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Gaussian effective potential. III. ϕ^6 theory and bound states

P. M. Stevenson
T. W. Bonner Laboratories, Physics Department, Rice University, Houston, Texas 77251

I. Roditi*
Département de Physique Théorique, Université de Genève, 1211 Genève 4, Switzerland
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Scalar theories with a $\lambda\phi^4+\xi\phi^6$ interaction are studied in $1+1$ and $2+1$ dimensions using the Gaussian-effective-potential method. Restrictions on the range of parameters are derived. In particular, the $(2+1)$-dimensional theory is unstable if the ϕ^6 coupling exceeds a critical value $\xi_c=0.255$. For certain ranges of parameters the approximation indicates the existence of a two-particle bound state, and yields expressions for its mass and wave function.

I. INTRODUCTION

The previous papers in this series[1,2] have discussed the motivation for the Gaussian-effective-potential (GEP) concept,[3–6] and have applied the method to $\lambda\phi^4$ field theories. This paper extends the results to ϕ^6 theories,

$$\mathscr{L}=\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi-\tfrac{1}{2}m_B{}^2\phi^2-\lambda_B\phi^4-\xi\phi^6\ , \qquad (1.1)$$

in 1 + 1 and 2 + 1 dimensions. The principal motivation is to illustrate how the GEP method can give information about bound states. In (1 + 1)- and (2 + 1)-dimensional $\lambda\phi^4$ theories, studied earlier, there are no bound states, since the interaction must be repulsive ($\lambda>0$) if the theory is to be stable. In ϕ^6 theories, however, one may have an attractive two-particle interaction, giving bound states, without compromising the theory's stability which is ensured by a positive ξ.[7–9]

The (2 + 1)-dimensional ϕ^6 theory is also interesting in its own right as a renormalizable, not superrenormalizable, theory. It is known that the $1/N$ expansion indicates that O(N)-symmetric ϕ^6 has a nontrivial ultraviolet fixed point at $\xi^*=O(1/N)$ (Refs. 10–12). Our results support the conjecture that this fixed point persists in the $N=1$ theory: we find that the stability of the theory requires

$$0<\xi<0.254\,916\ . \qquad (1.2)$$

Being a variational approximation, the GEP provides an upper bound on the true effective potential, and so we may definitely conclude that Eq. (1.2) is a *necessary* condition for stability. We cannot say for sure, however, that a theory with sufficiently small, finite ξ *will be* stable. It may be that beyond the Gaussian approximation the ϕ^6 coupling requires renormalization, with the bare ξ being infinitesimal. (In the $1/N$ expansion, for instance, ξ is finite at leading order but requires renormalization in higher orders.[12])

We shall only say a few words about the important question of the reliability of the GEP approximation, since this topic is discussed in detail in papers I and II. We are principally motivated by the considerable success of the GEP approach in a wide variety of (0 + 1)-dimensional systems,[1] including strong-coupling situations. A method for systematically improving the GEP results was proposed in I, and it is hoped that such calculations will be pursued in the future, so as to give a quantitative indication of the accuracy of the present results.

The plan of the paper is as follows. In Sec. II we calculate the GEP, and introduce the renormalized parameters. Section III analyzes the allowed ranges of the parameters, including the stability requirements. The general features of the GEP are discussed in Sec. IV, and some illustrative graphs are presented. Bound states are discussed in Sec. V, and the conclusions are summarized in Sec. VI. The Appendix deals with the evaluation of matrix elements. Our notation follows II, with which we assume the reader is acquainted.

II. CALCULATION OF THE GEP

As explained in II, one calculates the GEP by taking the Hamiltonian density

$$\mathscr{H}=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_B{}^2\phi^2+\lambda_B\phi^4+\xi\phi^6\ , \qquad (2.1)$$

and substituting

$$\begin{aligned}\phi&=\phi_0+\hat{\phi}\\ &=\phi_0+\int(dk)_\Omega[a_\Omega(\mathbf{k})e^{-ik\cdot x}+a_\Omega^\dagger(\mathbf{k})e^{ik\cdot x}]\ ,\end{aligned} \qquad (2.2)$$

thereby evaluating

$$V_G(\phi_0,\Omega)\equiv{}_\Omega\langle 0\,|\,\mathscr{H}\,|\,0\rangle_\Omega\ , \qquad (2.3)$$

where $|\,0\rangle_\Omega$ has the defining property

$$a_\Omega(\mathbf{k})\,|\,0\rangle_\Omega=0\ . \qquad (2.4)$$

Since the result for $\lambda\phi^4$ theory has been calculated in II, we need only to evaluate the contribution of the $\xi\phi^6$ term:

$$\begin{aligned}{}_\Omega\langle 0\,|\,\xi\phi^6\,|\,0\rangle_\Omega=\xi(\phi_0{}^6+15\phi_0{}^4\langle\hat{\phi}^2\rangle_\Omega\\ +15\phi_0{}^2\langle\hat{\phi}^4\rangle_\Omega+\langle\hat{\phi}^6\rangle_\Omega)\ ,\end{aligned} \qquad (2.5)$$

where

$$\langle\hat{\phi}^{2N}\rangle_\Omega \equiv {}_\Omega\langle 0|\hat{\phi}^{2N}|0\rangle_\Omega = [2^{-N}(2N)!/N!][I_0(\Omega)]^N . \quad (2.6)$$

(The calculation of such matrix elements is discussed in more detail in the Appendix.) The net result is

$$V_G(\phi_0,\Omega) = I_1 + \tfrac{1}{2}(m_B{}^2-\Omega^2)I_0 + \tfrac{1}{2}m_B{}^2\phi_0{}^2 + \lambda_B\phi_0{}^4 + \xi\phi_0{}^6 + 6\lambda_B I_0\phi_0{}^2 + 3\lambda_B I_0{}^2 + 15\xi(I_0\phi_0{}^4 + 3I_0{}^2\phi_0{}^2 + I_0{}^3) . \quad (2.7)$$

The GEP itself, $\bar{V}_G(\phi_0)$, is defined as the minimum with respect to Ω of $V_G(\phi_0,\Omega)$. Sometimes the minimum occurs at an end point of the range $0<\Omega<\infty$, but usually Ω is determined by the condition $dV_G(\phi_0,\Omega)/d\Omega=0$ which gives the "$\bar{\Omega}$ equation:"

$$\bar{\Omega}^2 = m_B{}^2 + 12\lambda_B(I_0+\phi_0{}^2) + 30\xi(\phi_0{}^4 + 6I_0\phi_0{}^2 + 3I_0{}^2) \quad (2.8)$$

[with $I_0 = I_0(\bar{\Omega})$ here].

The ϕ^6 theory has three free parameters: $m_B{}^2, \lambda_B, \xi$. In order to have the results in manifestly finite form, free of divergent integrals, we need to reparametrize the theory in terms of a set of three finite parameters. As $m_B{}^2, \lambda_B, \xi$ are essentially the second, fourth, and sixth derivatives of the original potential at the origin, an obvious strategy would be to use a set of "renormalized parameters" $m_R{}^2, \lambda_R, \xi_R$, defined through the second, fourth, and sixth derivatives of the GEP at the origin.[5] We shall follow this program in spirit, though our final choice of a convenient set of parameters is based on a desire to keep the final results as compact as possible, as well as manifestly finite.

