PHYSICAL REVIEW D VOLUME 33, NUMBER 8 15 APRIL 1986

Self-consistent, Poincaré-invariant and unitary three-particle scattering theory

James V. Lindesay, Alexander J. Markevich, H. Pierre Noyes, and George Pastrana
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
(Received 26 August 1985)

A Poincaré-invariant formalism for the scattering of three distinguishable scalar particles is developed. Lorentz invariance in the form of velocity conservation and a parametric relation between the two- and three-body off-shell continuations in energy are introduced in order to satisfy unitarity and physical clustering. The three-body-invariant probability amplitude is derived from the two-body transition matrix elements.

I. INTRODUCTION

We present a self-consistent, relativistic scattering theory for three distinguishable scalar particles of finite mass. From arbitrary pairwise interactions satisfying Lorentz invariance, individual particle mass conservation, and unitarity we derive integral equations leading to the probability amplitude for scattering in the full three-body system. The treatment satisfies several important criteria.[1,2]

(1) Relativistic invariance and four-momentum conservation. The equations, derived in an arbitrary Lorentz frame defined by an overall velocity, lead to an invariant probability amplitude. Four-momentum conservation is recovered in the on-shell limit as the product of energy conservation and velocity conservation.

(2) Two- and three-particle unitarity. The two-body input is constrained to satisfy unitarity. The form of the off-shell continuation guarantees that three-body unitarity follows.

(3) Unambiguous off-shell continuation. A set of parameters, corresponding to asymptotic single-particle energies, is introduced in order to write the relation between two- and three-body off-shell variables in terms of external quantities, independent of the integration over intermediate states. Both systems are then effectively dispersed in terms of the same variable, the three-body total energy.

(4) Proper cluster decomposition. Clustering, in the physical sense, is satisfied. If the interaction of one particle with each of the others vanishes, the solution decomposes into the product of a spectator and a two-particle scattering state.

(5) Correct nonrelativistic limit. In the low-energy limit the equations satisfy the same physical criteria as the nonrelativistic Faddeev equations.

The conditions of relativistic invariance, clustering, and unitarity place severe restrictions on the form of a scattering theory. In the three-body problem, the occurrence of successive pairwise interactions in different center-of-momentum frames leads to a consideration of the Lorentz transformation properties of off-mass-diagonal matrix elements. Clustering and unitarity point to the need for a parametric relation between the two- and three-body off-energy-shell dispersion variables. These considerations are treated here in the simplest possible context—the scattering of scalar particles.

Dirac[3] first showed that several different forms of Poincaré-invariant relativistic dynamics are possible. These dynamics are distinguished by the choice of invariant hypersurfaces on which initial conditions are specified. The usual choice is the "instant form" in which the hypersurface is $t=\text{const}$. In this case the generators of space translations and rotations are kinematic operators, while the dynamics is contained in the generator of time translation and the generators of Lorentz boosts. We utilize here the "point form" corresponding to the hypersurface $t^2-x^2=\text{const}>0$. Then the six generators of the Lorentz group are kinematic, while the dynamics is contained in the four-vector P. As a result, interactions are Lorentz invariant but do not commute with the generators of space-time translations. Since we are constructing a scattering theory that connects two-body t matrices to three-body t matrices, without explicit reference to the spatial form of the two-body potentials, the "point form" is the most natural for our purposes.

The various basis states needed to develop the scattering theory are defined in Sec. II. Section III reviews the fundamental operator relations and the Faddeev operator decomposition used to ensure well-defined integral equations. Section IV establishes the crucial connection to the two-body input. Here velocity conservation of the transition operator matrix elements is introduced in order to separate Lorentz invariance from the off-shell continuation in energy. Two-body dispersion is related parametrically to three-body dispersion. Section V presents the resulting integral equations. The connection to physical observables is described in Sec. VI, where the invariant probability amplitude is obtained from the solutions of the integral equations. Section VII summarizes the conclusions.

II. COVARIANT STATES

Consider a system of three distinguishable scalar particles with conserved, nonzero real masses and no internal degrees of freedom. States within this system transform via a unitary representation of the ten-dimensional Poincaré group $U(l,a)$ for Lorentz transformations l and space-time translations a:[4]

$$U(l_2,a_2)U(l_1,a_1)=U(l_2l_1,a_2+l_2a_1)\ . \tag{2.1}$$

For convenience, we write $U(l)$ for $U(l,0)$.

A general Lorentz transformation l can be written in terms of a pure Lorentz boost b and a pure rotation r. The boost is characterized by a velocity $\boldsymbol{\beta}$, from which we define a relativistic velocity

$$\mathbf{u}\equiv\frac{1}{(1-|\boldsymbol{\beta}|^2)^{1/2}}\boldsymbol{\beta}\ . \tag{2.2}$$

With $u^0=(1+|\mathbf{u}|^2)^{1/2}$, we define a four-vector velocity

$$u\equiv(u^0,\mathbf{u}) \tag{2.3}$$

which satisfies $u\cdot u=1$.

A general quantum state $|\psi\rangle$ can be used to define a new, boosted state

$$|\psi(\mathbf{u})\rangle=U(b(\mathbf{u}))|\psi\rangle\ . \tag{2.4}$$

A. Single-particle states

Quantities pertaining to a particular particle are labeled with a lower case roman letter or numerical subscript. A single-particle momentum eigenstate of mass m_a and velocity $\mathbf{u}_a$ is defined from a standard rest state of the same mass:[5]

$$|m_a,\mathbf{u}_a\rangle\equiv U(b(\mathbf{u}_a))|m_a,\mathbf{0}\rangle\ . \tag{2.5}$$

The four-momentum of this state is

$$k_a=m_au_a\ . \tag{2.6}$$

Since the individual particle masses are fixed, we adopt the convenient notation

$$|k_a\rangle\equiv|m_a,\mathbf{u}_a\rangle\ . \tag{2.7}$$

We choose the normalization

$$\langle k_a|k_a'\rangle=\frac{2u_a^0}{{m_a}^2}\delta^3(\mathbf{u}_a-\mathbf{u}_a')=2\epsilon_a\delta^3(\mathbf{k}_a-\mathbf{k}_a') \tag{2.8}$$

and completeness

$$\begin{aligned}1&=\int_{-\infty}^{+\infty}\frac{{m_a}^2d^3u_a}{2u_a^0}|k_a\rangle\langle k_a|\\ &=\int_{-\infty}^{+\infty}d^4k_a\delta({k_a}^2-{m_a}^2)\theta(k_a^0)|k_a\rangle\langle k_a|\ ,\end{aligned} \tag{2.9}$$

where $\epsilon_a=({m_a}^2+|\mathbf{k}_a|^2)^{1/2}$.

B. Three-particle states

Three particles can be grouped into a spectator a and a pair $(a+,a-)$, with $(a,a+,a-)$ cyclic. The subscript A is used to label quantities pertaining to the pair.

The 9 degrees of freedom of the three-body system can be represented in terms of collective variables such as the invariant mass of the system

$$W=[(k_1+k_2+k_3)\cdot(k_1+k_2+k_3)]^{1/2}\ ,$$

the relativistic four-velocity of the system

$$u=(k_1+k_2+k_3)/W\ ,$$

the two-body invariant masses

$$w_A=[(k_{a+}+k_{a-})\cdot(k_{a+}+k_{a-})]^{1/2}\ ,$$

and the two-body relativistic four-velocities

$$u_A=(k_{a+}+k_{a-})/w_A\ .$$

v_A is used to represent the four-vector u_A as observed from the three-body center-of-momentum frame. In particular,

$$v_A^0=u\cdot u_A\ . \tag{2.10}$$

p_A is used to represent the magnitude and $\hat{\mathbf{p}}_A$ the direction of the three-momentum of particle $a+$ as observed from the center of momentum of the $(a+,a-)$ subsystem.

The specification of any nine independent variables is sufficient to select a unique three-body momentum-space configuration. The remaining variables are then fixed as functions of these original nine variables and the three conserved individual particle masses. This functional dependence is not shown explicitly when it is clear from the context.

