Unsharp reality and joint measurements for spin observables

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The Einstein-Podolsky-Rosen (EPR) reality criterion is generalized to fit with the notion of positive-operator-valued observables occurring in quantum optics, stochastic quantum mechanics, and other fields of quantum physics. The resulting concept of unsharp reality for quantum systems is illustrated within stochastic spin space where it leads to a notion of unsharp spin property. Finally we investigate the possibility of joint spin measurements and give a brief discussion of the EPR-Bell argument for unsharp spin properties.

I. INTRODUCTION

In recent years a generalized notion of quantum-mechanical observables has been developed and found successful applications in various fields of quantum physics such as quantum stochastic processes, quantum optics, (non)relativistic quantum theory, or even the interpretation of quantum mechanics. Whereas the usual quantum theory describes observables as self-adjoint operators or, equivalently, in terms of their [projection valued (PV)] spectral measures, in the new, extended frame certain positive operators, the effects (self-adjoint with spectrum within the interval [0,1]), are included to build positive-operator-valued (POV) measures as generalized observables.

POV observables occur in a natural way in the abovementioned topics: as devices for the description of sequential measurements on open quantum systems[1] they allow implementation of information theory into quantum physics;[2,3] as nonorthogonal resolutions of the identity they are obtained in quantum optics[4] as well as in (non)relativistic quantum mechanics for phase-space representations of the canonical commutation relations[5] yielding, for instance, relativistically covariant probability currents; furthermore, the problem of photon localization has been solved by means of covariant POV measures.[6,7]

There is also a number of conceptual and interpretational problems which could only be tackled within the extended frame. In particular, phase-space observables give rise to proper probabilities for joint measurements of position and momentum. Such measurements yield values $(q,\delta q)$, $(p,\delta p)$ with unsharpnesses obeying the uncertainty relations $\delta q\,\delta p \geq \hbar/2$ (Ref. 8). In this way the original Heisenberg interpretation of the uncertainty relation could be justified within a systematic theory of measurement and the significance of these relations to joint measurements could be completely clarified.[9,10]

We see that *unsharp* values are *necessary* (and sufficient) for the possibility of joint measurements. Our final example shows that quantum measurements of observables possessing *sharp* values may be generally impossible. This fact goes back to a discovery by Wigner[11] who proved that predictable measurements of a quantity A are impossible whenever A does not commute with a conserved quantity. Since any measurement is reducible to some localization, and since sharp (predictable) position measurements are forbidden by momentum conservation[12] we must conclude that no observable can be determined sharply. Thus, in order to maintain consistency between theoretical concepts (here, observable) and experimental possibilities one has to give up the PV observables in favor of POV observables.

In the preceding examples it has become clear that POV observables have to do with unsharply defined measurement values or value sets; for that reason they have been called approximate,[1] fuzzy,[5] stochastic,[3] or unsharp[8] observables. In the present paper I shall try to show that one can make precise sense of this way of speaking. One aim of the so-called quantum logic approach is to establish the lattice $\mathscr{P}(\mathscr{H})$ of Hilbert-space projections as a language for quantum systems and their (sharp) properties.[13,14] Similarly one can start within the convexity approach (see Ref. 9 for further references) to show that the partially ordered set $\mathscr{E}(\mathscr{H})$ of Hilbert-space effects contains elements representing *unsharp* properties of quantum systems. Projections $\mathscr{P}(\mathscr{H})$ are characterized through measurement-theoretic notions such as predictability, repeatability, preparatory (ideal first kind) measurements which, as we shall see in the next section, can be relaxed to select the unsharp properties from $\mathscr{E}(\mathscr{H})$. To this end I propose a slight modification of the famous Einstein-Podolsky-Rosen[15] reality criterion into a definition of "elements of unsharp reality."

The present work is a direct continuation of the investigations performed in Ref. 9; the general concepts are further elaborated and applied to obtain results on spin observables similar to the results of Ref. 9 on position and momentum. In particular, I shall discuss measurements of unsharp spin properties in Sec. III. In Sec. IV criteria for the coexistence (i.e., existence of joint probability distributions) of different spin components are derived which show that noncommuting unsharp spin properties may well possess simultaneous (unsharp) reality. Working within stochastic spin space[16] leads to a thorough generalization and a simple geometric interpretation of previous results by Margenau and Hill[17] and by Prugovecki.[18] Finally (Sec. V) a brief discussion of the Einstein-Podolsky-Rosen- (EPR) Bell argument is given for the case of unsharp spin properties which shows the "stability" of that argument against the introduction of reasonably small unsharpness.

II. UNSHARP REALITY OF QUANTUM SYSTEMS

A. Motivation

In view of its mathematical (probabilistic) structure quantum theory is widely seen merely as statistical theory. Nevertheless physicists are thinking in terms of individual elementary particles possessing definite properties. Our aim is to show that this is more than a convenient way of speaking: quantum theory allows formulation and tests of statements referring to *individual systems*—just in the way that one has learned in classical physics or even in every day life to constitute systems (objects) from empirical data. [In the following I prefer to use the term "(physical) system" rather than the more specific "object" or "particle."[14]] The fact that "an electron" is not a classical particle does not imply that quantum theory does not refer to individuals. On the contrary, the well-known difficulties in ascribing an unobservable ("hidden-variables") deterministic path reality to an electron in the double-slit experiment seems to call for a realistic use of quantum theory: the electron should be regarded as a (nonclassical) system which is localized at most to the extent defined by the union of both slits. Accordingly, the models of joint measurements of quantum position and momentum of Ref. 8 were devised to justify the uncertainty relations as statements about *individual* systems (cf. Sec. I). The first ideas of a *general* reconstruction of quantum theory as a language for *individual* systems were presented in a conference recently.[19] Now I try to give a systematic formulation of the relevant notions within the statistical framework of the convexity approach. One may hope that this procedure will shed light on the operational background of the Hilbert space structure of quantum mechanics.