We proceed, therefore, to investigate the derivatives of $\bar{V}_G(\phi_0)$ at the origin. The first derivative of $\bar{V}_G(\phi_0)$ can be obtained from a *partial* differentiation of (2.7), because of the fact that $\partial V_G/\partial\Omega$ vanishes for $\Omega=\bar{\Omega}$. Hence,

$$\frac{d\bar{V}_G}{d\phi_0} = \phi_0[m_B{}^2 + 4\lambda_B(\phi_0{}^2+3I_0) + 6\xi(\phi_0{}^4 + 10I_0\phi_0{}^2 + 15I_0{}^2)] . \quad (2.9)$$

Differentiating again, we shall need to allow for the ϕ_0 dependence of $\bar{\Omega}$, which from (2.8) is

$$\frac{d\bar{\Omega}^2}{d\phi_0{}^2} = \frac{12[\lambda_B + 5\xi(3I_0+\phi_0{}^2)]}{1+6I_{-1}[\lambda_B + 15\xi(I_0+\phi_0{}^2)]} . \quad (2.10)$$

Using this, one finds for the second derivative

$$\frac{d^2\bar{V}_G}{d\phi_0{}^2} = m_B{}^2 + 12\lambda_B(I_0+\phi_0{}^2) + 30\xi(\phi_0{}^4 + 6I_0\phi_0{}^2 + 3I_0{}^2) - \phi_0{}^2 I_{-1}\frac{\{12[\lambda_B + 5\xi(3I_0+\phi_0{}^2)]\}^2}{1+6I_{-1}[\lambda_B+15\xi(I_0+\phi_0{}^2)]} . \quad (2.11)$$

Evaluating this at the origin, and comparing with (2.8), shows that

$$m_R{}^2 \equiv \left.\frac{d^2\bar{V}_G}{d\phi_0{}^2}\right|_{\phi_0=0} = \bar{\Omega}^2|_{\phi_0=0} = m_B{}^2 + 12\lambda_B I_0(m_R) + 90\xi[I_0(m_R)]^2 . \quad (2.12)$$

As in the ϕ^4 case,[2,5] the parameter m_R, so defined, proves to be the physical particle mass (see Sec. V).

Differentiating (2.11) twice more, keeping only the terms which contribute at $\phi_0=0$, yields

$$\lambda_R \equiv \left.\frac{1}{4!}\frac{d^4\bar{V}_G}{d\phi_0{}^4}\right|_{\phi_0=0} = \lambda_r\frac{[1-12\lambda_r I_{-1}(m_R)]}{[1+6\lambda_r I_{-1}(m_R)]} , \quad (2.13)$$

where

$$\lambda_r \equiv \lambda_B + 15\xi I_0(m_R) . \quad (2.14)$$

Unlike I_0, in $\nu+1$ dimensions ($\nu=1,2$) the integral I_{-1} is convergent:

$$I_{-1}(m_R) = \begin{cases} 1/(2\pi m_R{}^2), & \nu=1 , \\ 1/(4\pi m_R), & \nu=2 . \end{cases} \quad (2.15)$$

Consequently, as far as the removal of divergences is concerned λ_r is as good a parameter as λ_R. We choose to work with λ_r to avoid unnecessary algebraic complications.

Similarly, we find that the sixth derivative of $\bar{V}_G(\phi_0)$ at the origin is finitely related to ξ, λ_r, and m_R, so that we may choose to keep ξ as one of our set of parameters.

We now proceed to reparametrize the results (2.7) and (2.8) in terms of the parameter set m_R, λ_r, ξ. We use Eqs. (2.12) and (2.14) to eliminate $m_B{}^2$ and λ_B, and we isolate the vacuum-energy constant term

$$D \equiv \bar{V}_G(\phi_0=0) = I_1(m_R) - 3\lambda_r[I_0(m_R)]^2 + 15\xi[I_0(m_R)]^3 . \quad (2.16)$$

Algebraic manipulations, utilizing the $I_1(\Omega)-I_1(m_R)$ formula from Table II of II, then lead straightforwardly to

$$V_G(\phi_0,\Omega) = D + \tfrac{1}{2}m_R{}^2\phi_0{}^2 + \lambda_r\phi_0{}^4 + \xi\phi_0{}^6 - m_R{}^{\nu+1}L_2(x)/(8\pi) + \tfrac{1}{2}(m_R{}^2-\Omega^2)\Delta I_0 + 3\lambda_r(\Delta I_0)^2 + 6\lambda_r(\Delta I_0)\phi_0{}^2 + 15\xi(\Delta I_0)[\phi_0{}^4 + 3\phi_0{}^2\Delta I_0 + (\Delta I_0)^2] , \quad (2.17)$$

with

$$\bar{\Omega}^2 = m_R{}^2 + 12\lambda_r(\Delta I_0 + \phi_0{}^2) + 30\xi[\phi_0{}^4 + 6\phi_0{}^2\Delta I_0 + 3(\Delta I_0)^2] , \quad (2.18)$$

where

$$\Delta I_0 \equiv I_0(\Omega) - I_0(m_R) = -m_R{}^{\nu-1}L_1(x)/(4\pi) . \quad (2.19)$$

It is amusing that these equations are exactly what one would obtain by replacing I_1 and I_0 by their finite parts (defined by a subtraction at $\Omega=m_R$) in Eqs. (2.7) and (2.8), while ignoring the distinction between m_B,λ_B and m_R,λ_r. That procedure would, of course, have no *a priori* justification, but it does provide a useful mnemonic for the result of the proper substitution procedure.

At this stage it is convenient to introduce some new notation, so as to work with dimensionless variables. We thus define, for the GEP and the ϕ_0 field,

$$\mathscr{V}_G=(V_G-D)/m_R^{\nu+1}\,,\qquad F=4\pi\Phi_0^2,\quad \Phi_0^2=\phi_0^2/m_R^{\nu-1} \tag{2.20}$$

for the ϕ^4 and ϕ^6 couplings

$$\alpha=\frac{3\hat{\lambda}_r}{2\pi},\quad \hat{\lambda}_r=\frac{\lambda_r}{m_R^{3-\nu}}\,,\qquad \beta=\frac{45}{8\pi^2}\hat{\xi},\quad \hat{\xi}=\frac{\xi}{m_R^{4-2\nu}}\,, \tag{2.21}$$

and for the Ω variable

$$z=L_1(x)=\begin{cases}\ln x, & \nu=1\,,\\ (\sqrt{x}-1), & \nu=2\,,\end{cases}\qquad x=\Omega^2/m_R^2\,. \tag{2.22}$$

In terms of these new variables we can rewrite the results in a form suitable for further analysis and numerical evaluation:

$$\mathscr{V}_G(\phi_0,\Omega)=\frac{1}{8\pi}[(F+\tfrac{1}{3}\alpha F^2+\tfrac{1}{45}\beta F^3)+g(z)+\alpha z(z-2F)-\tfrac{1}{3}\beta z(F^2-3Fz+z^2)]\,, \tag{2.23}$$

where

$$g(z)=-[L_2(x)+(1-x)L_1(x)]=\begin{cases}e^z-(1+z), & \nu=1\,,\\ z^2(1+\tfrac{1}{3}z), & \nu=2\,.\end{cases} \tag{2.24}$$

The $\overline{\Omega}$ equation becomes

$$x(z)-1=2\alpha(F-z)+\tfrac{1}{3}\beta(F^2-6Fz+3z^2)\,, \tag{2.25}$$

where, from (2.22),

$$\frac{\Omega^2}{m_R^2}=x(z)=\begin{cases}e^z, & \nu=1\,,\\ (1+z)^2, & \nu=2\,.\end{cases} \tag{2.26}$$

III. THE α,β PARAMETER SPACE

A. Summary

The ϕ^4 and ϕ^6 coupling parameters α and β [Eq. (2.21)] can, *a priori*, take on any real values. However, not all values lead to stable theories. More subtly, not all distinct points in the α,β plane correspond to distinct theories, and to avoid duplications one should restrict the parameters to certain regions. These matters are studied in detail in this section. The results are summarized by Figs. 1 and 2, and the impatient reader may decide to inspect the figures and omit the rest of this section.

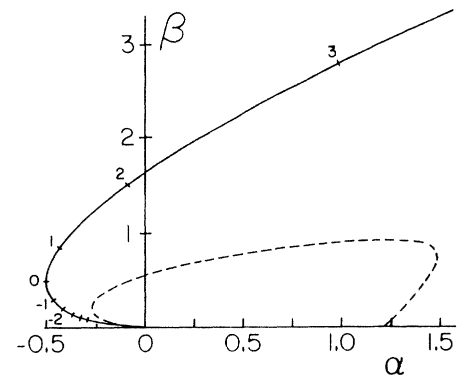

FIG. 1. The α,β parameter space for $(\phi^6)_{1+1}$ theory. The allowed region lies to the right of the solid curve and above the α axis. [The numbers along the curve indicate how z_0 varies in the parametric form (3.15).] Outside the region enclosed by the dashed curve the $\phi_0=0$ vacuum is no longer the global minimum of the GEP.