The full three-body Hamiltonian H is assumed to decompose into a noninteracting term plus a sum over three asymptotically pairwise interactions[6–8]

$$H=H^{(0)}+\sum_A H_A^{(I)}\ . \tag{2.11}$$

This leads to the use of several different types of three-particle states in our treatment.

An eigenstate of the noninteracting Hamiltonian $H^{(0)}$ is the direct product of three noninteracting single-particle states, one for each particle in the three-body system:

$$\begin{aligned}&H^{(0)}|k_1,k_2,k_3\rangle=E^{(0)}|k_1,k_2,k_3\rangle,\quad E^{(0)}=\epsilon_1+\epsilon_2+\epsilon_3\ ,\\ &\langle k_1,k_2,k_3|k_1',k_2',k_3'\rangle=\prod_{a=1}^{3}\frac{2u_a^0}{{m_a}^2}\delta^3(\mathbf{u}_a-\mathbf{u}_a')=u^0\zeta(w_A,v_A^0)\delta^3(\mathbf{u}-\mathbf{u}')\delta^3(\mathbf{v}_A-\mathbf{v}_A')\delta(w_A-w_A')\delta^2(\hat{\mathbf{p}}_A-\hat{\mathbf{p}}_A')\ ,\end{aligned} \tag{2.12}$$

where[9]

$$\zeta(w_A,v_A^0)=\frac{8v_A^0(W-w_Av_A^0)}{W^3{w_A}^2p_A}\ .$$

Corresponding to each free state is the equivalent boundary state of three widely separated, asymptotically noninteracting particles:

$$H^{(0)}|\Phi_0(k_1,k_2,k_3);Wu\rangle=E^{(0)}|\Phi_0(k_1,k_2,k_3);Wu\rangle\ , \tag{2.13}$$

where the total four-momentum $P=Wu$ is a convenient label[10] and $E^{(0)}=Wu^0$.

The eigenstates of $H_A=H^{(0)}+H_A^{(I)}$ form a complete set of clustered channel states. Specifying each state by its overall velocity u and the characteristics of the asymptotically interacting two-body subsystem gives

$$\begin{aligned} &H_A \mid u,u_A,\psi_A(w,\eta)\rangle = E_A \mid u,u_A,\psi_A(w,\eta)\rangle \ , \\ &E_A = Wu^0 \ , \\ &W = w_A v_A^0 + (m_a{}^2 + w_A{}^2 \mid \mathbf{v}_A \mid^2)^{1/2} \ . \end{aligned} \tag{2.14}$$

$\psi_A(w,\eta)$ represents a two-particle state of invariant mass w_A and internal quantum numbers summarized by the single parameter η_A. For two-particle scattering states the mass w is a continuous variable. For two-particle bound states w is one of a discrete set of bound-state masses μ_A. The two types of states are orthogonal, with normalizations

$$\begin{aligned} &\langle u,u_A,\psi_A(w,\eta) \mid u',u_A',\psi_A(w',\eta')\rangle \\ &\quad = u^0 \bar{\omega}(w_A,v_A^0)\delta^3(\mathbf{u}-\mathbf{u}')\delta^3(\mathbf{v}_A-\mathbf{v}_A')\delta(w_A,w_A')\delta_{\eta_A,\eta_A'} \ , \end{aligned} \tag{2.15}$$

where

$$\bar{\omega}(w_A,v_A^0) = \begin{cases} \frac{1}{2} p_A \zeta(\mu_A,v_A^0) & \text{bound} \ , \\ \frac{1}{4}(p_A/w_A)\zeta(w_A,v_A^0) & \text{scattering} \ , \end{cases}$$

$$\delta(w_A,w_A') = \begin{cases} \delta_{\mu_A,\mu_A'} & \text{bound} \ , \\ \delta(w_A-w_A') & \text{scattering} \ . \end{cases}$$

The overlap of these clustered channel states with noninteracting states defines wave functions

$$\begin{aligned} &\langle k_1,k_2,k_3 \mid u',u_a',\psi_A(w',\eta')\rangle \\ &\quad = u^0[\zeta(w_A,v_A^0)\bar{\omega}(w_A',v_A^0)]^{1/2}\delta^3(\mathbf{u}-\mathbf{u}') \\ &\qquad \times \delta^3(\mathbf{v}_A-\mathbf{v}_A')\psi_A(w,\hat{\mathbf{p}} \mid w',\eta') \ , \end{aligned} \tag{2.16}$$

where

$$\psi_A(w,\hat{\mathbf{p}} \mid w',\eta') = \begin{cases} \psi_A^b(w,\hat{\mathbf{p}} \mid \mu',\eta') & \text{bound} \ , \\ \psi_A^s(w,\hat{\mathbf{p}} \mid w',\eta') & \text{scattering} \ . \end{cases}$$

The wave functions are complete and orthonormal. With $\sum\!\!\!\!\!\!\int dw_A$ representing a sum over the bound-state masses and an integral from $m_A \equiv (m_{a+}+m_{a-})$ to ∞ over the scattering state energies,

$$\begin{aligned} &\sum_{\eta_A''} \sum\!\!\!\!\!\!\int dw_A'' \psi_A(w,\hat{\mathbf{p}} \mid w'',\eta'')\psi_A^*(w',\hat{\mathbf{p}}' \mid w'',\eta'') = \delta(w_A-w_A')\delta^2(\hat{\mathbf{p}}_A-\hat{\mathbf{p}}'_A) \ , \\ &\int_{m_A}^\infty dw_A'' \int d\hat{\mathbf{p}}_A'' \psi_A^{b*}(w'',\hat{\mathbf{p}}'' \mid \mu,\eta)\psi_A^b(w'',\hat{\mathbf{p}}'' \mid \mu',\eta') = \delta_{\mu_A,\mu_A'}\delta_{\eta_A,\eta_A'} \ , \\ &\int_{m_A}^\infty dw_A'' \int d\hat{\mathbf{p}}_A'' \psi_A^{s*}(w'',\hat{\mathbf{p}}'' \mid w,\eta)\psi_A^s(w'',\hat{\mathbf{p}}'' \mid w',\eta') = \delta(w_A-w_A')\delta_{\eta_A,\eta_A'} \ , \\ &\int_{m_A}^\infty dw_A'' \int d\hat{\mathbf{p}}_A'' \psi_A^{s*}(w'',\hat{\mathbf{p}}'' \mid w,\eta)\psi_A^b(w'',\hat{\mathbf{p}}'' \mid \mu',\eta') = 0 \ . \end{aligned} \tag{2.17}$$

Below the scattering threshold the summation over η_A'' extends only over quantum numbers corresponding to existing bound states.

Boundary states containing a bound pair of particles are equivalent to the bound clustered channel states

$$\begin{aligned} &H_A \mid \Phi_A(u,u_A,\psi_A^b(\mu,\eta)); Wu\rangle \\ &\quad = E_A \mid \Phi_A(u,u_A,\psi_A^b(\mu,\eta)); Wu\rangle \ , \\ &E_A = Wu^0 \ . \end{aligned} \tag{2.18}$$

The eigenstates of the full Hamiltonian H represent the solution of the physical problem

$$\begin{aligned} &H \mid \Psi_0(k_1,k_2,k_3); Wu\rangle = E \mid \Psi_0(k_1,k_2,k_3); Wu\rangle \ , \\ &H \mid \Psi_A(u,u_A,\psi_A^b(\mu,\eta)); Wu\rangle \\ &\quad = E \mid \Psi_A(u,u_A,\psi_A^b(\mu,\eta)); Wu\rangle \ , \\ &E = Wu^0 \ . \end{aligned} \tag{2.19}$$

Here k_1, k_2, and k_3 are the asymptotic momenta of individual particles and u_A is the asymptotic relativistic velocity of the bound pair.

The symbols $\mid \Phi_\alpha; Wu\rangle$ and $\mid \Psi_\alpha; Wu\rangle$ are used to represent general boundary and fully interacting states, respectively, with asymptotic limits containing either three free particles ($\alpha=0$) or a bound pair with a free spectator ($\alpha=A$).