B. Short sketch of the convexity frame

In the convexity approach an experiment is divided into several stages, the preparation (of "input data") and the measurement (registration of "output data"). In short (for details see Refs. 9 and 20), the preparations are represented by the set of *states* V^+ which is a norm-closed-generating ($V=V^+-V^+$) cone for a complete base norm space (V,B). V is an ordered real Banach space, B a base: $V^+=\cup(\lambda B:\lambda>0)$ defining a strictly positive linear functional $e:V\to\mathbb{R}$ such that $e(\alpha)=1$ for α in B. The partial order $\le$ on V is connected with the positive cone V^+ via $\alpha\le\beta$ iff $\beta-\alpha\in V^+$. The extreme elements $\mathrm{Ex}(B)$ of B (convex) are the *pure* states. The states are intended to represent the experimental situation after some preparation procedure. Any further preparational manipulation and, in particular, measurements are described as certain state transformations: an *operation* $\phi:V\to V,\alpha\mapsto\phi\alpha$ is a positive, norm-nonincreasing linear map. A measurement operation ϕ leads to an *effect* $e\circ\phi$ which is detected in the final stage of the experiment. The set $O(V)$ of operations exhausts the whole set $E(V)$ of effects, i.e., all elements a of the dual space (V^*,e) of (V,B) satisfying $0\le a\le e$. The ordering $a\le b$ iff $(b-a)(\alpha)\ge 0$ for all $\alpha\in V^+$ makes $E(V)$ a positive bounded by $0,e$. On $E(V)$ a complement operation is defined by $a\mapsto a':=e-a$ [obeying $(a')'=a, a\le b\Longrightarrow b'\le a'$]. Finally we note the concepts of instrument and observable. An *instrument* (Σ,I) is an operation-valued measure $I:\Sigma\to 0(V)$ on a Boolean ring Σ; it defines an *observable* $(\Sigma,A),A:\Sigma\to E(V)$ [via $e(I(X)(\alpha))=A(X)(\alpha)$ for all α in V,X in Σ] as an effect-valued measure.

In Hilbert-space quantum theory a realization of this abstract scheme is given in the following way. The states are the positive trace class operators $(V,B)=(\mathcal{T}_s(\mathcal{H}),\mathcal{T}_s(\mathcal{H})_1^+)$. The effects are represented via the trace functional by positive bounded self-adjoint operators

$$E(V)=\mathcal{E}(\mathcal{H})$$
$$=\{E\in L_s(\mathcal{H}):0\le E\le I\}\ :$$
$$e(\alpha)=\mathrm{tr}(\alpha),\quad E(\alpha)=\mathrm{tr}(\alpha\cdot E)\ .$$

[Note: $\mathcal{T}_s(\mathcal{H})^*=L_s(\mathcal{H})$.] In this way we arrive at the *extended* Hilbert-space frame of quantum theory mentioned in the Introduction and needed for incorporation of the POV measures. In the *standard* Hilbert-space frame the measurements are restricted to those bearing extreme effects $\mathrm{Ex}(E(V))=\mathcal{P}(\mathcal{H})$, i.e., projections.

C. Some measurement-theoretic notions

A priori a measurement result may refer either to the past (the preparation) or to both past and future. In the first case the measurement is only *determinative*, its statistics reflects certain features of the preparation. In the second case we have a *preparatory* measurement. Indeed, the map $\Psi:0(V)\to E(V)$, $\phi\mapsto\Psi(\phi)$, $\Psi(\phi)(\alpha)=e\circ\phi(\alpha)$, $\alpha\in B$, is surjective but not injective so that the set $\Psi^{-1}(a)$, $a\in E(V)$ will contain elements which may or may not be "preparatory" with respect to the effect a. Let us start with some definitions.

Definition 2.1. (1) A *measurement* of an effect $a\in E(V)$ is any operation ϕ in $\Psi^{-1}(a):=\{\phi\in O(V):e\circ\phi=a\}=\mathcal{M}(a)$. (2) A (joint) test of a number of effects $a_1,\dots,a_n$ is any element ϕ of $\cup[\mathcal{M}(c):c\le a_i,i=1,\dots,n]=\mathcal{T}(a_1,\dots,a_n)=\cap_i\mathcal{T}(a_i)$. Joint tests provide evidence for the effects a_i if the outcomes of the c measurements are positive. This notion is a relaxation of the concept of coexistence.

Definition 2.2. (1) A set of effects $a_1,\dots,a_n$ is *coexistent* if it is contained in the range of some observable A. (2) Observables $A_1,\dots,A_n$ are called *coexistent* if their ranges are contained in the range of some observables A, $\cup_i R(A_i)\subseteq R(A)$. Such an A is called a *joint observable* for the a_i,A_i. A characterization of the determinative power of a measurement is given by the following.