B. Stability considerations

We begin by showing that β must be positive. For β negative in 1 + 1 dimensions the GEP would be governed by the $\Omega\to 0$ $(z\to-\infty)$ end point of (2.23). This would give the infrared catastrophe that $\overline{V}_G(\Phi_0)\to-\frac{1}{3}\beta z^3\to-\infty$, at all Φ_0. In 2 + 1 dimensions the infrared behavior is much milder, and the $\Omega\to 0$ end point, which now corresponds to $z\to-1$, yields a finite result for $\overline{\mathscr{V}}_G(\Phi_0)$. However, the GEP is not bounded below, since the βF^3 term in (2.23) dominates at large F, rendering the theory unstable.

More surprising is that, in the (2 + 1)-dimensional case, β must not be too large. The existence of such a bound is obvious if one considers the $\Omega\to\infty$ $(z\to\infty)$ end point, which gives

$$\mathscr{V}_G(\Phi_0,\Omega\to\infty)=\tfrac{1}{3}(1-\beta)z^3\,, \tag{3.1}$$

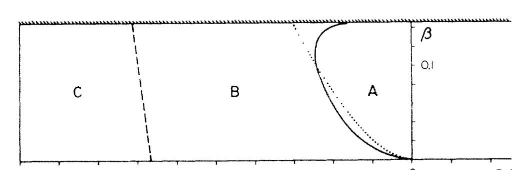

FIG. 2. The α,β parameter space for $(\phi^6)_{2+1}$ theory. The allowed region is the semi-infinite strip $\alpha>-1$, and $0<\beta<\beta_c=0.145\ldots$ The solid curve marks the onset of symmetry breaking: In region A the symmetric vacuum is stable; in region B there are degenerate vacua at $\phi_0=\pm c$ $(c\neq 0)$, with the origin being a local minimum; in region C the $\Omega=0$ end point is operative, making the origin a local maximum. [See Eq. (4.3).] The dotted curve indicates where the $\Omega=0$ end point first becomes relevant. [See Eq. (4.7).] Not shown is the fact that the unbroken-symmetry region extends only up to $\alpha\simeq 1.5$. Thereafter, as in $(\phi^4)_{2+1}$ theory, spontaneous symmetry breaking sets in.

implying that $\overline{\mathscr{V}}_G(\Phi_0)=-\infty$ at all Φ_0 if $\beta>1$. The reason is that the contribution from the kinetic term, $g(z)$, diverges only like $\frac{1}{3}z^3$ and can be overwhelmed by the $-\frac{1}{3}\beta z^3$ term from the ϕ^6 interaction. [This cannot happen in 1 + 1 dimensions, where $g(z)$ diverges exponentially as $z\to\infty$.] The same instability was observed in O(N)-symmetric $(\phi^6)_{2+1}$ theory in the $1/N$-expansion analysis of Bardeen, Moshe, and Bander.[11]

The bound on β is actually stronger than $\beta<1$, and arises from consideration of the large-Φ_0 behavior of $\overline{\mathscr{V}}_G(\Phi_0)$. As $F, \equiv 4\pi\Phi_0^2, \to\infty$, the solution of the $\overline{\Omega}$ equation (2.25) behaves as

$$z\to AF\ , \tag{3.2}$$

where the constant A satisfies the quadratic equation

$$A^2=\tfrac{1}{3}\beta(1-6A+3A^2)\ . \tag{3.3}$$

Substituting into (2.23), one finds the large-Φ_0 behavior of the GEP to be

$$\begin{aligned}\overline{\mathscr{V}}_G(\Phi_0)&\to\frac{F^3}{24\pi}[\tfrac{1}{15}\beta+A^3-\beta A(1-3A+A^2)]\\&=\frac{F^3}{8\pi}\frac{\beta}{45}\frac{1-4[\frac{1}{3}\beta(1+2\beta)]^{1/2}}{6[\frac{1}{3}\beta(1+2\beta)]^{1/2}+1+5\beta}\ ,\end{aligned} \tag{3.4}$$

i.e., $\Phi_0^6\times$const. Stability of the theory requires this constant to be positive, and hence

$$\beta<\beta_c=(\sqrt{10}-2)/8=0.145\,285\ , \tag{3.5}$$

which translates into

$$\xi<\xi_c=\frac{8\pi^2}{45}\frac{\sqrt{10}-2}{8}=0.254\,916\ . \tag{3.6}$$

[In the critical case $\beta=\beta_c$ the large-Φ_0 behavior of the GEP is governed by what is normally the subleading Φ_0^4 term. Carrying out a more detailed version of the above analysis, we have found the subleading term in (3.4) to be $(F^2/8\pi)A^2(\alpha+\beta)/\beta$. Hence, in the critical case $\beta=\beta_c$, the theory is only stable if $\alpha>-\beta_c$. This implies that for β's just less than β_c, the onset of spontaneous symmetry breaking (SSB) must occur near $\alpha=-\beta$. See the solid curve in Fig. 2.]

C. Avoiding duplications (1 + 1 dimensions)

A root of the $\overline{\Omega}$ equation will only correspond to a *minimum* of V_G if $d^2V_G/d\phi_0^2|_{\Omega=\overline{\Omega}}>0$. This condition requires

$$1+6I_{-1}(\overline{\Omega})[\lambda_B+15\xi(I_0(\overline{\Omega})+\phi_0^2)]>0\ . \tag{3.7}$$

We consider first the (1 + 1)-dimensional case, where this becomes

$$\tfrac{1}{2}e^z+\alpha+\beta(F-z)>0\ . \tag{3.8}$$

In particular, we see that at the origin ($F=0$) the $z=0$ root (corresponding to $\Omega=m_R$) is not in fact a minimum unless $\alpha>-\frac{1}{2}$. If $\alpha<-\frac{1}{2}$ we are in a mess because m_R^2 no longer corresponds to its definition as $d^2\overline{V}_G/d\phi_0^2|_{\phi_0=0}=\overline{\Omega}^2|_{\phi_0=0}$, since the real $\overline{\Omega}^2|_{\phi_0=0}$ must be a different root of the $\overline{\Omega}$ equation. Exactly the same situation was encountered in $(\phi^4)_{3+1}$, and the story that follows parallels Sec. V B of II.

When $\alpha<-\frac{1}{2}$, the true particle mass squared, denoted $m_R'^2$, differs from the "fake" m_R^2 by a factor e^{z_0}, where z_0 is a nonzero root of the $\overline{\Omega}$ equation at the origin:

$$e^{z_0}=1-2\alpha z_0+\beta z_0^2\ . \tag{3.9}$$

(In fact, there are two nonzero roots when $\alpha<-\frac{1}{2}$. The relevant one is that which gives the *global* minimum of $\mathscr{V}_G$.) To correct for the misidentification of m_R, we need to rescale all the variables so that they are measured in units of the true mass m_R':

$$\begin{aligned}&z'=\ln(\Omega^2/m_R'^2)=z-z_0\ ,\\&F'=F\ ,\\&\alpha'=(m_R^2/m_R'^2)\alpha=\alpha e^{-z_0}\ ,\\&\beta'=(m_R^2/m_R'^2)\beta=\beta e^{-z_0}\ ,\\&\mathscr{V}_G'+\text{const}=(m_R^2/m_R'^2)\mathscr{V}_G=\mathscr{V}_G e^{-z_0}\ .\end{aligned} \tag{3.10}$$

[A constant term must be added to $\mathscr{V}_G'$ so that the zero of the energy scale corresponds to $\overline{\mathscr{V}}_G(\Phi_0=0)=0$ with $\overline{\Omega}|_{\phi_0=0}=m_R'$, rather than m_R.] Inserting (3.10) into the $\overline{\Omega}$ equation (2.25) yields

$$\begin{aligned}e^{z'}-1=&\,2(\alpha'-\beta' z_0)(F'-z')\\&+\tfrac{1}{3}\beta'(F'^2-6F'z'+3z'^2)\ .\end{aligned} \tag{3.11}$$

We observe that this has the same form as the original $\overline{\Omega}$ equation, except that the effective α' parameter is

$$\alpha'_{\text{eff}}=\alpha'-\beta' z_0=(\alpha-\beta z_0)e^{-z_0}\ . \tag{3.12}$$

In the same way, inserting (3.10) into (2.23) leads to $\mathscr{V}_G'$ taking the same form as $\mathscr{V}_G$, but with α replaced by α'_{eff}, and all the other variables replaced by their primed counterparts.