III. SCATTERING OPERATORS

To solve the physical scattering problem, the exact eigenstates of the full three-body Hamiltonian are expressed in terms of the asymptotic boundary states with corresponding momenta. The standard techniques of scattering theory give

$$|\Psi_0^{(\pm)}(k_1,k_2,k_3);Wu\rangle = \lim_{\epsilon\to 0}(\mp i\epsilon)R(E\pm i\epsilon)|\Phi_0(k_1,k_2,k_3);Wu\rangle ,$$

$$|\Psi_A^{(\pm)}(u,u_A,\psi_A^b(w,\eta));Wu\rangle = \lim_{\epsilon\to 0}(\mp i\epsilon)R(E\pm i\epsilon)|\Phi_A(u,u_A,\psi_A^b(w,\eta));Wu\rangle , \tag{3.1}$$

where

$$R(Z)=\frac{1}{H-Z} \tag{3.2}$$

is the fully interacting resolvent. We also define the noninteracting resolvent

$$R^{(0)}(Z)=\frac{1}{H^{(0)}-Z} \tag{3.3}$$

and the channel resolvents

$$R_A(Z)=\frac{1}{H_A-Z} . \tag{3.4}$$

All resolvents satisfy the Hilbert identity

$$R(Z_1)-R(Z_2)=(Z_1-Z_2)R(Z_1)R(Z_2) \tag{3.5}$$

and

$$R^{\dagger}(Z)=R(Z^*) . \tag{3.6}$$

Several relations follow directly from the resolvent definitions:

$$R(Z)=R^{(0)}(Z)-R^{(0)}(Z)\sum_A H_A^{(I)}R(Z) , \tag{3.7}$$

$$R(Z)=R_A(Z)-R_A(Z)\sum_B \bar{\delta}_{AB}H_B^{(I)}R(Z) , \tag{3.8}$$

$$R_A(Z)=R^{(0)}(Z)-R^{(0)}(Z)H_A^{(I)}R_A(Z) , \tag{3.9}$$

where $\bar{\delta}_{AB}=1-\delta_{AB}$.

The three-body transition operator $T(Z)$ is defined to satisfy a Lippmann-Schwinger[11]-type equation:

$$T(Z)=\sum_A H_A^{(I)}-\sum_A H_A^{(I)}R^{(0)}(Z)T(Z) . \tag{3.10}$$

Then

$$R(Z)=R^{(0)}(Z)-R^{(0)}(Z)T(Z)R^{(0)}(Z) , \tag{3.11}$$

$$T(Z)=\sum_A H_A^{(I)}-\sum_A H_A^{(I)}R(Z)\sum_B H_B^{(I)} . \tag{3.12}$$

The Hilbert identity for the resolvents (3.5) leads to a unitarity relation for $T(Z)$

$$T(Z_1)-T(Z_2)=(Z_2-Z_1)T(Z_1)R^{(0)}(Z_1)R^{(0)}(Z_2)T(Z_2). \tag{3.13}$$

As it stands, the Lippmann-Schwinger-type equation (3.10) for $T(Z)$ yields an integral equation with a noncompact kernel and therefore has no unique solutions. In order to proceed, $T(Z)$ is decomposed using Faddeev's method[12] into

$$T(Z)=\sum_{A,B} T_{AB}(Z) . \tag{3.14}$$

The components satisfy

$$T_{AB}(Z)=\delta_{AB}T_A(Z)-\sum_D \bar{\delta}_{AD}T_A(Z)R^{(0)}(Z)T_{DB}(Z) , \tag{3.15}$$

where $T_A(Z)$, the transition operator for the scattering problem generated by the Hamiltonian $H_A=H^{(0)}+H_A^{(I)}$, satisfies

$$T_A(Z)=H_A^{(I)}-H_A^{(I)}R^{(0)}(Z)T_A(Z) , \tag{3.16}$$

$$R_A(Z)=R^{(0)}(Z)-R^{(0)}(Z)T_A(Z)R^{(0)}(Z) , \tag{3.17}$$

$$T_A(Z)=H_A^{(I)}-H_A^{(I)}R_A(Z)H_A^{(I)} , \tag{3.18}$$

$$T_A(Z_1)-T_A(Z_2) =(Z_2-Z_1)T_A(Z_1)R^{(0)}(Z_1)R^{(0)}(Z_2)T_A(Z_2) . \tag{3.19}$$

The Freedman, Lovelace, and Namyslowski (FLN) proof of unitarity[13] demonstrates that the unitarity of $T(Z)$ (3.13) follows from the unitarity of $T_A(Z)$ (3.19).

$T_A(Z)$ expresses the scattering of two particles in the presence of a third, asymptotically noninteracting particle. The relation of $T_A(Z)$ to the purely two-body scattering problem is the central issue of this treatment. It is discussed in Sec. IV.

To obtain integral equations with fully connected kernels, Eq. (3.15) for $T_{AB}(Z)$ is iterated once. Defining the operator $W(Z)$ and its components $W_{AB}(Z)$ through

$$W(Z)=\sum_{A,B} W_{AB}(Z) , \tag{3.20}$$

$$T_{AB}(Z)=\delta_{AB}T_A(Z)+W_{AB}(Z) , \tag{3.21}$$

gives

$$W_{AB}(Z)=-\bar{\delta}_{AB}T_A(Z)R^{(0)}(Z)T_B(Z) -\sum_D \bar{\delta}_{AD}T_A(Z)R^{(0)}(Z)W_{DB}(Z) . \tag{3.22}$$

The solution of this equation yields $T(Z)$, which then through (3.11) gives the full resolvent $R(Z)$. The connection to the physical probability amplitude is discussed in Sec. VI.

IV. TWO-BODY INPUT

The solution to the two-body problem is the input for this formalism. The transition operator $t(z)$, generated by a Hamiltonian $h=h^{(0)}+h^{(I)}$ acting in a two-body space, satisfies

$$t(z)=h^{(I)}-h^{(I)}r^{(0)}(z)t(z) , \tag{4.1}$$

$$r(z)=r^{(0)}(z)-r^{(0)}(z)t(z)r^{(0)}(z) , \tag{4.2}$$

$$t(z)=h^{(I)}-h^{(I)}r(z)h^{(I)} , \tag{4.3}$$

$$t(z_1)-t(z_2)=(z_2-z_1)t(z_1)r^{(0)}(z_1)r^{(0)}(z_2)t(z_2) , \tag{4.4}$$

where

$$r(z)=\frac{1}{h-z}, \quad r^{(0)}(z)=\frac{1}{h^{(0)}-z} .$$

The connection between $t_A(z)$ and $T_A(Z)$ cannot be written in operator form, since these two operators act in different Hilbert spaces. Instead, a matrix element relation is sought which satisfies covariance, unitarity, and clustering.

At the two-body level, Lorentz invariance and unitarity restrict the form of the matrix elements of $t(z)$ (Ref. 14). Lorentz invariance requires that the scattering process not alter the velocity of the center of momentum (see the Appendix). Extracting phase-space factors gives

$$\langle k_{a+},k_{a-} \mid t_A(z) \mid k'_{a+},k'_{a-} \rangle = (u_A^0)^2 (w_A w'_A)^{-3/2} (16 w_A w'_A / p_A p'_A)^{1/2} \delta^3(\mathbf{u}_A - \mathbf{u}'_A) \tau_A(w,\hat{\mathbf{p}} \mid w',\hat{\mathbf{p}}';\mathscr{Z}) , \tag{4.5}$$

where

$$\mathscr{Z}_A = z/u_A^0 .$$

The function τ_A depends on the indicated center-of-momentum variables, the off-shell parameter $\mathscr{Z}_A$, and the conserved individual particle masses m_{a+},m_{a-}.

