{Z}\,|\,\frac{1}{2}\mathbf{Z}\rangle}\ . \tag{8.7}$$

For the parameter sets I–III, the baryon M^*'s are, respectively, 4.824, 6.886, and 7.926. The M^* of the meson for set III is 6.883. These numbers should be compared with the energies in the third lines of Tables III–VI. The differences between these two masses illustrates the "Peierls-Yoccoz problem" referred to in Sec. IV. Projection onto states of different momenta does not give states with the same internal structure, and so their energies are not related by the appropriate Lorentz boosts. Hence we have adopted the procedure described in Sec. IV, of boosting the zero-momentum state to produce approximate four-momentum eigenstates. These do have the correct energy-momentum relationship, at least for expectation values.

Figures 1 and 2 show the effects of projection after variation for a range of parameter sets. For comparison, they show also the mass obtained from the MFA results with Eq. (8.2). For parameter sets with $c\gtrsim 10^4$ the c.m. corrections are sufficiently large as to make the approximations we use questionable. In fact, for parameters with $c\sim 10^5$ variation after projection can lead to negative values for the baryon mass. Hence we restrict our attention to sets with smaller values of c, corresponding to "softer" bags. A similar preference for soft parameters was found in calculations which include gluon-exchange interactions.[37] These sets lead to values of m_σ which are comparable to estimates of the 0^{++} glueball mass from lattice gauge calculations.[42]

We have carried out projection followed by variation for both coherent states and general single-mode (GSM) states. The results of calculations of the energy and three virial theorems are shown in Tables III–VI, for the pa-

TABLE IV. As Table III, but for parameter set II.

Approximation		Energy	%	V_1	V_2	V_3
MFA		6.431		0.0	0.0	0.0
$(\langle H\rangle^2-\langle P^2\rangle)^{1/2}$		5.042	21.6			
Projection	CSA	5.232	18.6	−0.145	3.589	−0.074
	GSM	4.303	33.1	−0.395	−2.271	−0.494
Variation	CSA	4.914	23.6	0.215	−9.138	−0.682
	GSM	4.731	26.4	0.208	−6.784	−0.602
δm^2 only	CSA	4.118	36.0	0.173	7.105	0.042
	GSM	3.324	48.3	−0.171	1.314	−0.078
Full variation	CSA	3.785	41.1	0.189	−9.111	−0.406
	GSM	3.683	42.7	0.169	−7.545	−0.525

TABLE V. As Table III, but for parameter set III.

Approximation		Energy	%	V_1	V_2	V_3
MFA		6.460		0.0	0.0	0.0
$(\langle H\rangle^2-\langle P^2\rangle)^{1/2}$		5.044	21.9			
Projection	CSA	5.406	16.3	−0.156	4.092	0.018
	GSM	4.639	28.2	−0.400	−1.078	−0.408
Variation	CSA	5.187	19.7	0.239	−6.506	−0.508
	GSM	5.093	21.2	0.221	−5.430	−0.503
δm^2 only	CSA	4.184	35.2	0.182	7.658	0.099
	GSM	3.417	47.1	−0.113	0.615	0.062
Full variation	CSA	3.886	39.8	0.195	−6.744	−0.260
	GSM	3.847	40.4	0.199	−5.935	−0.308

rameter sets of Table II. Also, we give the results of an unprojected GSM calculation for parameter set I. The energy is reduced from the MFA result by $\sim 5\%$, indicating that the GSM state is very similar to a coherent state. This is confirmed by the expansion coefficients shown in Table I. In the calculations we used terms with up to 12 σ quanta; this is more than adequate for convergence.

Simple projection without variation reduced the energy by about 15—30 % in the coherent state approximation. This is comparable to the estimates of c.m. corrections using (8.2). Much of this reduction comes from the term in the energy which depends on the conjugate momentum:

$$\langle \int d^3r{:}\pi(\mathbf{r})^2{:}\rangle \leq 0 \ . \tag{8.8}$$

This term vanishes in the unprojected coherent state but after projection it gives a negative contribution. [Although $\pi(\mathbf{r})^2$ is a positive-definite operator, it contains a divergence. After removal of this by normal ordering, the net contribution from this term is negative.] Variation of the projected energy with respect to the four parameters defined in (7.1) leads to a further 10—15 % reduction.

For the GSM state, solution of the projected equations (5.9) implicitly includes variation with respect to a parameter analogous to ξ in (7.1a). Hence the reduction in the energy due to the projection is greater than for the coherent state: $\sim$30—45 %. Further variation with respect to the other three parameters of (7.1) leads to relatively small changes in the energy.

The results just discussed were calculated keeping the frequencies in the expansions (3.1) and (3.2) fixed to their vacuum values, (3.4). Tables III—VI also give results of calculations in which the frequencies Ω_k were treated as variational parameters, as discussed in Sec. VII. A Gaussian form was used, as in Eq. (7.8). The results shown are for $k_c = 16\ \text{fm}^{-1}$, but the results are essentially independent of k_c for $k_c \gtrsim 8\ \text{fm}^{-1}$. This variation leads to a further reduction in the energy of 10—20 %, giving a total reduction from the MFA energy of 40—55 %.