This means that a theory with parameters m_R,α,β with $\alpha<-\frac{1}{2}$ (for which m_R does *not* correspond to the particle mass) is equivalent to a theory with parameters $m_R',\alpha'_{\text{eff}},\beta'$ (where m_R' *does* correspond to the particle mass). It is easy to see that $\alpha'_{\text{eff}}>-\frac{1}{2}$ whenever $\alpha<-\frac{1}{2}$, as follows. Equation (3.9) corresponds graphically to the intersection of an exponential with a parabola. When the slope of the parabola exceeds that of the exponential at the origin (as is the case when $\alpha<-\frac{1}{2}$), the reverse must be true at the other two roots. Therefore,

$$e^{z_0}>-2(\alpha-\beta z_0),\ \text{ i.e., } \alpha'_{\text{eff}}>-\tfrac{1}{2}\ . \tag{3.13}$$

Thus, any theory with an α parameter less than $-\frac{1}{2}$ is simply a duplicate of an equivalent theory with an α parameter greater than $-\frac{1}{2}$. No generality is lost by restricting the α parameter to the region $\alpha>-\frac{1}{2}$.

However, there is more to the story, since even if $z=0$ is a local minimum of $\mathscr{V}_G$, it may not give the global minimum. (In 1 + 1 dimensions the $\overline{\Omega}$ equation can have three roots, two of which correspond to local minima of $\mathscr{V}_G$.) Hence, a transformation of the form (3.10) and

(3.12) will be necessary in this case also. This consideration effectively restricts the α parameter to the region to the right of the solid curve in Fig. 1. The boundary curve corresponds to the case where the two local minima of $\mathcal{V}_G(0,\Omega)$ one at $z_0=0$, one at $z_0\neq 0$, are exactly degenerate. The condition for this, obtained using (3.9) in (2.23) for $F=0$, is

$$1+2\alpha-(\alpha+\beta)z_0+\tfrac{1}{3}\beta z_0^2=0 . \tag{3.14}$$

From this equation, together with (3.9) for z_0, one can obtain a parametric form of the boundary curve:

$$\begin{aligned} \alpha &= [2z_0+3+(z_0-3)e^{z_0}]/z_0^2 , \\ \beta &= 3[z_0+2+(z_0-2)e^{z_0}]/z_0^3 , \end{aligned} \tag{3.15}$$

enabling one to plot it straightforwardly.

Insight into the equivalence between superficially different theories can be gained by a closer consideration of the relations between bare and renormalized parameters. Returning to (2.14) and (2.12), and eliminating $I_0(m_R)$ between them, yields the intriguing result

$$5\xi m_R^2-2\lambda_r^2=5\xi m_B^2-2\lambda_B^2 , \tag{3.16}$$

which implies that this peculiar combination of bare parameters is finite. In the notation of Eq. (2.21) this becomes

$$(\beta-\alpha^2)m_R^4=F_1(\text{bare}) , \tag{3.17}$$

where F_1(bare) is a function of the bare parameters only. Trivially, we have also that

$$\beta m_R^2=\frac{45}{8\pi^2}\xi=F_2(\text{bare}) . \tag{3.18}$$

Finally, at the expense of introducing an arbitrary mass scale μ, we can write (2.14) as

$$[\alpha+\beta\ln(m_R^2/\mu^2)]m_R^2=F_3(\text{bare},\mu), \quad \forall\mu . \tag{3.19}$$

In this way we have expressed the three renormalized parameters m_R^2,α,β in terms of the bare parameters in the Lagrangian with all renormalized parameters on the left-, and all bare parameters on the right-hand side.

Now, if z_0 is a root of (3.9), then the transformation

$$\begin{aligned} &\alpha\to\alpha'_{\text{eff}}=(\alpha-\beta z_0)e^{-z_0} , \\ &\beta\to\beta'=\beta e^{-z_0} , \\ &m_R^2\to m_R'^2=m_R^2e^{z_0} , \end{aligned} \tag{3.20}$$

leaves the left-hand sides of all three equations invariant. Hence, the primed and unprimed parameter sets correspond to one and the same Lagrangian. They ought to produce the same theory, and indeed they do. Since (3.9) has at most three real roots, the transformation (3.20) relates up to three versions of the same theory. Only for one of the three parametrizations will the m_R parameter correspond to the particle mass. We may avoid the less-convenient duplicate parametrizations simply by staying in the region prescribed by Fig. 1.

D. Avoiding duplications (2 + 1 dimensions)

A similar story occurs in the (2 + 1)-dimensional case. The condition for a root of the $\overline{\Omega}$ equation to be a *minimum* of V_G is now

$$(1+z)+\alpha+\beta(F-z)>0 . \tag{3.21}$$

Hence, m_R^2 will be the wrong root for $\overline{\Omega}\mid_{\phi_0=0}$ whenever $\alpha<-1$. If $\alpha<-1$, the real m_R^2 will be $m_R'^2=m_R^2(1+z_0)^2$, where z_0 is the nonzero root of the $\overline{\Omega}$ equation at the origin:

$$(1+z_0)^2=1-2\alpha z_0+\beta z_0^2 . \tag{3.22}$$

Rescaling the variables to express them in units of m_R', we have

$$\begin{aligned} &z'=\Omega/m_R'-1=(z-z_0)/(1+z_0) , \\ &F'=(m_R/m_R')F=F/(1+z_0) , \\ &\alpha'=(m_R/m_R')\alpha=\alpha/(1+z_0) , \\ &\beta'=\beta , \\ &\mathcal{V}_G'+\text{const}=(m_R/m_R')^3\mathcal{V}_G=\mathcal{V}_G/(1+z_0)^3 . \end{aligned} \tag{3.23}$$

Inserting these into the $\overline{\Omega}$ equation (2.25), and into Eq. (2.23) for $\mathcal{V}_G$, we find that those equations regain their original forms, now in terms of the primed variables, except that the effective α' parameter becomes

$$\begin{aligned} \alpha'_{\text{eff}}&=\alpha'-\beta' z_0/(1+z_0) \\ &=(\alpha-\beta z_0)/(1+z_0) . \end{aligned} \tag{3.24}$$

Again, by noting that whichever side of (3.22) has the greater slope at the $z_0=0$ root will have the lesser slope at the other root, one can see that $\alpha'_{\text{eff}}>-1$ whenever $\alpha<-1$.

Of course, in this case we can solve (3.22) explicitly for the nonzero root:

$$z_0=-2(1+\alpha)/(1-\beta) , \tag{3.25}$$

but this does not seem to provide any new insights, and we have proceeded in direct analogy with the (1 + 1)-dimensional case. Unlike that case, however, the $\overline{\Omega}$ equation has only two roots, one a maximum and one a minimum of $\mathcal{V}_G$, so it suffices to restrict α to $\alpha>-1$ to avoid duplications.

The (2 + 1)-dimensional analogues of (3.17)—(3.19) are

$$\begin{aligned} &(\beta-\alpha^2)m_R^2=F_1(\text{bare}) , \\ &\beta=F_2(\text{bare}) , \\ &(\alpha+\beta)m_R=F_3(\text{bare}) . \end{aligned} \tag{3.26}$$

If z_0 is a root of (3.22), or equivalently (3.25), then the transformation

$$\begin{aligned} &\alpha\to\alpha'_{\text{eff}}=(\alpha-\beta z_0)/(1+z_0) , \\ &\beta\to\beta'=\beta , \\ &m_R^2\to m_R'^2=m_R^2(1+z_0)^2 , \end{aligned} \tag{3.27}$$

leaves the left-hand sides invariant, implying that both sets of parameters describe the same theory.

Finally, we note that the restriction $\alpha > -1$, and the corresponding $\alpha > -\frac{1}{2}$ requirement in the (1 + 1)-dimensional case, both correspond to

$$1+6\lambda_r I_{-1}(m_R)>0 , \qquad (3.28)$$

which is just (3.7) for $\phi_0=0$. This has an interesting parallel with $(\phi^4)_{3+1}$ theory. It means that λ_r always lies to the right of the pole in the λ_R,λ_r relation (2.13).

IV. THE GEP FOR ϕ^6 THEORIES

A. 1 + 1 dimensions

Using the formulas (2.23)—(2.26) we have explored numerically the shape of the GEP for a wide variety of parameters. Some illustrative results are shown in Fig. 3. Basically, the GEP may either be U shaped, with a single minimum at the origin, or it may have two additional minima symmetrically placed on each side of the origin. Such behavior is familiar from the ϕ^4 case.[2] Clearly, there is a change in the qualitative behavior of the theory once the $\phi_0\neq 0$ minima become deeper than the $\phi_0=0$ minimum. We have investigated numerically where this transition occurs in terms of the α,β parameters, and the result is shown by the dashed curve in Fig. 1.