Unitarity (4.4) requires

$$\tau_A(w,\hat{\mathbf{p}} \mid w',\hat{\mathbf{p}}';\mathscr{Z}_1) - \tau_A(w,\hat{\mathbf{p}} \mid w',\hat{\mathbf{p}}';\mathscr{Z}_2) = (\mathscr{Z}_2 - \mathscr{Z}_1) \int_{m_A}^{\infty} dw''_A \int d\hat{\mathbf{p}}''_A \tau_A(w,\hat{\mathbf{p}} \mid w'',\hat{\mathbf{p}}'';\mathscr{Z}_1) \times \frac{1}{w''_A - \mathscr{Z}_1} \frac{1}{w''_A - \mathscr{Z}_2} \tau_A(w'',\hat{\mathbf{p}}'' \mid w',\hat{\mathbf{p}}';\mathscr{Z}_2) . \tag{4.6}$$

The three-body unitarity condition (3.19) must reduce to this same restriction.

Clustering is satisfied if the exact physical solution for the case of a noninteracting third particle decomposes into the product of a spectator plane-wave and a two-body scattering state. When $T_{(a,a+)}(Z)$ and $T_{(a-,a)}(Z)$ both vanish, Eqs. (3.14), (3.21), and (3.22) give

$$T(Z) \Longrightarrow T_A(Z) . \tag{4.7}$$

Therefore, the product of an energy-conserving δ function with a matrix element of $T_A(Z)$ must conserve both the momentum of the spectator and the momentum of the pair.

Both clustering and unitarity require the matrix elements of $T_A(Z)$ to be proportional to the function τ_A. Lorentz invariance requires the conservation of u. The conservation of u_A is also necessary to ensure the independent Lorentz invariance of the decoupled spectator and the interacting two-body state in the clustering limit.

In order to connect three-body unitarity with two-body unitarity a parametric relation must exist between Z and $\mathscr{Z}$. This relation must reduce the three-body off-shell behavior to that of the two-body problem. Defining ϵ_a^{par} to be a parameter equal to the physical asymptotic energy of the spectator in the three-body center-of-momentum frame, we write

$$\mathscr{Z}_A = (Z^c - \epsilon_a^{\text{par}})/v_A^0 , \tag{4.8}$$

where

$$Z^c = Z/u^0 .$$

This gives the correct on-shell limit:

$$w_A = (W - \epsilon_a^{\text{par}})/v_A^0 . \tag{4.9}$$

The linear nature of the resolvent denominator gives

$$\begin{aligned} \langle k_1,k_2,k_3 \mid R^{(0)}(Z) \mid k'_1,k'_2,k'_3 \rangle &= \frac{1}{Wu^0 - Z} \langle k_1,k_2,k_3 \mid k'_1,k'_2,k'_3 \rangle \\ &= \frac{1}{u^0 v_A^0} \frac{1}{\tilde{w}_A - \mathscr{Z}_A} \langle k_1,k_2,k_3 \mid k'_1,k'_2,k'_3 \rangle , \end{aligned} \tag{4.10}$$

where

$$\tilde{w}_A = (W - \epsilon_a^{\text{par}})/v_A^0 .$$

By expressing the off-diagonal dependence of the matrix elements of $T_A(Z)$ on τ_A through $\tilde{w}_A$ instead of w_A, the restriction (4.6) on τ_A can be used to guarantee three-body unitarity (3.13) through (3.19).

The three-body phase-space element can be written as

$$\prod_{i=1}^{3} \frac{m_i{}^2 d^3 u_i}{2u_i^0} = (W^3/u^0)[\rho(W,v_A^0)]^{-1} dW\, d^3u\, d^3v_A d\hat{\mathbf{p}}_A , \tag{4.11}$$

where

$$\rho(W,v_A^0) = \frac{8 v_A^0 (W v_A^0 - w_A)}{w_A{}^2 p_A} .$$

The form of the matrix elements of $T_A(Z)$ which satisfies all the required conditions is

$$\langle k_1,k_2,k_3 \mid T_A(Z) \mid k'_1,k'_2,k'_3 \rangle = (u^0)^2 (WW')^{-3/2} [\rho(W,v_A^0)\rho(W',v_A^0)]^{1/2} \delta^3(\mathbf{u}-\mathbf{u}')\delta^3(\mathbf{v}_A - \mathbf{v}'_A) \times \theta(W - \epsilon_a^{\text{par}} - m_A v_A^0)\theta(W' - \epsilon_a^{\text{par}} - m_A v_A^0) \tau_A(\tilde{w},\hat{\mathbf{p}} \mid \tilde{w}',\hat{\mathbf{p}}';\mathscr{Z}) , \tag{4.12}$$

where

$$\tilde{\omega}_A = (W - \epsilon_a^{\text{par}})/v_A^0 ,$$
$$\tilde{\omega}'_A = (W' - \epsilon_a^{\text{par}})/v_A^0 ,$$
$$\mathscr{Z}_A = (Z^c - \epsilon_a^{\text{par}})/v_A^0 .$$

Since the matrix elements conserve both u and v_A^0, the relation between $\mathscr{Z}_A$ and Z is parametric. Using (4.10) and (4.12) to evaluate the matrix elements of (3.19) reproduces the two-body unitarity condition (4.6) written in terms of the variables $\tilde{\omega}_A$, $\tilde{\omega}'_A$, and $\tilde{\omega}''_A$, instead of w_A, w'_A, and w''_A. The θ functions in (4.12) provide the correct lower integration limit.

Because of the three-body nature of the interaction $H_A^{(I)}$ (see Ref. 6), $T_A(Z)$ does not conserve the velocity of the spectator off-energy-diagonal. However, as will be shown in Sec. VI, (4.12) does lead to a Poincaré-invariant S matrix which clusters properly. In the limit (4.7), $\mathscr{W}_{AB}^{(+)}=0$ in (6.20). Then the substitution of (6.20) into (6.1) gives an S matrix which conserves both the momentum of the spectator and the momentum of the interacting pair.

V. INTEGRAL EQUATIONS

Given the two-body inputs, (4.12) can be used in (3.22) to generate a coupled set of integral equations for the matrix elements of the components of $W(Z)$. In these equations the ϵ^{par} factors are formally treated as fixed parameters. In Sec. VI we will show that the resulting matrix elements are related to the physical probability amplitude only for a unique choice of values for these parameters.

To simplify the calculation, define the functions $\mathscr{W}_{AB}$ by

$$\langle k_1,k_2,k_3 \mid W_{AB}(Z) \mid k'_1,k'_2,k'_3 \rangle = (u^0)^2 (WW')^{-3/2} [\rho(W,v_A^0)\rho(W',v_B^{0'})]^{1/2} \delta^3(\mathbf{u}-\mathbf{u}')\theta(W-\epsilon_a^{\text{par}}-m_A v_A^0)$$
$$\times \theta(W'-\epsilon_b^{\text{par}'}-m_B v_B^{0'}) \mathscr{W}_{AB}(W,\mathbf{v},\hat{\mathbf{p}} \mid W',\mathbf{v}',\hat{\mathbf{p}}';Z^c) . \quad (5.1)$$

In addition to the indicated variables, $\mathscr{W}_{AB}$ depends parametrically on the individual particle masses and the factors ϵ_1^{par}, ϵ_2^{par}, ϵ_3^{par}, and $\epsilon_b^{\text{par}'}$.