Definition 2.3. An observable (Σ,A) is *informationally complete* if $A(X)(\alpha_1)=A(X)(\alpha_2)$ for all $X\in\Sigma$ implies $\alpha_1=\alpha_2(\alpha_i\in B)$.

The next definition will be most important for our subsequent considerations.

Definition (2.4). Let ϵ be a real number, $0\le\epsilon<1$, $\phi\in O(V)$, $a\in E(V)$. (1) ϕ is ϵ-predictable (ϵ-pred) if there exists a state $\alpha \in B$ such that $e(\phi\alpha) \geq 1-\epsilon$; the corresponding effect $e \circ \phi$ is called *ϵ-actualizable* and is ϵ-actual in α. (2) ϕ is *ϵ-repeatable* (ϵ-rep) iff ϕ is ϵ-pred and $e(\phi\alpha) \geq 1-\epsilon$ implies $e(\phi^2\alpha) \geq (1-\epsilon)e(\phi\alpha)$ for all $\alpha \in B$. ϕ is ϵ-rep with respect to a iff a is ϵ-actualizable and $a(\alpha) \geq 1-\epsilon$ implies $e(\phi\alpha) \neq 0$ and $a(\phi\alpha) \geq (1-\epsilon)e(\phi\alpha)$ for all $\alpha \in B$. Such effects are called *ϵ-reproducible.* (3) ϕ is *ϵ-preparatory* (ϵ-prep) iff $e(\phi^2\alpha) \geq (1-\epsilon)e(\phi\alpha)$ for all $\alpha \in B$. ϕ is ϵ-prep with respect to a iff $a(\phi\alpha) \geq (1-\epsilon)e(\phi\alpha)$. Such effects are called *ϵ-preparable.*

Among these notions the following relations hold: ϵ-prep $\Longrightarrow$ ϵ-rep $\Longrightarrow$ ϵ-pred. For $\epsilon=0$ we obtain the usual concepts[9] of predictability, repeatability, and preparatory measurements: ϕ is predictable iff $e(\phi\alpha)=1$ for some $\alpha \in B$; ϕ is repeatable iff it is predictable and $e(\phi\alpha)=1$ implies $e(\phi^2\alpha)=1$ for $\alpha \in B$; ϕ is preparatory iff $e(\phi^2\alpha)=e(\phi\alpha)$ for $\alpha \in B$. According to the introductory remarks predictable ($\epsilon=0$) measurements seem to be impossible. Therefore the case $\epsilon \neq 0$ will be employed as a straightforward relaxation. The number $r=1-\epsilon$ represents a positive lower bound for the probability of obtaining or reproducing certain measurement results: r will be interpreted as an estimate of the *reality content* of an effect; in actual experiments one will try to make ϵ as small as possible if $\epsilon=0$ is not realizable.

D. Quantum systems

Now we are ready to formulate necessary and sufficient conditions for inferring the presence of a physical system in a given experimental context. Let us start with a verbal description of the most general features characteristic of any "system." A system is a part of the "whole" of physical reality which, in spite of its interactions and connections with the rest of the world, is recognizable as a separate entity. Thus a system must be observable, that is, have a regular influence on its surroundings: it must produce observable *effects;* in this way systems give rise to the *event structure* of physical reality. Moreover, systems themselves are sequences of events. These events must be *causally connected,* at least in the sense of a probability law, in order to infer the presence of a "system." This first minimal requirement is already incorporated in the convexity framework: certain preparation and registration *events* are assumed to be connected by *probabilities* $a(\alpha)$. Further, in order to test these probabilities statistically one must be able to perform a large number of repetitions of the "same" experiment. This implies that the experiment must be invariant under the group T of space-time translations, i.e., $a^g(\alpha^g)=a(\alpha)$ where α^g and a^g denote the state and effect of the translated experiment. Since an objective system constitution has to be observer independent, the above invariance requirement must be extended to the full Galilei or Lorentz group. This shows that the usual procedure of characterizing a system type (i.e., the *essential properties)* as unitary representations of an invariance group is based on necessary preconditions of the objective constitution of individual systems. (In this argumentation I follow a similar exposition made in Ref. 14 where it is applied to derive certain important features of the abstract quantum language.) Once the *state space* (V,B) of a *system type* has been established along these lines one can proceed to formulate necessary and sufficient conditions for the possibility of interpreting an event sequence as being caused by an *individual,* permanently existing system. An individual system must be recognizable and identifiable, at least during a certain period of time, by means of some of its *accidental properties.*[13] In their famous attempt to criticize the completeness of quantum mechanics Einstein, Podolsky, and Rosen[15] propose the following as a *sufficient* reality criterion: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

Here the presence of the system is taken for granted. According to the above discussion the EPR criterion turns out to be even a *necessary* condition for the *presence* of a *real* system.