However, in the (1 + 1)-dimensional case, one should not immediately conclude that this represents a true phase transition to a spontaneous-symmetry-breaking (SSB) phase. There are theorems which forbid a *first-order* phase transition,[13] though a higher-order transition is a possibility.[14] The point is that in 1 + 1 dimensions, as in 0 + 1 dimensions, there is the possibility of mixing between the two degenerate vacua. This point is discussed in Secs. IV A and IV B of II.

B. 2 + 1 dimensions

Numerical calculations in the (2 + 1)-dimensional case are simplified by the fact that the $\overline{\Omega}$ equation (2.25) is quadratic and can be solved analytically. The relevant root [see (3.21)] is

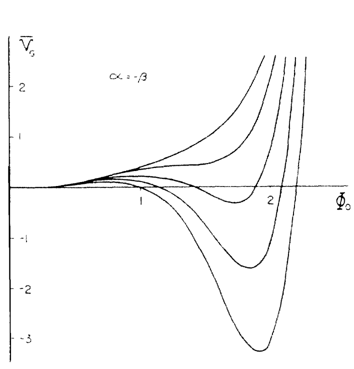

FIG. 3. Illustrative results for the GEP for $(\phi^6)_{1+1}$ theory, with $\alpha=-\beta$, and α varying from -0.1 to -0.5 in steps of -0.1.

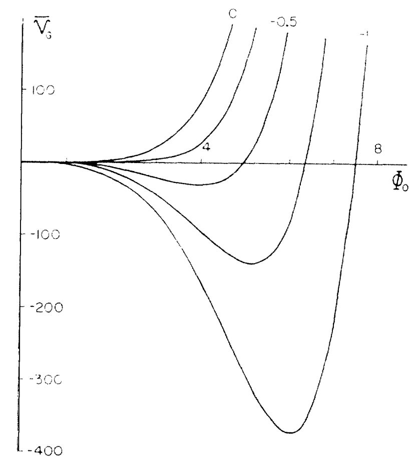

FIG. 4. Illustrative results for the GEP for $(\phi^6)_{2+1}$ theory, with $m_R{}^2=1$, $\beta=0.1$, and α varying from 0 to -1 in steps of -0.25. See also Fig. 5.

$$z=\{[(1+\alpha)^2+2(\alpha+\beta)F+\tfrac{1}{3}\beta(1+2\beta)F^2]^{1/2} -(1+\alpha+\beta F)\}/(1-\beta) . \qquad (4.1)$$

The GEP can then be evaluated directly from (2.23). Some illustrative results are shown in Fig. 4. Spontaneous symmetry breaking sets in when α is sufficiently negative (see the solid curve in Fig. 2).

A complication is that sometimes the minimum of $V_G(\phi_0,\Omega)$ is given, not by the $\overline{\Omega}$ equation, but by the $\Omega=0$ end point. The $\Omega=0$ (hence $z=-1$) result is

$$\mathscr{V}_G(\phi,\Omega=0)=\frac{1}{24\pi}[(2+3\alpha+\beta)+3(1+2\alpha+\beta)F +(\alpha+\beta)F^2+\tfrac{1}{15}\beta F^3] , \qquad (4.2)$$

and this gives the GEP in regions of ϕ_0 where the z from (4.1) is complex, or real but less than -1 (Ref. 15). This complication only arises if the ϕ^4 coupling parameter α is sufficiently negative [see Eq. (4.7) below, and the dotted curve in Fig. 2]. An illustration of what happens then is given in Fig. 5, which shows a detailed view of three of the curves in Fig. 4. We find that the $\Omega=0$ portion of the GEP never contains a minimum, so that there is never any vacuum (stable or metastable) governed by the $\Omega=0$ end point. In fact, the $\Omega=0$ part of the GEP curve is usually far from the true vacuum, and so is essentially irrelevant for the physics. The only exceptions arise very near to the SSB phase transition, which is anyway where the GEP is least reliable in detail.

Figure 5(c) also illustrates the fact that for sufficiently negative α the $\Omega=0$ end point governs the behavior at the origin. This happens when

$$\alpha < -\tfrac{1}{3}(2+\beta) \qquad (4.3)$$

because Eq. (4.2) is then negative at $F=0$, and so is lower than the $\overline{\Omega}$-equation result $\overline{\mathscr{V}}_G(\phi_0=0)=0$. The condition (4.3) appears as the dashed line in Fig. 2. For larger α's, the $\overline{\Omega}$ equation applies and one has $d^2\overline{V}_G/d\phi_0{}^2$

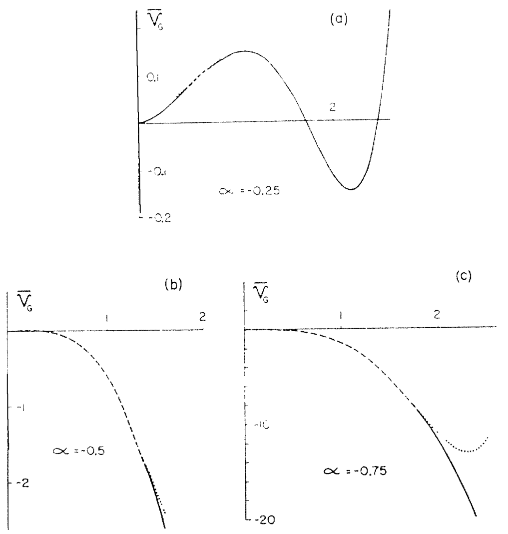

FIG. 5. As Fig. 4, showing details of the behavior near the origin for three cases: (a) $\alpha=-0.25$, (b) $\alpha=-0.5$, (c) $\alpha=-0.75$. The GEP is represented by a solid or a dashed line, depending on whether Ω is given by the $\overline{\Omega}$ equation or by $\Omega=0$. The continuation of the $\Omega=0$ result, when inoperative, is indicated by the dotted line. In cases (a) and (b), the origin is a local minimum; in case (c) it is a maximum.

$=\overline{\Omega}^2|_{\phi_0=0}\geq 0$, so that the origin is always a local minimum. However, once the $\Omega=0$ end point takes over, the origin becomes a maximum, since the coefficient of F in (4.2) is then negative. Thus, for $\alpha<-\frac{1}{3}(2+\beta)$ *there ceases to be even a metastable state with unbroken symmetry.*

Finally, we return to the question of when the $\Omega=0$ end point first becomes relevant. Basically, the $\Omega=0$ end point is not relevant provided the $\overline{\Omega}$ equation always has a solution with $\Omega>0$.[16] This requires $z>-1$ in (4.1), and hence

$$[(1+\alpha)^2+2(\alpha+\beta)F+\tfrac{1}{3}\beta(1+2\beta)F^2]^{1/2}>\alpha+\beta+\beta F\ . \tag{4.4}$$

Squaring each side, rearranging, and extracting a factor of $(1-\beta)$, this simplifies to

$$(1+2\alpha+\beta)+2(\alpha+\beta)F+\tfrac{1}{3}\beta F^2>0\ . \tag{4.5}$$

Clearly, the inequality always holds if $\alpha>-\beta$. If, however, $\alpha<-\beta$, then the left-hand side has a minimum at

$$F=-3(\alpha+\beta)/\beta\ \ (>0)\ . \tag{4.6}$$

If the inequality is to hold even here, we shall need

$$\alpha>-\tfrac{1}{3}\{2\beta+[\beta(3-2\beta)]^{1/2}\}\ , \tag{4.7}$$

and this condition is shown as the dotted curve in Fig. 2. For α's less than this there will be a region of F, around the value in (4.6), where the $\Omega=0$ result takes over. [Note that, for F given by (4.6), the right-hand side of (4.4) is positive, so that it was in fact legitimate to square each side of the inequality.]