In order to write the integral equations satisfied by $\mathscr{W}_{AB}$ another phase-space element is needed. Define the functions

$$\omega(W,v_I^0,m_i^2) = Wv_I^0 - \{m_i^2 + W^2[(v_I^0)^2-1]\}^{1/2} ,$$
$$\kappa(W,v_I^0,v_J^0,u_I\cdot u_J) = m_i^2 + m_j^2 - W^2 + 2\omega(W,v_I^0,m_i^2)\omega(W,v_J^0,m_j^2)u_I\cdot u_J , \quad (5.2)$$

and let W_{ij} be the largest real root of the fourth-order equation in W^2

$$\kappa(W_{ij},v_I^0,v_J^0,u_I\cdot u_J) - m_k^2 = 0, \quad k \neq i,j . \quad (5.3)$$

The phase-space element can then be written as

$$\prod_{i=1}^{3} \frac{m_i^2 d^3u_i}{2u_i^0} = (W_{ab}^3/u^0)\iota(W_{ab},v_A^0,v_B^0) d^3u\, d^3v_A d^3v_B , \quad (5.4)$$

where

$$\iota(W_{ab},v_A^0,v_B^0) = \frac{1}{4m_a m_b} \frac{[\omega(W_{ab},v_A^0,m_a^2)]^3}{v_A^0} \frac{[\omega(W_{ab},v_B^0,m_b^2)]^3}{v_B^0} \frac{\partial\omega(W_{ab},v_A^0,m_a^2)}{\partial m_a}$$
$$\times \frac{\partial\omega(W_{ab},v_B^0,m_b^2)}{\partial m_b} \left[\frac{\partial\kappa(W_{ab},v_A^0,v_B^0,u_A\cdot u_B)}{\partial W_{ab}} \right]^{-1} .$$

The driving terms in the integral equations have the form

$$\mathscr{D}_{AB}(W,\mathbf{v},\hat{\mathbf{p}} \mid W',\mathbf{v}',\hat{\mathbf{p}}';Z^c) = \iota(W_{ab}^{(I)},v_A^0,v_B^{0'})[\rho(W_{ab}^{(I)},v_A^0)\rho(W_{ab}^{(I)},v_B^{0'})]^{1/2}\theta(W_{ab}^{(I)}-\epsilon_a^{\text{par}}-m_A v_A^0)\theta(W_{ab}^{(I)}-\epsilon_b^{\text{par}'}-m_B v_B^{0'})$$
$$\times \tau_A(\tilde{\omega},\hat{\mathbf{p}} \mid \tilde{\omega}^{(I)},\hat{\mathbf{p}}^{(I)};\mathscr{Z}) \frac{1}{W_{ab}^{(I)}-Z^c} \tau_B(\tilde{\omega}^{(I)},\hat{\mathbf{p}}^{(I)} \mid \tilde{\omega}',\hat{\mathbf{p}}';\mathscr{Z}') , \quad (5.5)$$

where $W_{ab}^{(I)}$ is the largest real root of

$$\kappa_{ab}(W_{ab}^{(I)},v_A^0,v_B^{0'},u_A\cdot u'_B) - m_c^2 = 0, \quad c \neq a,b ,$$

and

$$\tilde{\omega}_A^{(I)} = (W_{ab}^{(I)}-\epsilon_a^{\text{par}})/v_A^0, \quad \tilde{\omega}_B^{(I)} = (W_{ab}^{(I)}-\epsilon_b^{\text{par}'})/v_B^{0'}, \quad \tilde{\omega}'_B = (W'-\epsilon_b^{\text{par}'})/v_B^{0'}, \quad \mathscr{Z}'_B = (Z^c-\epsilon_b^{\text{par}'})/v_B^{0'} .$$

$\hat{\mathbf{p}}_A^{(I)}$ and $\hat{\mathbf{p}}_B^{(I)}$ are specified through the four-vector

$$\mathscr{P}(W,\mathbf{v}_I,\mathbf{v}_J,m_i^2,m_j^2) = b^{-1}(\mathbf{v}_I)\{-\epsilon_{ijk}[\omega(W,v_J^0,m_j^2)v_J + \tfrac{1}{2}\omega(W,v_I^0,m_i^2)v_I]\} ,$$

where ϵ_{ijk} is the antisymmetric permutation symbol and $b(\mathbf{v}_I)$ is a boost from the center of momentum of the $(i+,i-)$

system to the three-body center of momentum. Define

$$\hat{\mathscr{P}}(W,\mathbf{v}_I,\mathbf{v}_J,m_i{}^2,m_j{}^2)=\frac{\mathscr{P}(W,\mathbf{v}_I,\mathbf{v}_J,m_i{}^2,m_j{}^2)}{|\mathscr{P}(W,\mathbf{v}_I,\mathbf{v}_J,m_i{}^2,m_j{}^2)|}\ .$$

Then

$$\hat{\mathbf{p}}_A^{(I)}=\hat{\mathscr{P}}(W_{ab}^{(I)},\mathbf{v}_A,\mathbf{v}'_B,m_a{}^2,m_b{}^2),\quad \hat{\mathbf{p}}_B^{(I)}=\hat{\mathscr{P}}(W_{ab}^{(I)},\mathbf{v}'_B,\mathbf{v}_A,m_b{}^2,m_a{}^2)\ .$$

With integration to occur over $d^3v''_D$, the kernels have the form

$$\mathscr{K}_{AD}(W,\mathbf{v},\hat{\mathbf{p}}\,|\,W'',\mathbf{v}'',\hat{\mathbf{p}}'';Z^c)=\iota(W'',v_A^0,v_D^{0''})[\rho(W'',v_A^0)\rho(W'',v_D^{0''})]^{1/2}$$
$$\times\theta(W''-\epsilon_a^{\text{par}}-m_Av_A^0)\theta(W''-\epsilon_a^{\text{par}}-m_Dv_D^{0''})\tau_A(\tilde{\omega},\hat{\mathbf{p}}\,|\,\tilde{\omega}'',\hat{\mathbf{p}}'';\mathscr{Z})\frac{1}{W''-Z^c}\ , \tag{5.6}$$

where W'' is the largest real root of

$$\kappa_{ad}(W'',v_A^0,v_D^{0''},u_A\cdot u''_D)-m_e{}^2=0,\quad e\neq a,d\ ,$$

and

$$\hat{\mathbf{p}}''_A=\hat{\mathscr{P}}(W'',\mathbf{v}_A,\mathbf{v}''_D,m_a{}^2,m_d{}^2),\quad \hat{\mathbf{p}}''_D=\hat{\mathscr{P}}(W'',\mathbf{v}''_D,\mathbf{v}_A,m_d{}^2,m_a{}^2),\quad \tilde{\omega}''_A=(W''-\epsilon_a^{\text{par}})/v_A^0\ .$$

Thus, the integral equations generated by the matrix elements of (3.22) have the form

$$\mathscr{W}_{AB}(W,\mathbf{v},\hat{\mathbf{p}}\,|\,W',\mathbf{v}',\hat{\mathbf{p}}';Z^c)=-\bar{\delta}_{AB}\mathscr{D}_{AB}(W,\mathbf{v},\hat{\mathbf{p}}\,|\,W',\mathbf{v}',\hat{\mathbf{p}}';Z^c)$$
$$-\sum_D\bar{\delta}_{AD}\int d^3v''_D[\mathscr{K}_{AD}(W,\mathbf{v},\hat{\mathbf{p}}\,|\,W'',\mathbf{v}'',\hat{\mathbf{p}}'';Z^c)\mathscr{W}_{DB}(W'',\mathbf{v}'',\hat{\mathbf{p}}''\,|\,W',\mathbf{v}',\hat{\mathbf{p}}';Z^c)]\ , \tag{5.7}$$

where the dependence on the conserved single-particle masses and the ϵ^{par} factors has been suppressed.