Definition 2.5. An effect $a \in E(V)$ is an *(accidental) unsharp (ϵ-) property* (potential or actual=ϵ-real) of some system of type (V,B) if it satisfies the following conditions for some ϵ, $0 \leq \epsilon < \frac{1}{2}$:$(P_\epsilon 1)$ both a and the complement $a'=e-a$ are ϵ-actualizable, $(P_\epsilon 2)$ a admits pure tests ϕ which leave (at least) all pure states $\alpha \in Ex(B)$ with $a(\alpha) \geq 1-\epsilon$ almost (ϵ) unchanged, i.e.,

$$a(\alpha) \geq 1-\epsilon \Longrightarrow a(\phi\alpha) \geq (1-\epsilon)e(\phi\alpha)$$

and

$$\forall b \in E(V): |\, b(\phi\alpha) - b(\alpha)e(\phi\alpha) \,| \leq \mu(\epsilon)$$

for some small $\mu(\epsilon) \geq 0$.

Now we can formulate the *criterion of unsharp reality* (R_ϵ): There exists a system $S=(\alpha,a)$ of type (V,B) possessing in state $\alpha=\alpha(S)$ the *ϵ-real (unsharp) property* $a=a(S)$ if a $E(V)$ is an unsharp property and is ϵ-actual in $\alpha \in B$. Such effects represent "elements of unsharp reality" in α.

An operation ϕ is called *pure* if it transforms pure states into pure states. Nonpure states may be produced by mixing pure states; to exclude complications arising with this kind of subjective ignorance we have introduced in $(P_\epsilon 2)$ the tentative restriction to pure states. Actually, it seems possible to admit "unsharpness" also with respect to subjective uncertainties by including "almost pure" states: we call $\alpha \in B$ *almost pure* if it possesses a decomposition $\alpha=\Sigma_i \lambda_i \alpha_i$ into pure states α_i with $\lambda_i \geq 0, \Sigma_i \lambda_i = 1$ and one of the λ_i being close to unity.

The *nondisturbance* postulate $(P_\epsilon 2)$ makes the concept of unsharp property a proper relaxation of the notion of (sharp) property ($\epsilon=0$) used in quantum logic or in standard Hilbert space quantum theory: $\epsilon=0, \mu(\epsilon)=0$ characterizes the pure, ideal, first kind measurements of the projections $P \in \mathcal{P}(\mathcal{H})$.

The restriction $\epsilon < \frac{1}{2}$ is necessary for avoiding the situation that an effect and its complement may be ϵ-actual in the same state. The property postulate $(P_\epsilon 1)$ guarantees the premise of $(P_\epsilon 2)$ to be realizable and represents a relaxation of the EPR condition ("probability equal to unity") of predictability; it excludes the effect intervals $[0,\frac{1}{2}e]$ and $[\frac{1}{2}e,e]$ from the set of (nontrivial) properties. (For a further specification of the set of unsharp properties one might introduce additional requirements such as closedness of the set of properties under complementation, but we shall not pursue this here.)

The following result shows that the set $\mathscr{E}(\mathscr{H})$ of Hilbert-space effects contains unsharp properties which are not projections.

Proposition 2.2. Any ϵ-actualizable effect $E\in\mathscr{E}(\mathscr{H})$ is an unsharp property; in particular, the Lüders operation $\phi_L:\alpha\mapsto\phi_L\alpha=E^{1/2}\alpha E^{1/2}$ is ϵ-repeatable and satisfies the nondisturbance postulate $(P_\epsilon 2)$ with $\mu(\epsilon)=\sqrt{\epsilon}$.

Proof. It is straightforward to calculate the spectrum of $A=P_\psi E(P_\psi)-\phi_L(P_\psi)$ and to estimate $||A||\le\mu=\sqrt{\epsilon}$ by using the assumption $E(P_\phi)\ge 1-\epsilon$; this gives $|b(\phi_L P_\phi)-b(P_\psi)e(\phi_L P_\psi)|\le\mu(\epsilon)$ for all $b\in\epsilon(\mathscr{H})$. On the other hand, for any $\alpha\in B$ with $E(\alpha)\ge 1-\epsilon$,

$$E(\phi_L\alpha)=\mathrm{tr}(\alpha E^2)\ge[\mathrm{tr}(\alpha E)]^2$$
$$=E(\alpha)^2\ge(1-\epsilon)E(\alpha)\ ,$$

i.e., ϕ_L is ϵ-rep. This completes the proof.

The Lüders operations are a generalization of the ideal first kind measurements $\alpha\mapsto P\alpha P$ of the projections $P=P^2\in\mathscr{P}(\mathscr{H})$. The above statement proves that the POV observables admit an interpretation in terms of unsharp properties of (unsharply constituted) systems in the sense of the modified EPR reality criterion (R_ϵ). As an application of this conception of unsharp (quantum) reality it was shown in Ref. 19 that phase-space observables admit ϵ-prep measurements; thus, position and momentum can be simultaneously ϵ-real unsharp properties: in this sense bubble-chamber tracks are seen to be elements of unsharp reality for individual elementary particles. In the remaining sections we shall establish similar results for spin observables. It was mentioned in Sec. I that quantities which do not commute with conserved quantities do not possess predictable measurements: they never can represent elements of *sharp* reality; but they *can* be elements of *unsharp* reality.[19] Moreover, there exist quantities such as momentum which, by their very operational definition (as dynamical quantities) can only be measured by disturbing their values. In the case of momentum we have to distinguish it from velocity which is defined via time-of-flight measurements: momentum is a property of particles which characterizes their behavior in collisions—thus it will be changed during its measurement. This shows that particles with rather well-defined momenta, as they are produced by an accelerator cannot be considered to be real in the sharp $\epsilon=0$ sense; but they *are* constituted in the sense of our notion of *unsharp* reality. These examples illustrate some of the advantages of the generalized reality concept which corresponds to the generalized, POV measure description of observables.