C. The $m_R{}^2=0$ case

In the (2 + 1)-dimensional case one may have $m_R{}^2=0$ because ΔI_0 in (2.19) is infrared finite:

$$\Delta I_0=-\frac{m_R}{4\pi}\left[\frac{\Omega}{m_R}-1\right]\to\frac{-\Omega}{4\pi}\quad\text{as } m_R\to 0\ . \tag{4.8}$$

Equations (2.17) and (2.18) then become

$$V_G(\phi_0,\Omega)=\frac{1}{8\pi}\left[\alpha(\Omega^2-2\Omega F+\tfrac{1}{3}F^2)+\frac{\beta}{45}F^3+\frac{\Omega^3}{3}-\tfrac{1}{3}\beta\Omega(\Omega^2-3F\Omega+F^2)\right]\ , \tag{4.9}$$

with Ω given by

$$\Omega^2=-2\alpha(\Omega-F)+\beta(\Omega^2-2\Omega F+\tfrac{1}{3}F^2)\ . \tag{4.10}$$

The notation here follows (2.20) and (2.21), except that we have not, of course, divided through by powers of m_R. Thus, for this section only, α and F have the dimensions of mass.

There are three cases to consider: $\alpha<0$, $\alpha>0$, and $\alpha=0$. If α is negative then the correct root of the $\overline{\Omega}$ equation at the origin is not $\overline{\Omega}|_{\phi_0=0}=0$, but

$$\overline{\Omega}|_{\phi_0=0}=-2\alpha/(1-\beta)\ \ (>0)\ . \tag{4.11}$$

Hence, the true m_R is not zero. This case is identical to the case $\alpha=-\frac{1}{2}(1+\beta)$ with $m_R{}^2=1$, already included in the previous analysis.

If $\alpha>0$, we may choose units such that $\alpha=1$. We then obtain the results shown in Fig. 6. Note that all these theories exhibit SSB. In fact, the origin is unstable—so that there is not even a metastable state with massless particles—because the fourth derivative there is negative. We see this from (2.13), in which $I_{-1}(m_R)\to\infty$ for $m_R\to 0$, so that

$$\frac{1}{4!}\frac{d^4\overline{V}_G}{d\phi_0{}^4}\bigg|_{\phi_0=0}=-2\lambda_r=-\frac{4\pi}{3}\alpha<0\ . \tag{4.12}$$

The exceptional case is $\alpha=0$, which is a scale-invariant

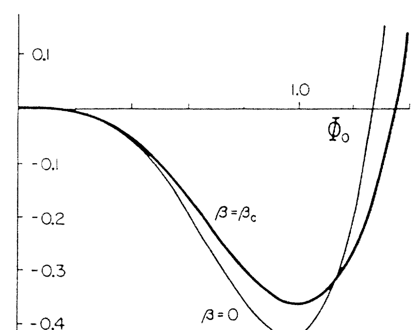

FIG. 6. The GEP for $(\phi^6)_{2+1}$ theory in the case $m_R{}^2=0$, $\alpha=1$. The two extreme values for β are shown; intermediate values show intermediate behavior. For $\beta>\beta_c$ an instability appears at very large ϕ_0.

theory, since it has no parameters with dimensions of mass. [The study of Bardeen, Moshe, and Bander[11] has previously discovered a scale-invariant phase of O(N)-symmetric ϕ^6 theory in the $N\rightarrow\infty$ limit.] The GEP thus has a pure $\phi_0{}^6$ behavior, and in fact is given exactly by Eq. (3.4), which we obtained previously as the large-ϕ_0 behavior in the general case.

D. The $m_R{}^2=-1$ case

A peculiar feature of the (2 + 1)-dimensional case is that it appears possible to choose $m_R{}^2$ to be negative, i.e.,

$$m_R=\pm i\mid m_R\mid . \tag{4.13}$$

Normally, this is impossible for two reasons: (i) $m_R{}^2$ is the solution to the $\overline{\Omega}$ equation at $\phi_0=0$, and Ω must be positive for the Gaussian wave functional to be normalizable, and (ii) a negative $m_R{}^2$ gives rise to imaginary parts in the I_N integrals, implying that the bare parameters are complex, so that the Hamiltonian is not Hermitian. Here, the first objection is nullified because the GEP at the origin will turn out to be governed by the $\Omega=0$ end point, not by the $\overline{\Omega}$ equation. The second objection is circumvented by arranging that the imaginary parts cancel out. Since

$$\mathrm{Im}[I_0(m_R)]=\mp\mid m_R\mid/(4\pi) , \tag{4.14}$$

we see from Eqs. (2.12) and (2.14) that the bare parameters will be real if

$$\lambda_r=-\frac{15\xi}{4\pi}m_R . \tag{4.15}$$

That is, λ_r must also be pure imaginary. Defining

$$\alpha=\frac{3}{2\pi}\frac{\lambda_r}{m_R},\quad \beta=\frac{45\xi}{8\pi^2} , \tag{4.16}$$

as usual, we see that the necessary condition is

$$\alpha=-\beta . \tag{4.17}$$

One may also derive this by considering Eq. (3.26). (The latter relations also imply that the $m_R{}^2=-1$ case is not equivalent to any of the $m_R{}^2=1$, or $m_R{}^2=0$ cases discussed earlier.)

Returning to the GEP in Eqs. (2.17) and (2.18), we obtain, for $m_R{}^2=-1$, $\alpha=-\beta$,

$$\mathscr{V}_G=\frac{1}{24\pi}[(1-\beta)(x\sqrt{x}+3\sqrt{x}-3F)+\beta F(3x-F\sqrt{x}+\tfrac{1}{15}F^2)] , \tag{4.18}$$

with the $\overline{\Omega}$ equation becoming

$$\sqrt{x}=\{[-(1-\beta)^2+\tfrac{1}{3}\beta(1+2\beta)F^2]^{1/2}-\beta F\}/(1-\beta) , \tag{4.19}$$

where

$$\mathscr{V}_G=(V_G-D)/\mid m_R\mid^3,\quad x=\Omega^2/\mid m_R\mid^2 \tag{4.20}$$

in this subsection. (The expected cancellation of imaginary parts in V_G was explicitly checked.)

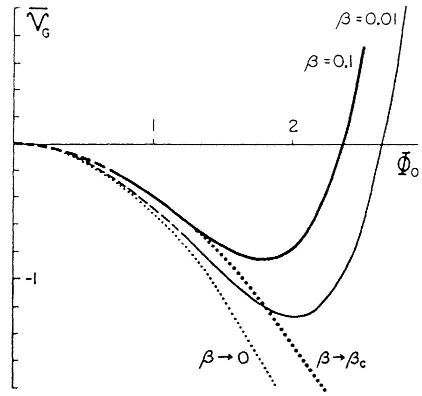

FIG. 7. The GEP for $(\phi^6)_{2+1}$ theory in the case $m_R{}^2=-1$, $\alpha=-\beta$. Two examples, $\beta=0.01$ and 0.1 are shown. The dashed portion of the curves indicates where the $\Omega=0$ end point is operative. The dotted curves correspond to the limiting behavior as $\beta\rightarrow 0$ or $\beta\rightarrow\beta_c$.

Clearly, there is no real solution for x at the origin, and in fact an acceptable solution does not appear until $F^2>3(1-\beta)/\beta$. Thus, the $\Omega=0$ end point will govern the behavior near the origin. If β is very small, the $\Omega=0$ end point dominates almost everywhere, giving $\mathscr{V}_G(\Phi_0)\simeq-\frac{1}{2}\Phi_0{}^2$ until the $\Phi_0{}^6$ behavior finally sets in at very large Φ_0. The large-Φ_0 behavior is, in fact, the same as always. Thus, the restriction $\beta\leq\beta_c$ still applies. When $\beta=\beta_c$, one can show that, except near the origin, $\mathscr{V}_G(\Phi_0)\simeq-\frac{3}{8}\Phi_0{}^2$. Thus, for β near either end of its range $0<\beta<\beta_c$, the GEP has a deep minimum, very far from the origin. This SSB minimum is shallower, and much nearer the origin, in the intermediate cases. See Fig. 7.