VI. PROBABILITY AMPLITUDE

The physical cross section is related in a well-known manner to the invariant probability amplitude $\mathscr{A}^{(+)}(\Phi_a\,|\,\Phi'_b;W)$ defined by

$$\langle\Psi_\alpha^{(-)};Wu\,|\,\Psi_\beta^{(+)'};W'u'\rangle=\delta_{\alpha\beta}\langle\Phi_\alpha;Wu\,|\,\Phi'_\beta;W'u'\rangle+2\pi i\delta^4(Wu-W'u')\mathscr{A}^{(+)}(\Phi_\alpha\,|\,\Phi'_\beta;W)\ ,$$
$$\delta^4(Wu-W'u')=(u^0/W^3)\delta(W-W')\delta^3(\mathbf{u}-\mathbf{u}')\ . \tag{6.1}$$

From (3.1) and (3.6)

$$\langle\Psi_\alpha^{(-)};Wu\,|\,\Psi_\beta^{(+)'};W'u'\rangle=\lim_{\epsilon\to0}\lim_{\epsilon'\to0}(-\epsilon\epsilon')\langle\Phi_\alpha;Wu\,|\,R(E+i\epsilon)R(E'+i\epsilon')\,|\,\Phi'_\beta;W'u'\rangle\ . \tag{6.2}$$

From (3.11) and (3.13)

$$R(Z_1)R(Z_2)=R^{(0)}(Z_1)[1-T(Z_1)R^{(0)}(Z_1)][1-R^{(0)}(Z_2)T(Z_2)]R^{(0)}(Z_2)\ ,$$
$$R(Z_1)R(Z_2)=R^{(0)}(Z_1)\left[1+T(Z_1)\left[\frac{1}{Z_2-Z_1}-R^{(0)}(Z_1)\right]-\left[\frac{1}{Z_2-Z_1}+R^{(0)}(Z_2)\right]T(Z_2)\right]R^{(0)}(Z_2)\ . \tag{6.3}$$

A. Elastic and rearrangement scattering

Consider first the case of elastic and rearrangement scattering. Each boundary state consists of one asymptotically free particle and a bound pair. Define a set of operators Q_{AB} satisfying

$$R^{(0)}(Z)W_{AB}(Z)R^{(0)}(Z)=R_A(Z)Q_{AB}(Z)R_B(Z)\ . \tag{6.4}$$

Then, by writing (3.5) as

$$R(Z_2)=R(Z_1)[1-(Z_1-Z_2)R(Z_2)]$$

and using (3.17) and (3.21), the second term on the right-hand side of (6.3) can be written as

$$R^{(0)}(Z_1)T(Z_1)\left[\frac{1}{Z_2-Z_1}-R^{(0)}(Z_1)\right]R^{(0)}(Z_2)=\frac{1}{Z_2-Z_1}R^{(0)}(Z_1)T(Z_1)R^{(0)}(Z_1)$$
$$=\frac{1}{Z_2-Z_1}\sum_A R^{(0)}(Z_1)T_A(Z_1)R^{(0)}(Z_1)$$
$$+\frac{1}{Z_2-Z_1}\sum_{A,B}R_A(Z_1)Q_{AB}(Z_1)R_B(Z_1)$$
$$=-\frac{1}{Z_2-Z_1}\sum_A[R_A(Z_1)-R^{(0)}(Z_1)]$$
$$+\sum_{A,B}R_A(Z_1)Q_{AB}(Z_1)\left[\frac{1}{Z_2-Z_1}-R_B(Z_1)\right]R_B(Z_2)\,. \tag{6.5}$$

With a similar manipulation on the third term, (6.3) becomes

$$R(Z_1)R(Z_2)=\sum_A[R_A(Z_1)R_A(Z_2)]-2R^{(0)}(Z_1)R^{(0)}(Z_2)$$
$$+\sum_{A,B}R_A(Z_1)\left[Q_{AB}(Z_1)\left[\frac{1}{Z_2-Z_1}-R_B(Z_1)\right]-\left[\frac{1}{Z_2-Z_1}+R_B(Z_2)\right]Q_{AB}(Z_2)\right]R_B(Z_2)\,. \tag{6.6}$$

Substituting this into (6.2) with $Z_1=E+i\epsilon$ and $Z_2=E'+i\epsilon'$, and using the principal value relation

$$\lim_{\epsilon\to 0}\frac{1}{x\pm i\epsilon}=\mathbf{P}\left[\frac{1}{x}\right]\mp i\pi\delta(x)\,, \tag{6.7}$$

gives

$$\langle\Psi_A^{(-)};Wu\mid\Psi_B^{(+)'};W'u'\rangle=\delta_{ab}\langle\Phi_A;Wu\mid\Phi_B';W'u'\rangle-2\pi i\delta(E-E')\langle\Phi_A;Wu\mid Q_{AB}^{(+)}(E)\mid\Phi_B';W'u'\rangle\,, \tag{6.8}$$

where

$$Q_{AB}^{(+)}(E)\equiv\lim_{\epsilon\to 0}Q_{AB}^{(+)}(E+i\epsilon)\,.$$

Equation (6.4) relates Q_{AB} to W_{AB}. Taking the matrix element between noninteracting states and using completeness in clustered channel states, along with the wave-function definitions (2.16), gives

$$\frac{1}{E-Z}\langle k_1,k_2,k_3\mid W_{AB}(Z)\mid k_1',k_2',k_3'\rangle\frac{1}{E'-Z}=[\zeta(w_A,v_A^0)\zeta(w_B',v_B^{0'})]^{1/2}$$
$$\times\sum_{\eta_A'',\eta_B'''}\sum\!\!\!\!\!\!\int dw_A''dw_B'''[\bar\omega(w_A'',v_A^0)\bar\omega(w_B''',v_B^{0'})]^{-1/2}$$
$$\times\psi_A(w,\hat{\mathbf{p}}\mid w'',\eta'')\psi_B^*(w',\hat{\mathbf{p}}'\mid w''',\eta''')\frac{1}{E''-Z}\,\frac{1}{E'''-Z}$$
$$\times\langle u,u_A,\psi_A(w'',\eta'')\mid Q_{AB}(Z)\mid u',u_B',\psi_B(w''',\eta''')\rangle\,. \tag{6.9}$$

Consider a scattering process characterized by a physical energy E^p. The parameters E^p, u, and u_A together specify a unique invariant mass μ_A^p for the $(a+,a-)$ system. Similarly E^p, u', and u_B' specify a unique invariant mass $\mu_B^{p'}$ for the $(b+,b-)$ system

$$\mu_A^p=\omega(W^p,v_A^0,m_a{}^2),\quad \mu_B^{p'}=\omega(W^{p'},v_B^{0'},m_b{}^2)\,, \tag{6.10}$$

where

$$W^p=E^p/u^0,\quad W^{p'}=E^p/u^{0'}\,.$$

In (6.9) set $Z=E^p+i\epsilon$, multiply both sides by $(-i\epsilon)^2$, and take the limit $\epsilon\to 0$. Since the wave functions and the matrix element of Q_{AB} are nonsingular, the right-hand side will vanish unless the invariant masses (6.10) correspond to actual two-body bound-states masses. Assume, for simplicity, that the spectrum of two-body bound states is nondegenerate. Then

$$\frac{1}{E-E^p}\lim_{\epsilon\to 0}(-i\epsilon)^2\langle k_1,k_2,k_3\,|\,W_{AB}(E^p+i\epsilon)\,|\,k_1',k_2',k_3'\rangle\frac{1}{E'-E^p}$$
$$=[\zeta(w_A,v_A^0)\zeta(w_B',v_B^{0'})]^{1/2}[\varpi(\mu_A^p,v_A^0)\varpi(\mu_B^{p'},v_B^{0'})]^{-1/2}\psi_A^b(w,\hat{\mathbf{p}}\,|\,\mu^p,\eta^p)\psi_B^{b*}(w',\hat{\mathbf{p}}'\,|\,\mu_B^{p'},\eta_B^{p'})$$
$$\times\langle u,u_A,\psi_A^b(\mu^p,\eta^p)\,|\,Q_{AB}^{(+)}(E^p)\,|\,u,u_B',\psi_B^b(\mu^{p'},\eta^{p'})\rangle\ . \tag{6.11}$$

We have chosen $u=u'$, since this is the only case which will contribute to the probability amplitude.

u, u_A, and u_B' are parameters of the particular physical process under consideration. Their values restrict the range of bras and kets which can appear on the left-hand side of (6.11). The requirement that the left-hand side of (6.11) vanish unless μ_A^p and $\mu_B^{p'}$ correspond to existing two-particle bound-state masses uniquely determines the values of the $\epsilon^{\rm par}$ factors in terms of the physical parameters. To see this we must consider the singularity structure of $W(Z)$.