III. MEASUREMENT OF UNSHARP SPIN PROPERTIES

In the following we are dealing with the two-dimensional $(\mathscr{H}=\mathscr{H}_2)$ representation of the spin algebra:

$$\sigma_j\sigma_k=i\epsilon_{jkl}\sigma_l,\quad \sigma_j\sigma_k+\sigma_k\sigma_j=2\delta_{jk}I\ .$$

Any linear operator A on $\mathscr{H}_2$ can be written in the form

$$A=\alpha_0 I+\boldsymbol{\alpha}\cdot\boldsymbol{\sigma},\quad (\alpha_0,\boldsymbol{\alpha})\in\mathbb{C}^4\ .$$

In particular, effects $F\in\epsilon(\mathscr{H})$ are uniquely represented as

$$F=\tfrac{1}{2}\gamma(I+\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})=:F(\boldsymbol{\lambda},\gamma)$$

with

$$0\le\gamma,\ \lambda=||\boldsymbol{\lambda}||\le 1\quad(\text{positivity } F\ge 0)$$

and

$$\gamma\le\frac{2}{1+\lambda}\quad(F\le I)\ .$$

For later use we denote

$$F(\boldsymbol{\lambda},\gamma)=\gamma E(\boldsymbol{\lambda}),\quad E(\boldsymbol{\lambda})=\tfrac{1}{2}(I+\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})\ ,$$

$$T_{\hat{\boldsymbol{\lambda}}}=E(\hat{\boldsymbol{\lambda}}),\quad \hat{\boldsymbol{\lambda}}=\boldsymbol{\lambda}/\lambda\quad(\text{for } \lambda\neq 0)\ .$$

The $T_{\mathbf{n}}$ are the one-dimensional spin-projection operators on $\mathscr{H}_2$. The complement of $F=F(\boldsymbol{\lambda},\gamma)=\gamma E(\boldsymbol{\lambda})$ is

$$F'=I-F=(2-\gamma)E\left[-\frac{\gamma}{2-\gamma}\boldsymbol{\lambda}\right]\ .$$

A ("simple") spin observable shall be determined through a direction $\hat{\boldsymbol{\lambda}}$ which is (unsharply) defined by orientation of the measuring device. If a positive outcome for $F(\boldsymbol{\lambda},\gamma)$ corresponds to a spin-up result then this should be equivalent with a spin-down result for $F(-\boldsymbol{\lambda},\gamma)$ since the apparatus for the latter effect results from that for the former simply by changing the orientation from $\boldsymbol{\lambda}$ to $-\boldsymbol{\lambda}$. On the other hand, the results "spin up" and "spin down" should be complementary to each other; thus we select from the set of possible unsharp properties F those obeying $F'=F(-\boldsymbol{\lambda},\gamma)$ as unsharp *spin* properties. But this requirement implies $\gamma=1$, i.e., only the $E(\boldsymbol{\lambda})$ may be "spin properties."

The spectral decomposition of $F(\boldsymbol{\lambda},\gamma)$ is

$$F(\boldsymbol{\lambda},\gamma)=\frac{\gamma}{2}(I+\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})=\frac{\gamma}{2}(1+\lambda)T_{\hat{\boldsymbol{\lambda}}}+\frac{\gamma}{2}(1-\lambda)T_{-\hat{\boldsymbol{\lambda}}}\ .$$

The eigenvalues of F, $r=(\gamma/2)(1+\lambda)$ and $u=(\gamma/2)(1-\lambda)$ (which are just the values of the confidence measure of fuzzy spin space[18]), represent the (maximal) *reality degree* (r) and the (minimal) *unsharpness* (u) of F. The following simple statements give a characterization of the *unsharp (spin) properties*

Proposition 3.1. (1) $F(\boldsymbol{\lambda},\gamma)$ is an unsharp property iff

$$r=\frac{\gamma}{2}(1+\lambda)>\frac{1}{2}\quad\text{and}\quad u=\frac{\gamma}{2}(1-\lambda)<\frac{1}{2}$$

which implies $\frac{1}{2}<\gamma<\frac{3}{2}$. (2) $F(\boldsymbol{\lambda},\gamma)$ is an unsharp property for *all* λ $(0\le\lambda\le 1)$ iff $\gamma=1$. (3) Let r,u and r',u' be reality degree and unsharpness of F and F', respectively. Then $r=r'$ iff $u=u'$ iff $\gamma=1$. (4) The "contrasts" $r+u$ of F and $r'+u'$ of F' are simultaneously optimal (i.e., their product maximal) iff $r+u=r'+u'=1$ iff $\gamma=1$.

The *contrast* $c(E)$ of an effect E is an overall measure of its ability to distinguish between different states:

$$c(E)=\sup\{\,|E(\alpha_1)-E(\alpha_2)|:\alpha_1,\alpha_2\in B\}\ .$$

We have $c(F)=r+u=\gamma$ and $c(F')=r'+u'=2-\gamma$.