Results for the meson state (Table VI) are qualitatively similar, but the overall reduction in the energy is somewhat larger than for the baryon with the same parameter set.

The stationary points of the energy are found by using a multidimensional Newton-Raphson method. As noted above, these stationary points are not necessarily minima, and in some cases we find maxima or inflection points. In most cases the method converges quickly to the stationary point. However, inflection points can present problems for the Newton method. The changes in the variational parameters from their mean-field values can be significant, up to $\sim 40\%$. Some representative results are shown in Table VII.

TABLE VI. Meson energies obtained by projection and variation, for parameter set III. For explanation see Table III.

Approximation		Energy	%	V_1	V_2	V_3
MFA		4.920		0.0	0.0	0.0
$(\langle H\rangle^2-\langle P^2\rangle)^{1/2}$		3.467	29.4			
Projection	CSA	3.895	20.8	−0.296	3.510	−0.073
	GSM	2.810	42.9	−0.782	−1.183	0.621
Variation	CSA	3.523	28.4	0.393	−6.136	−0.741
	GSM	3.387	31.3	0.428	−4.293	−0.624
δm^2 only	CSA	2.689	45.3	0.157	5.815	0.069
	GSM	1.718	65.1	−0.467	1.339	0.014
Full variation	CSA	2.787	43.4	0.248	−8.153	−0.050
	GSM	2.719	44.7	0.381	−7.143	0.077

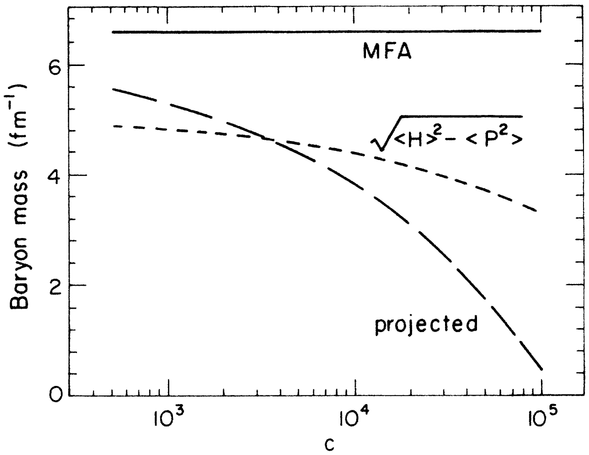

FIG. 1. Dependence of the projected nucleon energy on the model parameter c (long-dashed curve). For comparison, the unprojected MFA energy is shown by the solid line. Also shown is the nucleon mass calculated from Eq. (8.2) (short-dashed curve). The parameter sets used all have $b^2/ac=3.0$.

We are free to readjust the model parameters a, b, c, and g in order to obtain physically meaningful results. It is easy to rescale all lengths in the model in order to obtain agreement with (say) the baryon mass (the mean of nucleon and Δ masses). The model parameters have the following dimensions: $a \sim L^{-2}$, $b \sim L^{-1}$, c and g are dimensionless, and the energy is, of course, L^{-1}. In a subsequent paper, the parameters will be adjusted to yield the experimental baryon mass and nucleon size, as well as best fits to other properties. Therefore we are not concerned with absolute energies here, but seek rather "good" solutions to the field equations. The virial theorems provide a control (necessary but not sufficient) on the quality of the solutions.

As noted, the three virial theorems are satisfied identically in the MFA. Projection destroys this satisfaction. Variation of the energy with respect to the various parameters lowers the energy but does not solve the virial problem. Variation of $\omega_k{}^2$ through δm^2, as in Eq. (7.8), leads to a distinct improvement in the virial tests. Furthermore, as can be seen in Fig. 3, all three virial tests pass through zero in the vicinity of the (very flat) energy minimum. Unfortunately, the three do not pass through zero simultaneously. One can hope that they would have a coincidence in a higher-dimensional parameter space. Note that $\delta m^2=0$ is neither an energy minimum nor a particularly good point for the virial tests.

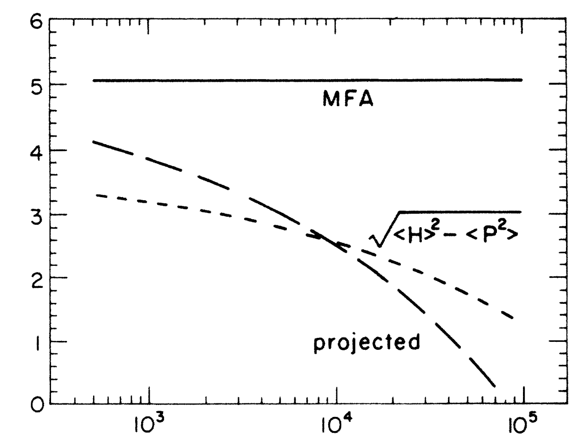

FIG. 2. As Fig. 1, but for the meson state.