The "primary singularities" due to the τ functions in the driving terms (5.5) occur to all orders of iteration of the integral equations (5.7) for the components of $W(Z)$. The singularity structure of the τ functions follows directly from (4.3). Taking a matrix element of (4.3) between free states, using $h^{(I)}=h-h^{(0)}$, and inserting completeness in terms of exact eigenstates of h shows that $\tau_A(\mathscr{Z})$ has poles at $\mathscr{Z}=\mu_A$, for each two-body bound state μ_A, and a scattering cut extending from $\mathscr{Z}=m_A$ to $+\infty$ along the real axis. Because of (4.8), this means that the matrix elements of $\mathscr{W}_{AB}(Z)$ have "primary singularities" at

$$Z^c=\epsilon_a^{\rm par}+\mu_A v_A^0,\quad Z^c=\epsilon_a^{\rm par}+m_A v_A^0\ ,$$
$$Z^c=\epsilon_b^{\rm par'}+\mu_B v_B^{0'},\quad Z^c=\epsilon_b^{\rm par'}+m_B v_B^{0'}\ . \tag{6.12}$$

These singularities must correspond to poles at $Z^c=W^p$ for values of W^p which satisfy

$$\mu_A=\omega(W^p,v_A^0,m_a^2),\quad \mu_B=\omega(W^p,v_B^{0'},m_b^2)\ . \tag{6.13}$$

Therefore, the left-hand side of (6.11) has the correct behavior in the $\epsilon\to 0$ limit only if

$$\epsilon_i^{\rm par}=W^p-\omega(W^p,v_I^0,m_i^2)v_I^0\ ,$$
$$\epsilon_i^{\rm par'}=W^p-\omega(W^p,v_I^{0'},m_i^2)v_I^{0'}\ . \tag{6.14}$$

The $\epsilon^{\rm par}$ factors are independent of the off-diagonal integration used in the coupled integral equations (5.7). All six $\epsilon^{\rm par}$ factors are fixed by (6.14) because matrix elements of each of the components of $W(E^p+i\epsilon)$ between the same free-particle bra and ket correspond to possible physical processes in different channels.

Having established the values of the $\epsilon^{\rm par}$ factors in terms of the physical problem under consideration, we return to the relation between $W_{AB}(Z)$ and $Q_{AB}(Z)$ in the case of physically realizable asymptotic states. With μ_A^p and $\mu_B^{p'}$ now particular bound-state masses in the outgoing and incoming channels, the wave functions in (6.11) can be expressed in terms of the two-body input (4.5). Using $H_A^{(I)}=H_A-H^{(0)}$ in (3.18), completeness in the clustered channel states, and the wave-function definitions gives

$$\langle k_1,k_2,k_3\,|\,T_A(Z)\,|\,k_1,k_2,k_3\rangle$$
$$=u^0\zeta(w_A,v_A^0)\delta^3(\mathbf{u}-\mathbf{u})\delta^3(\mathbf{v}_A-\mathbf{v}_A)\sum_{\eta_A''}\,\,\SumInt dw_A''\,|\,\psi_A(w,\hat{\mathbf{p}}\,|\,w'',\eta'')\,|^2\left[(E''-E)-(E''-E)\frac{1}{E''-Z}(E''-E)\right]\ . \tag{6.15}$$

Substituting (4.12), setting $Z=E^p+i\epsilon$, multiplying by $(-i\epsilon)$, and taking the limit as $\epsilon\to 0$ gives

$$v_A^0W^{-3}\rho(W,v_A^0)\lim_{\epsilon_A\to 0}(-i\epsilon_A)\tau_A(\tilde{\omega},\hat{\mathbf{p}}\,|\,\tilde{\omega},\hat{\mathbf{p}};\mu_A^p+i\epsilon_A)=-\zeta(w_A,v_A^0)(W-W^p)^2\,|\,\psi_A(w,\hat{\mathbf{p}}\,|\,\mu^p,\eta^p)\,|^2\ , \tag{6.16}$$

where

$$\epsilon_A=\epsilon^c/v_A^0,\quad \epsilon^c=\epsilon/u^0,\quad \tilde{\omega}_A=(W-W^p)/v_A^0+\omega(W^p,v_A^0,m_a^2)\ .$$

Similar considerations hold in the incoming channel.

Define

$$\chi(W,v_I^0,\hat{\mathbf{p}}_I,\mu_I^p)=[v_I^0W^3\varpi(\mu_I^p,v_I^0)]^{1/2}[-\lim_{\epsilon_I\to 0}(-i\epsilon_I)\tau_I(\tilde{\omega},\hat{\mathbf{p}}\,|\,\tilde{\omega},\hat{\mathbf{p}};\mu_I^p+i\epsilon_I)]^{-1/2}\ . \tag{6.17}$$

Then the comparison of (6.1) with (6.8), along with substitutions from (5.1), (6.11), and (6.16), yields the elastic and rearrangement scattering probability amplitude:

$$\mathscr{A}^{(+)}(\Phi_A(u,u_A,\psi_A^b(\mu^p,\eta^p))\,|\,\Phi_B(u,u_B',\psi_B^b(\mu^{p'},\eta^{p'}));W^p)$$
$$=-\chi(W,v_A^0,\hat{\mathbf{p}}_A,\mu_A^p)\chi(W',v_B^{0'},\hat{\mathbf{p}}_B',\mu_B^{p'})\lim_{\epsilon_A\to 0}\lim_{\epsilon_B'\to 0}(-\epsilon_A\epsilon_B')\mathscr{W}_{AB}(W,\mathbf{v},\hat{\mathbf{p}}\,|\,W',\mathbf{v}',\hat{\mathbf{p}}';W^p+i\epsilon^c)\ . \tag{6.18}$$

The $\epsilon^{\rm par}$ factors needed to evaluate this expression are fixed by (6.14), with v_I^0 determined by u, u_A, W, and $\hat{\mathbf{p}}_A$.

B. Free-particle scattering

In the case of free-particle scattering each boundary state consists of three free particles. Substituting (6.3) into (6.2) with $Z_1=E+i\epsilon$ and $Z_2=E'+i\epsilon'$ gives

$$\langle \Psi_0^{(-)};Wu \mid \Psi_0^{(+)'};W'u'\rangle=\langle \Phi_0;Wu \mid \Phi_0';W'u'\rangle-2\pi i\delta(E-E')\langle \Phi_0;Wu \mid T^{(+)}(E)\mid \Phi_0';W'u'\rangle \ . \tag{6.19}$$

The comparison of (6.19) with (6.1), along with substitutions from (3.14), (3.21), (4.12), and (5.1), gives

$$\mathscr{A}^{(+)}(\Phi_0(k_1,k_2,k_3)\mid \Phi_0(k_1',k_2',k_3');W)$$
$$=-\sum_{A,B}[\rho(W,v_A^0)\rho(W,v_B^{0'})]^{1/2}[\delta_{AB}\delta^3(\mathbf{v}_A-\mathbf{v}_A')\tau_A^{(+)}(w,\hat{\mathbf{p}}\mid w,\hat{\mathbf{p}}';w)+\mathscr{W}_{AB}^{(+)}(W,\mathbf{v},\hat{\mathbf{p}}\mid W,\mathbf{v}',\hat{\mathbf{p}}';W)] \ . \tag{6.20}$$

The ϵ^{par} factors needed to evaluate this expression are fixed by (6.14), with $W^p=W$. The result is that each ϵ^{par} factor is equal to the corresponding asymptotic single-particle energy, as observed from the three-body center-of-momentum frame.