The interpretation of $u=\gamma/2(1-\lambda)$ as "unsharpness" can be illustrated by the following (always possible, nonunique[21]) representation of an effect $F=F(\boldsymbol{\lambda},\gamma)$:

$$F(\boldsymbol{\lambda},\gamma)=\int_\Omega d\mu(\mathbf{n})T_{\mathbf{n}},\quad \gamma=\mu(\Omega)\ ,$$
$$\boldsymbol{\lambda}=\langle \mathbf{n}\rangle_\mu=\int_\Omega d\mu(\mathbf{n})\mathbf{n}/\mu(\Omega)\ ,$$

with μ being a positive measure on the unit sphere Ω of $\mathbb{R}^3$. This shows that $\boldsymbol{\lambda}$ may arise from the effective orientation of the apparatus defining $F(\boldsymbol{\lambda},\gamma)$, and $\lambda<1$ is due to unsharpness expressed by μ.

Now we turn to the discussion of measurements in stochastic spin space.[16] Let $\Sigma=\mathscr{B}(\Omega)$ be the Borel sets on Ω. Then the following is a reasonably large class (if not exhaustive) of instruments (Σ,I) defining the same observable (Σ,A):

$$I(\Delta)(\alpha)=\int_\Delta d\nu(\mathbf{n})U(\mathbf{n})E(\lambda_{\mathbf{n}}\mathbf{n})^{1/2}\alpha\cdot E(\lambda_{\mathbf{n}}\mathbf{n})^{1/2}U(\mathbf{n})^+ \qquad (\Delta\in\Sigma,\alpha\in B)$$

with unitary $U(\mathbf{n})$, ν a positive measure on Ω:

$$A(\Delta)=\int_\Delta d\nu(\mathbf{n})E(\lambda_{\mathbf{n}}\mathbf{n})$$
$$=F_\Delta=F(\boldsymbol{\lambda}_\Delta,\gamma_\Delta)\quad(\Delta\in\Sigma)$$
$$\gamma_\Delta=\int_\Delta d\nu(\mathbf{n})=\nu(\Delta)\ ,$$
$$\boldsymbol{\lambda}_\Delta=\int_\Delta d\nu(\mathbf{n})\lambda_{\mathbf{n}}\mathbf{n}/\nu(\Delta)\ .$$

These instruments are a generalization of the case $U(\mathbf{n})=I$, $E(\lambda_{\mathbf{n}}\mathbf{n})=T_{\mathbf{n}}$ studied in Ref. 16 which we shall refer to as "maximal" measurements ϕ_m^Δ:

$$\phi_m^\Delta(\alpha)=\int_\Delta d\nu(\mathbf{n})T_{\mathbf{n}}\alpha\cdot T_{\mathbf{n}}\ .$$

If we reduce the Borel sets $\Sigma=\mathscr{B}(\Omega)$ to a simple Boolean algebra $\mathscr{B}(\Delta_1,\Delta_{\bar 1})$ generated by $\Delta_1,\Delta_{\bar 1}=\Delta_1'\in\Sigma$ we obtain a *simple* (i.e., two-valued) observable; further division of $\Delta_1=\Delta_{12}\cup\Delta_{1\bar 2}$ and $\Delta_{\bar 1}=\Delta_{\bar 1 2}\cup\Delta_{\bar 1\bar 2}$ gives rise to a joint observable (Σ_{12},A) for the observables (Σ_1,A) and (Σ_2,A) with

$$\Sigma_1=\mathscr{B}(\Delta_1,\Delta_{\bar 1}),\quad \Sigma_2=\mathscr{B}(\Delta_2,\Delta_{\bar 2})\ ,$$
$$\Delta_2=\Delta_{12}\cup\Delta_{\bar 1 2},\quad \Delta_{\bar 2}=\Delta_{1\bar 2}\cup\Delta_{\bar 1\bar 2}\ ,$$
$$\Sigma_{12}=\mathscr{B}(\Sigma_1,\Sigma_2)\ .$$

All these discrete observables are reductions of the original observable (Σ,A) to certain Boolean subalgebras. By this procedure of dividing into N,S,E,W hemispheres Schroeck[16] arrived at a stochastic spin space interpretation of Prugovecki's[18] joint spectral densities for pairs of orthogonal directions $\mathbf{n}_1,\mathbf{n}_2$. In the next section we shall establish a generalization of those results to arbitrary pairs of unsharp spin properties $E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)$.

To conclude the present section I shall state some properties of Lüders operations and of the above-mentioned maximal operations. The Lüders operation ϕ_L^Δ for an effect $F_\Delta=E(\boldsymbol{\lambda})$ is obtained from the general instrument by taking a (two-) point measure $d\nu_{\boldsymbol{\lambda}}(\mathbf{n})=[\delta(\hat{\boldsymbol{\lambda}}-\mathbf{n})+\delta(\hat{\boldsymbol{\lambda}}+\mathbf{n})]d\Omega$, $U(\mathbf{n})=I$, $\lambda_{\mathbf{n}}=\lambda,\Delta=N$ (north hemisphere with respect to the pole $\hat{\boldsymbol{\lambda}}$):

$$\phi_L^\Delta(\alpha)=E(\boldsymbol{\lambda})^{1/2}\alpha\cdot E(\boldsymbol{\lambda})^{1/2}=I_L(\Delta)(\alpha)\quad(\alpha\in B)\ .$$

Proposition 3.2. (1) The Lüders measurement ϕ_L of the unsharp spin property $E(\boldsymbol{\lambda})=\mathrm{tr}\phi_L$ is nondisturbing in the sense of $(P_\epsilon 2)$ for *arbitrary* states $\alpha\in B$ in which $E(\boldsymbol{\lambda})$ is ϵ-actual. (2) The maximal operation ϕ_m is ϵ-preparatory for some $\epsilon<\frac{1}{2}$ *if* $\mathrm{tr}\phi_m=E(\boldsymbol{\lambda})$ is an unsharp spin property and the measure ν is rotation invariant around $\boldsymbol{\lambda}$ and $\Delta=N$ is the "north" hemisphere with respect to the "pole" $\hat{\boldsymbol{\lambda}}$.