TABLE VII. Effects of variation after projection on the nucleon wave function. The results shown are for a restricted variation in which δ was held at the same value as λ. For explanation of the variational parameters, see Eqs. (7.1) and (7.8). The parameter set III was used. No value of ξ is quoted for the GSM results, since the GSM equations implicitly include an analogous variation.

Approximation		ξ	$\lambda(=\delta)$	γ	δm^2 (fm^{-2})
MFA		1.0	1.0	1.0	0.0
Variation	CSA	1.174	1.077	1.354	0.0
	GSM		1.048	1.339	0.0
δm^2 only	CSA	1.0	1.0	1.0	−2.297
	GSM		1.0	1.0	−1.491
Full variation	CSA	0.962	1.461	1.158	−1.549
	GSM		1.464	1.111	−1.494

After variation with respect to the several parameters, there is little difference between the coherent state and general single-mode approximations.

IX. SUMMARY

In the context of the soliton bag model, the wave function for a localized state of quarks and the quantum σ field is constructed. The σ part of the wave function is either a coherent state or a general single-mode state. The former is closely analogous to the mean-field approxima-

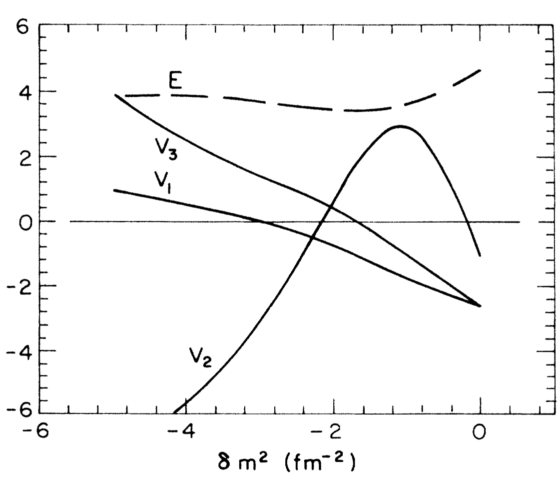

FIG. 3. Dependence of the projected energy and virial theorems on the parameter δm^2 [see Eq. (7.8)]. The results shown are for parameter set III. The cutoff k_c in Eq. (7.8) was taken to be 16 fm^{-1}.

tion, but it does contain quantum fluctuations.

An eigenstate of zero momentum is constructed by projection, which is effected by integration over the generator coordinate describing the center of the bag. The energy is evaluated by taking the expectation value of the Hamiltonian in the projected state. Since this expectation value is infinite, we must subtract off the energy of a uniform vacuum state to obtain the physical energy of the baryon. This energy difference is varied with respect to various parameters of the wave function. Stationary points in the energy variation are usually minima, but maxima and inflection points occur for some parameters, since the model Hamiltonian is not positive definite. (This is clear in the quark sector, where only valence quarks are considered.)

The variations which we find to be important include the radius and depth of the "well" in the σ field, and the radius and ratio of upper to lower components of the quark wave function. For the general single-mode state the Fock-space expansion coefficients are also variational parameters. In addition, we have studied variation of the frequencies of the plane-wave modes used to expand the quantum σ field. Although such a variation affects both the vacuum and hadron energies by a term proportional to the volume of all space, only the energy difference is evaluated and so the results are finite. The Ω_k variation is found to be important to the satisfaction of three virial theorems.

The objective has been to construct the best wave function of zero momentum. A measure of the quality of this wave function is the satisfaction of virial theorems. Three virial theorems are satisfied reasonably well, considering the limited space of variations. The final energy obtained shows a reduction of 40–50 % (depending on the model parameters) from the mean-field value. Since we have varied after projection, this reduction includes more than just recoil corrections.

Variation after projection, followed by boosting to a state of finite momentum, avoids the Peierls-Yoccoz problem: the relativistic energy-momentum relationship is guaranteed, at least for the expectation values.

The method described here improves on those in Refs. 5 and 35, since it involves construction of momentum eigenstates. The present paper outlines only the projection onto zero-momentum states and the variational method applied to these states. A subsequent paper will explore hadron properties calculated using projected and boosted soliton states, in which case, recoil corrections are automatically included. The properties to be considered include electromagnetic form factors, in particular, charge radii and magnetic moments. Although we find large reductions in the nucleon energy, due to projection and variation, these should be taken only as a measure of the importance of "recoil corrections" to the energy. The full implications of our results for nucleon properties in the model will not be clear until this program is complete. In particular, the model parameters will need to be refitted, to obtain the correct nucleon energy and size. Other properties, such as magnetic moments and g_A/g_V, will then provide a test of whether the model can adequately describe physical nucleons.

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