V. BOUND STATES

We first consider the one-particle state built on our trial vacuum

$$\mid \mathbf{p}\rangle_{\Omega,\phi_0}=a^\dagger{}_\Omega(\mathbf{p})\mid 0\rangle_{\Omega,\phi_0} . \tag{5.1}$$

Its energy may be computed straightforwardly as

$$E_1(\phi_0,\mathbf{p})={}_{\phi_0,\Omega}\langle\mathbf{p}\mid H\mid\mathbf{p}\rangle_{\Omega,\phi_0}/{}_{\phi_0,\Omega}\langle\mathbf{p}\mid\mathbf{p}\rangle_{\Omega,\phi_0} , \tag{5.2}$$

which gives

$$E_1(\phi_0,\mathbf{p})=\omega_p(\Omega)+[m_B{}^2-\Omega^2+12\lambda_B(I_0+\phi_0{}^2)+30\xi(\phi_0{}^4+6I_0\phi_0{}^2+3I_0{}^2)]/(2\omega_p)+{}_{\phi_0,\Omega}\langle 0\mid H\mid 0\rangle_{\Omega,\phi_0} \tag{5.3}$$

(see the Appendix, and also Refs. 2–5). The second term vanishes by virtue of the $\overline{\Omega}$ equation (2.8). Subtracting off the vacuum energy, we see that the extra energy due to the presence of a particle is just $\omega_p(\Omega)=(\mathbf{p}^2+\Omega^2)^{1/2}$. Incidentally, it is clear from the formulas in the Appendix that this result will generalize to any polynomial potential. Thus, for any ϕ_0 which is a minimum of the GEP, we may identify the corresponding $\bar{\Omega}$ as being the particle mass in that vacuum.

Next we examine the possibility of having a two-particle bound state, using the ansatz[3-5]

$$|2\rangle_{\Omega,\phi_0} = \int (dp)_\Omega \sigma(\mathbf{p}) a_\Omega^\dagger(\mathbf{p}) a_\Omega^\dagger(-\mathbf{p}) |0\rangle_{\Omega,\phi_0} , \quad (5.4)$$

which describes an s-wave state, with $\sigma(\mathbf{p})$ being the Fourier transform of the spatial wave function. For convenience we work in the overall center-of-mass frame, so that the energy of this state, minus the vacuum energy, gives the bound-state mass M_2,

$$M_2 = \frac{\int (dp) \sigma^2(\mathbf{p}) 4\omega_p{}^2 + 12\lambda_{\text{eff}}(\phi_0) \left[\int (dp) \sigma(\mathbf{p}) \right]^2}{\int (dp) 2\omega_p \sigma^2(\mathbf{p})} . \quad (5.5)$$

The form of the result is exactly as in the ϕ^4 case,[2-5] except that λ_B is replaced by

$$\lambda_{\text{eff}}(\phi_0) = \lambda_B + 15\xi[I_0(\Omega) + \phi_0{}^2] , \quad (5.6)$$

a finite combination we have met before in (2.10) and (3.7).

The optimum form of the function $\sigma(\mathbf{k})$ is obtained by functionally minimizing Eq. (5.5). This leads to the integral equation

$$\sigma(\mathbf{k})\omega_k(2\omega_k - M_2) + 6\lambda_{\text{eff}} \int (dp) \sigma(\mathbf{p}) = 0 . \quad (5.7)$$

Since the last term is $\mathbf{k}$ independent, $\sigma(\mathbf{k})$ must have the form

$$\sigma(\mathbf{k}) = \frac{A}{\omega_k(2\omega_k - M_2)} , \quad (5.8)$$

where A is some normalization constant. Inserted back into (5.7) this yields

$$1 + 6\lambda_{\text{eff}} \int (dp) \frac{1}{\omega_p(2\omega_p - M_2)} = 0 , \quad (5.9)$$

which clearly allows bound-state solutions $[M < 2\Omega < 2\omega_p(\Omega)]$ only if λ_{eff} is negative.

We now show that, in the GEP approximation, *there are no bound states in any $\phi_0 \neq 0$ vacuum.* From (2.9), any minimum of $\bar{V}_G(\phi_0)$, except $\phi_0 = 0$, will satisfy

$$m_B{}^2 + 4\lambda_B(\phi_0{}^2 + 3I_0) + 6\xi(\phi_0{}^4 + 10I_0\phi_0{}^2 + 15I_0{}^2) = 0 . \quad (5.10)$$

Using the $\bar{\Omega}$ equation (2.8) to eliminate $m_B{}^2$, this means that

$$\Omega^2 = 8\lambda_B\phi_0{}^2 + 24\xi\phi_0{}^2(5I_0 + \phi_0{}^2) , \quad (5.11)$$

so that λ_{eff} at a nontrivial minimum can be expressed as

$$\lambda_{\text{eff}}(\phi_0^{\min} \neq 0) = \frac{\Omega^2}{8\phi_0{}^2} + 12\xi\phi_0{}^2 , \quad (5.12)$$

which is positive definite.

In the $\phi_0 = 0$ vacuum, however, λ_{eff} becomes λ_r, which may have either sign. As the single-particle mass in this vacuum is $\bar{\Omega}|_{\phi_0=0} = m_R$, we define

$$\eta = M_2/(2m_R) , \quad (5.13)$$

so that $0 < \eta < 1$ corresponds to a bound state. Using the dimensionless variables of (2.20) and (2.21) we may rewrite (5.9), in the (1 + 1)-dimensional case, as

$$1 + \alpha \int_0^\infty dx \frac{1}{(x^2+1)[(x^2+1)^{1/2} - \eta]} = 0 . \quad (5.14)$$

Performing the integration one obtains

$$-\alpha = \frac{\eta\sqrt{1-\eta^2}}{\sin^{-1}\eta + \frac{\pi}{2}[1 - (1-\eta^2)^{1/2}]} , \quad \eta < 1, \ \alpha < 0 \quad (5.15)$$

which, when inverted, gives the bound-state mass as a function of the coupling parameter α. (Note that β does not appear: the effect of the ϕ^6 coupling here is solely to renormalize the effective ϕ^4 interaction.) A weak-coupling expansion of (5.15) yields

$$\eta = 1 - \frac{\pi^2}{2}\alpha^2 + \cdots = 1 - \frac{9}{8}\left[\frac{\lambda_r}{m_R{}^2}\right]^2 + \cdots , \quad (5.16)$$

which agrees with the perturbative calculation of Dimock and Eckmann.[8] See also Refs. 7 and 9. The full formula is plotted in Fig. 8(a).

Since η decreases monotonically as $(-\alpha)$ increases, one might worry that, for sufficiently large, negative coupling, the bound-state mass could become zero or even negative. This is not so, because, as we explained in Sec. III, α is effectively bounded below by $-\frac{1}{2}$. Thus, in fact, η can never be less than $\simeq 0.60$. Moreover, if we require the $\phi_0 = 0$ vacuum to be stable, then we must stay within the dashed curve in Fig. 1, limiting $(-\alpha)$ to at most 0.27, and so η is at least 0.83. Thus, the bound state is never ultratightly bound, and can reasonably be described as two quasifree particles plus some binding energy. Hence, the simple ansatz (5.4), which ignores higher Fock states, is quite self-consistent.

The corresponding formula in 2 + 1 dimensions is

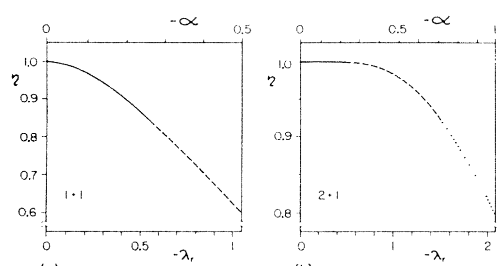

FIG. 8. The bound-state mass $\eta \equiv M_2/(2m_R)$ as a function of the coupling α in (a) 1 + 1 dimensions, (b) 2 + 1 dimensions. See Eqs. (5.15) and (5.18), respectively. The curves are shown as dashed lines if the $\phi_0 = 0$ vacuum is necessarily metastable. If no $\phi_0 = 0$ vacuum can exist (region C of Fig. 2) the curve is shown only by a dotted line.

$$1+\tfrac{1}{2}\alpha\int_0^\infty dx\frac{x}{(x^2+1)[(x^2+1)^{1/2}-\eta]}=0\ , \tag{5.17}$$

and hence

$$\alpha=2\eta/\ln(1-\eta),\ \ \eta<1,\ \ \alpha<0\ . \tag{5.18}$$

This formula is plotted in Fig. 8(b). Note that the binding is a purely nonperturbative effect, being exponentially small in the coupling constant α:

$$(1-\eta)=\exp\left[-\frac{2}{|\alpha|}\right]\left[1+\frac{2}{|\alpha|}\exp\left[-\frac{2}{|\alpha|}\right]+\cdots\right]. \tag{5.19}$$

In this case α is bounded below by -1, as we showed in Sec. III, so that $\eta\gtrsim 0.80$. Also, we found in Sec. IV B that, because the $\Omega=0$ end point prevails, the origin is not a local minimum if $\alpha<-\frac{1}{3}(2+\beta)$. Thus, for any $\alpha<-\frac{1}{3}(2+\beta_c)\simeq-0.7$, there is no $\phi_0=0$ vacuum on which to build a two-particle state. This implies that η's less than $\simeq 0.92$ are never realized, physically. Finally, if we want the $\phi_0=0$ vacuum to be globally stable, we must stay to the right of the solid curve in Fig. 2. This requires $\alpha\gtrsim-0.25$, even in the most favorable case, and implies a binding energy fraction $(1-\eta)$ no bigger than 3.4×10^{-4}. Thus, in 2 + 1 dimensions we are dealing with a very weakly bound system.