Proof. (1) Write

$$E=E(\boldsymbol{\lambda})=\tfrac{1}{2}(I+\boldsymbol{\lambda}\cdot\boldsymbol{\sigma})=rT_{\boldsymbol{\lambda}}+uT_{-\boldsymbol{\lambda}}\ ,$$
$$r=\tfrac{1}{2}(1+\lambda),\quad u=\tfrac{1}{2}(1-\lambda),\quad \alpha=\tfrac{1}{2}(I+\mathbf{a}\cdot\boldsymbol{\sigma})\ .$$

Then after some elementary calculations one obtains

$$\phi_L\alpha=E^{1/2}\alpha E^{1/2}$$
$$=\tfrac{1}{2}IE(\alpha)+\tfrac{1}{2}\boldsymbol{\sigma}\cdot\tfrac{1}{2}\{\sqrt{1-\lambda^2}[\mathbf{a}-\hat{\boldsymbol{\lambda}}(\mathbf{a}\cdot\hat{\boldsymbol{\lambda}})]+\hat{\boldsymbol{\lambda}}[\lambda+(\mathbf{a}\cdot\hat{\boldsymbol{\lambda}})]\}\ .$$

Further,

$$W\equiv\alpha E(\alpha)-\phi_L\alpha$$
$$=\boldsymbol{\sigma}\cdot\tfrac{1}{4}\{(1+\mathbf{a}\cdot\boldsymbol{\lambda}-\sqrt{1-\lambda^2})[\mathbf{a}-\hat{\boldsymbol{\lambda}}(\hat{\boldsymbol{\lambda}}\cdot\mathbf{a})]-\boldsymbol{\lambda}[1-(\mathbf{a}\cdot\hat{\boldsymbol{\lambda}})^2]\}$$
$$\equiv\boldsymbol{\sigma}\cdot\mathbf{A}\ .$$

Now let $F=\mu/2(I+\tilde{\boldsymbol{\lambda}}\cdot\boldsymbol{\sigma})$, then

$$|F(\phi_L\alpha)-F(\alpha)E(\alpha)|=\left|\mathrm{tr}\left[\frac{\mu}{2}(I+\tilde{\boldsymbol{\lambda}}\cdot\boldsymbol{\sigma})(\boldsymbol{\sigma}\cdot\mathbf{A})\right]\right|$$
$$=|\mu\tilde{\boldsymbol{\lambda}}\cdot\mathbf{A}|\le\mu\tilde{\lambda}\|\mathbf{A}\|\le\|\mathbf{A}\|$$
$$(\mu\tilde{\lambda}=r_F-u_F\le 1)\ .$$

But $\|\mathbf{A}\|^2\le\epsilon/2$ for $E(\alpha)\ge 1-\epsilon$; this proves the nondisturbance property for ϕ_L.

(2) Let $E(\boldsymbol{\lambda})=\mathrm{tr}\circ\phi_m=\int_\Delta d\nu(\mathbf{n})T_{\mathbf{n}}$, $\alpha=\frac{1}{2}(I+\mathbf{a}\cdot\boldsymbol{\sigma})$, then

$$\mathrm{tr}[\phi_m(\alpha)E(\boldsymbol{\lambda})]=\int_\Delta d\nu(\mathbf{n})\int_\Delta d\nu(\mathbf{n}')\mathrm{tr}(\alpha T_{\mathbf{n}})\mathrm{tr}(T_{\mathbf{n}}\cdot T_{\mathbf{n}'})$$
$$=\tfrac{1}{2}\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]+R_\Delta\ ,$$
$$4R_\Delta=\int_\Delta d\nu(\mathbf{n})(\mathbf{n}\cdot\mathbf{a})(\mathbf{n}\cdot\boldsymbol{\lambda})+(\boldsymbol{\lambda}\cdot\boldsymbol{\lambda})\quad[\nu(\Delta)=1]\ .$$

Now let ν be rotation invariant with respect to $\boldsymbol{\lambda}=\boldsymbol{\lambda}_\Delta=\lambda\cdot\hat{\boldsymbol{\lambda}}$ $(\lambda>0)$. If we choose Cartesian coordinates with $\mathbf{e}_3=\hat{\boldsymbol{\lambda}}$, then

$$4R_\Delta=\lambda^2+a_3\langle n_3\rangle_\Delta\langle n_3^2\rangle_\Delta$$
$$\ge\lambda[\langle n_3\rangle_\Delta-\langle n_3^2\rangle_\Delta]\equiv m,\quad 0<m<1$$
$$\left[\lambda=\langle n_3\rangle_\Delta=\int_\Delta d\nu(\mathbf{n})n_3,\text{etc.}\right]$$

since $a_3\ge -1$ and $n_3\ge 0$. Further, $m\ge m\,\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]$, thus

$$\mathrm{tr}[\phi_m(\alpha)E(\boldsymbol{\lambda})]=\tfrac{1}{2}\,\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]+R_\Delta$$
$$\geq \left[\frac{1}{2}+\frac{m}{4}\right]\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]$$
$$\equiv(1-\epsilon)\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]\ ,$$

where $\epsilon=\frac{1}{2}(1-m/2)<\frac{1}{2}$. This completes the proof.