C. Breakup and coalescence

Breakup and coalescence involve transitions between boundary states containing three asymptotically free particles and boundary states containing a spectator and a bound pair. Define the operators $K_{AB}(Z)$ for breakup and $\widetilde{K}_{AB}(Z)$ for coalescence through

$$R^{(0)}(Z)W_{AB}(Z)R^{(0)}(Z)=\sum_A R^{(0)}(Z)K_{AB}(Z)R_B(Z),\quad R^{(0)}(Z)W_{AB}(Z)R^{(0)}(Z)=\sum_B R_B(Z)\widetilde{K}_{AB}(Z)R^{(0)}(Z) \ . \tag{6.21}$$

Then an analysis similar to that of Secs. VI A and VI B gives the probability amplitude for breakup:

$$\mathscr{A}^{(+)}(\Phi_0(k_1,k_2,k_3)\mid \Phi_B(u',u_B',\psi_B^b(\mu^{p'},\eta^{p'}));W^p)=-\sum_A[\rho(W,v_A^0)]^{1/2}\chi(W',v_B^{0'},\hat{\mathbf{p}}_B',\mu_B^{p'})$$
$$\times \lim_{\epsilon_B'\to 0}(-i\epsilon_B')\mathscr{W}_{AB}(W,\mathbf{v},\hat{\mathbf{p}}\mid W',\mathbf{v}',\hat{\mathbf{p}}';W^p+i\epsilon^c) \tag{6.22}$$

and coalescence

$$\mathscr{A}^{(+)}(\Phi_A(u,u_A,\psi_A^b(\mu^p,\eta^p))\mid \Phi_0(k_1',k_2',k_3');W^p)$$
$$=-\sum_B\chi(W,v_A^0,\hat{\mathbf{p}}_A,\mu_A^p)[\rho(W',v_B^{0'})]^{1/2}\lim_{\epsilon_A\to 0}(-i\epsilon_A)\mathscr{W}_{AB}(W,\mathbf{v},\hat{\mathbf{p}}\mid W',\mathbf{v}',\hat{\mathbf{p}}';W^p+i\epsilon^c) \ . \tag{6.23}$$

VII. CONCLUSION

We have succeeded in deriving an explicitly invariant probability amplitude from considerations of the three-body problem in an arbitrary frame. Two ideas were central to this treatment. The first was the use of velocity conservation in place of momentum conservation in order to separate Lorentz invariance from the off-shell continuation in energy. The second was the introduction of ϵ^{par} factors into the connection between the two-body input and the three-body problem. The resulting equations exhibit exact unitarity and physical clustering.

ACKNOWLEDGMENTS

We are indebted to I. R. Afnan, E. O. Alt, and W. N. Polyzou for useful discussions of this problem at an earlier stage and to W. N. Polyzou for sending us unpublished papers showing the relationship between his work and that of S. N. Sokolov. We are also indebted to S. N. Sokolov for comments on a late draft of this paper. This work was supported by the Department of Energy under Contract No. DE-AC03-76SF00515.

APPENDIX

Poincaré invariance requires matrix elements which correspond to physical observables to remain unchanged under the action of the unitary operator $U(l,a)$. Exact solutions $|\psi\rangle$ are eigenstates of the four-momentum operator P, which forms four of the generators of the Poincaré group. Noninteracting states $|\phi\rangle$ are eigenstates of the noninteracting four-momentum operator $P^{(0)}$. This choice of noninteracting basis is made to ensure that the Poincaré boost generator is the same three-vector operator for both the fully interacting and the noninteracting systems. Since in the point form

$$[\mathbf{P},\mathbf{P}^{(0)}]\neq 0 \ ,$$

matrix elements such as

$$\langle \phi_1 \mid H \mid \phi_2\rangle$$

do not, in general, conserve three-momentum. Instead, they must conserve three-velocity in order that a transformation to a well-defined center-of-momentum frame be possible. Let U be such a transformation and Λ be the corresponding Lorentz matrix. Then

$$\langle \phi_1(W,\mathbf{u}) \mid H \mid \phi_2(W',\mathbf{u}') \rangle = \delta^3(\mathbf{u}-\mathbf{u}') f(W,W')$$

transforms into

$$\begin{aligned}\langle \phi_1(W,\mathbf{u}) \mid U^{-1}UHU^{-1}U \mid \phi_2(W',\mathbf{u}) \rangle &= \Lambda^0_\mu \langle \phi_1(W,\mathbf{0}) \mid P^\mu \mid \phi_2(W',\mathbf{0}) \rangle \\ &= u^0 \delta^3(\mathbf{0}-\mathbf{0}') g(W,W') .\end{aligned}$$

In the center-of-momentum frame both the three-momentum and the three-velocity vanish. Therefore, in this frame the conservation of one is equivalent (up to a Jacobian) to the conservation of the other.

In order to show a connection with the more common instant form, we consider the two-body potential. We refer here specifically to the operator which connects the generators of time translations in the interacting and the noninteracting systems

$$H = H^{(0)} + V .$$

A general instant form potential expressed in a momentum-space basis conserves three-momentum

$$\langle p_1,p_2 \mid V^I \mid p'_1,p'_2 \rangle = \delta^3(\mathbf{P}-\mathbf{P}') v^I(E,p \mid E',p') ,$$

where

$$P = p_1 + p_2, \quad p = \tfrac{1}{2}(p_1 - p_2) .$$

On the energy shell this becomes

$$\delta(E-E') \langle p_1,p_2 \mid V^I \mid p'_1,p'_2 \rangle = \delta^4(P-P') \tilde{v}^I(p \mid p'; W) ,$$

where

$$W^2 = P \cdot P$$

is an invariant.

A general point form potential conserves velocity

$$\langle p_1,p_2 \mid V^P \mid p'_1,p'_2 \rangle = \delta^3(\mathbf{U}-\mathbf{U}') v^P(W,p \mid W',p') ,$$

where

$$U = P/W .$$

On the energy shell this becomes

$$\delta(E-E') \langle p_1,p_2 \mid V^P \mid p'_1,p'_2 \rangle = W^3 [U^0]^{-2} \delta^4(P-P') \tilde{v}^P(p \mid p'; W) .$$

Thus, the two forms of the potential give the same on-shell result in the center-of-momentum frame ($U^0 = 1$) if

$$\tilde{v}^I(p \mid p'; W) = W^3 \tilde{v}^P(p \mid p'; W) .$$

The existence of different forms for the off-energy-shell extension reflects an ambiguity in the specification of this physically unobservable quantity. Each form preserves certain symmetries off the energy shell and breaks others.

---

[1]S. Morioka and I. R. Afnan, Phys. Rev. C **23**, 852 (1981).

[2]J. M. Namyslowski, Nuovo Cimento **57A**, 355 (1968).

[3]P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949). See also F. Coester and W. N. Polyzou, Phys. Rev. D **26**, 1348 (1982), and H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) **112**, 94 (1978). The latter two articles contain extensive references to other works on this problem.

[4]We adopt the notation a for a four-vector, $\mathbf{a}$ for a three-vector, and $a \cdot b = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$ for the four-vector dot product.

[5]The symbol $\mathbf{0}$ is used to refer to a three-vector with each component equal to zero.

[6]As Sokolov (Ref. 7) and Polyzou (Ref. 8) have noted, the separation of H into a noninteracting term plus a sum over three purely pairwise interactions (each of which is the direct product of a two-body interaction operator with a unit operator for the spectator) is not consistent with the commutation relations satisfied by the generators of the Poincaré group. A three-body correction term, which vanishes asymptotically, must be added. We absorb this correction term into the definition of the asymptotically pairwise interactions $H_A^{(I)}$.

[7]S. N. Sokolov, Report No. IHEP 74-134, Serpukhov, 1974 (unpublished). We are indebted to W. N. Polyzou for sending us this report.

[8]W. N. Polyzou (private communication).

[9]Several phase-space functions are introduced in the text. In each case the dependence on individual particle masses is not shown explicitly.

[10]A semicolon is used to separate variable dependence from parameter or label content of quantum states and functions.

[11]B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[12]L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1960) [Sov. Phys.—JETP **12**, 1014 (1961)]. See also L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in the Quantum Scattering Theory* (Davey, New York, 1965).

[13]D. Z. Freedman, C. Lovelace, and J. M. Namyslowski, Nuovo Cimento **43A**, 258 (1966). A more relevant operator form of the proof is provided by H. P. Noyes, Czech. J. Phys. B **24**, 1205 (1974). See also H. P. Noyes, Phys. Rev. C **26**, 1858 (1982).

[14]Angular momentum conservation is not discussed here. The treatment will be extended in the following article to include particle spin and exhibit angular momentum conservation.