The bound-state formulas given above are in agreement with those found in Ref. 17, which we received as this work was being completed. That reference deals with the $:\lambda(\phi^6-\phi^4):$ model, also studied in Refs. 7–9. This model corresponds to a one-parameter subset of theories selected rather arbitrarily from the full two-parameter set of possibilities.[9] We have slight objections to this model on aesthetic grounds: The "normal ordering" device seems rather ugly and artificial; it harks back to perturbation theory, which we want at all costs to avoid. In our view it is much more satisfactory to treat ϕ^6 theories in their full generality, as we have done here.

VI. SUMMARY AND CONCLUSIONS

The validity of most of our results depends, of course, on the reliability of the GEP approximation. However, the statement that the $(\phi^6)_{2+1}$ theory is unstable for $\xi>0.255$ is definitive, because the GEP provides an upper bound on the effective potential. For positive ξ's below this value it appears from our approximation that the theory is stable and well behaved. (Note that our ξ is a fixed, bare parameter and does not "run.") Consequently we are reluctant to believe—though we cannot disprove the possibility—that *all* $(\phi^6)_{2+1}$ theories are unstable.[11,12]

The varieties of ϕ^6 theories are labeled by three continuous parameters: $m_R{}^2,\alpha,\beta$. However, one has the freedom to choose mass units to suit one's convenience. In 1 + 1 dimensions since $m_R{}^2$ is necessarily positive, one may take $m_R{}^2=1$: α and β are then restricted to the region shown in Fig. 1. The (2 + 1)-dimensional case is more complicated. One may have $m_R{}^2=1$, with α and β restricted to the region in Fig. 2; i.e., $-1<\alpha<\infty$, $0<\beta\leq\beta_c$ (with $\beta=\beta_c$ allowed only if $\alpha>-\beta$). However, one may also have $m_R{}^2=0$, with $\alpha=1$ (by choice of units), and $0<\beta\leq\beta_c$. Also possible is $m_R{}^2=-1$, $\alpha=-\beta$, with $0<\beta<\beta_c$. Both of these possibilities always exhibit SSB, and do not seem in any way pathological: the origin does not correspond to even a metastable vacuum, so that no massless or tachyonic particles arise. Finally, there is the unique possibility that the theory is scale invariant: $m_R{}^2=0$, $\alpha=0$, $0<\beta<\beta_c$.

According to the GEP approximation bound states can occur in the $\phi_0=0$ vacuum, for certain ranges of parameters: one needs $m_R{}^2=1$; $\alpha<0$; and for the origin to be at least a local minimum. An SSB vacuum never has bound states. We conjecture that this is related to the steepness of the potential at these minima. In quantum mechanics one has a "bound state" $(E_2-E_1<E_1-E_0)$ only when the potential well rises less steeply than a parabola. Something similar seems to hold, at least qualitatively, in the field theory case.

A variational calculation of excited states is, of course, not rigorously justifiable. Nevertheless, we believe our bound-state results are worth taking seriously, for three reasons. First, the equivalent calculation for excited states of the anharmonic oscillator, and other (0 + 1)-dimensional models, yields very satisfactory results.[1] Second, our (1 + 1)-dimensional results agree with the perturbative calculation[8] and with a recent lattice calculation.[9] Third, our results seem very reasonable and self-consistent: potential embarrassments, such as the bound states becoming ultrarelativistic or even tachyonic (which would indicate an instability of the vacuum) never in fact arise.

Perhaps the most encouraging aspect of our work is that with a simple approximation, based on free-field theory, we are nonetheless able to find bound states, as in $(\phi^6)_{2+1}$ theory, where the binding is a purely nonperturbative effect.

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APPENDIX: CALCULATION OF MATRIX ELEMENTS

The GEP method involves calculating the matrix elements of $\hat{\phi}^{2N}$, where $\hat{\phi}=\phi-\phi_0$ [see Eq. (2.2)]. The necessary formulas are derived in this appendix.

The main difficulty lies in the combinatoric factors that arise. These are easiest to calculate in the (0 + 1)-dimensional case, where the formalism is simplest. Once the combinatoric factors have been identified it is very easy to determine which field-theoretic factors go with them.

Essentially we require the normal-ordered expression for $(a+a^\dagger)^n$, where $[a,a^\dagger]=1$. This is provided by Glimm and Jaffe,[18] who give

$$Q_n \equiv 2^{-n/2}(a^\dagger+a)^n = \sum_{j=0}^{[n/2]} c_{n,j} 2^{-j} :Q^{n-2j}: , \tag{A1}$$

where

$$c_{n,j}=\frac{n!}{(n-2j)!j!2^j} \tag{A2}$$

are the Hermite polynomial coefficients, and

$$:Q^n: = 2^{-n/2}\sum_{i=0}^{n}\binom{n}{i} a^{\dagger i}a^{n-i} \tag{A3}$$

by the binomial theorem.

Since we are interested in diagonal matrix elements, we need only those terms in $(a^\dagger+a)^n$ with equal numbers of a's and $a^\dagger$'s. This is zero for n odd. For n even, $=2N$, we obtain

$$(a^\dagger+a)^{2N}=\frac{(2N)!}{2^N}\sum_{i=0}^{N}\frac{2^i}{(N-i)!i!i!}a^{\dagger i}a^i + (\text{terms } a^{\dagger j}a^i \text{ with } j\neq i) . \tag{A4}$$

For vacuum matrix elements, only the $i=0$ term will contribute. Reinstating the full formalism, in which

$$\hat{\phi}=\int (dk)_\Omega [a_\Omega^\dagger(\mathbf{k})e^{-ik\cdot x}+a_\Omega(\mathbf{k})e^{ik\cdot x}] , \tag{A5}$$

we see that

$${}_\Omega\langle 0|\hat{\phi}^{2N}|0\rangle_\Omega=\frac{(2N)!}{2^N N!}[I_0(\Omega)]^N . \tag{A6}$$

For a one-particle excited state

$$|1\rangle_\Omega \equiv a_\Omega^\dagger(\mathbf{p})|0\rangle_\Omega , \tag{A7}$$

the $i=0,1$ terms in Eq. (A4) both contribute, and we obtain

$${}_\Omega\langle 1|\hat{\phi}^{2N}|1\rangle_\Omega=\frac{(2N)!}{2^N N!}({}_\Omega\langle 1|1\rangle_\Omega I_0^N+2NI_0^{N-1}) . \tag{A8}$$

For a two-particle state

$$|2\rangle_\Omega=\int (dp)_\Omega \sigma(\mathbf{p}) a_\Omega{}^\dagger(\mathbf{P}+\mathbf{p})a^\dagger(\mathbf{P}-\mathbf{p})|0\rangle_\Omega , \tag{A9}$$

the $i=0,1,2$ terms of (A4) contribute, giving

$${}_\Omega\langle 2|\hat{\phi}^{2N}|2\rangle_\Omega=\frac{(2N)!}{2^N N!}\left[{}_\Omega\langle 2|2\rangle_\Omega I_0{}^N+8NI_0{}^{N-1}\int (dp)_\Omega \sigma^2(\mathbf{p})+4N(N-1)I_0{}^{N-2}\left[\int (dp)_\Omega\sigma(\mathbf{p})\right]^2\right] . \tag{A10}$$

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*On leave of absence from Centro Brasileiro de Pesquisas Físicas (CBPF), Rio de Janeiro, 22290, Brazil.

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