The operations ϕ_m are called maximal since they produce a maximal destruction of coherence by transforming even pure states into continuous mixtures of pure states. This strong "disturbance" seems necessary for the preparatory influence as described above. On the other hand, the Lüders measurements with their "minimal" disturbance are not ϵ-preparatory for *arbitrary* initial states; still they are *probabilistically preparatory*;[9] i.e., they will generally improve the reality content of the corresponding effect:

$$a(\phi_L\alpha/a(\alpha))\geq a(\alpha)\quad (a=e\circ\phi_L)$$
$$\text{for } \alpha\in B \ \text{ with } a(\alpha)\neq 0\ .$$

It is straightforward to derive from this the following.

Proposition 3.3. The Lüders measurement ϕ_L of an unsharp spin property $E(\boldsymbol{\lambda})=\mathrm{tr}\circ\phi_L$ is *weakly ϵ-prep* with $\epsilon=\frac{1}{2}(1-\lambda^2)$, i.e.,

$$\mathrm{tr}[\phi_L(\alpha)E(\boldsymbol{\lambda})]\geq(1-\epsilon)\mathrm{tr}[\alpha E(\boldsymbol{\lambda})]$$
$$\text{for all } \alpha\in B \text{ with } \mathrm{tr}[\alpha E(\boldsymbol{\lambda})]\geq\tfrac{1}{2}\ .$$

This property of the Lüders operations turns out to be important for the Einstein-Podolsky-Rosen experiment (cf. Sec. V and Ref. 25).

IV. COEXISTENCE OF VARIOUS SPIN COMPONENTS

A. Coexistence and lower bounds

Among the "unsharp properties" of stochastic spin space the "unsharp *spin* properties" are characterized by $r+u=1$. The property condition $(P_\epsilon 1)$ is equivalent to $r>u$ (i.e., $\lambda>0$, or $\epsilon<\frac{1}{2}$). It will be shown that increase of unsharpness to a sufficiently large amount makes pairs or triples of different spin directions coexistent. The price is a fairly low reality degree.

A criterion for coexistence of a pair $\{E_1,E_2\}$ of arbitrary effects in $E(V)$ is given by the following.[22]

Proposition 4.1. Effects $\{E_1,E_2\}$ are coexistent iff there exists an effect E satisfying $E\leq E_1$, $E\leq E_2$, $E_1+E_2-E\leq e$. In that case a joint observable (Σ,A) on $\Sigma=\mathscr{B}(1,\bar{1},2,\bar{2})$ is defined by

$$(12)\mapsto E_{12}=E,\quad (1\bar{2})\mapsto E_{1\bar{2}}=E_1-E\ ,$$
$$(\bar{1}2)\mapsto E_{\bar{1}2}=E_2-E,\quad (\bar{1}\,\bar{2})\mapsto E_{\bar{1}\bar{2}}=e-E_1-E_2+E\ .$$

In particular, for $E_2=E_1'=e-E_1$ this reduces to a simple observable: $E=0,2\equiv\bar{1}$. As a simple consequence we derive the following.

Proposition 4.2. If a pair of effects $\{E_1,E_2\}$ is coexistent then there exists a pair $(E_1^\vee,E_2^\vee)$ from $\{E_1,E_{\bar{1}}\}\times\{E_2,E_{\bar{2}}\}$ possessing a positive (nonzero) lower bound, i.e., admitting joint tests.

Proof. Let $\{E_1,E_2\}$ be coexistent: $E^0\leq E_1$, $E^0\leq E_2$, $E_1+E_2-E^0\leq e$. If $E^0\neq 0$ then it is positive lower bound for (E_1,E_2); if $E^0=0$ then E_2 is a positive lower bound for $(E_{\bar{1}},E_2)$.

This statement motivates the search for joint tests; in the two-dimensional Hilbert space simple PV observables are noncoexistent if and only if they do not commute if and only if they are complementary (i.e., admit no joint tests).[23] Thus the (well known[17]) noncoexistence of spin projections T_x,T_z can be expressed in terms of the impossibility of joint tests. Both kinds of verdicts will be relaxed by introducing unsharpness. A similar investigation on joint lower bounds for position and momentum has been done in Ref. 24.

B. Lower bounds for unsharp spin properties

In Sec. III we introduced a one-to-one representation of spin-$\frac{1}{2}$ effects on the three-dimensional closed unit sphere S. For the following we denote by $S(\mathbf{a},\rho):=\{\mathbf{x}\in\mathbb{R}^3:||\mathbf{x}-\mathbf{a}||\leq\rho\}$ the closed sphere with radius ρ around the point $\mathbf{a}\in S$.

An effect $F=\gamma E(\mathbf{l})$ is a lower bound for $E(\boldsymbol{\lambda})$ if and only if $E(\boldsymbol{\lambda})-F\geq 0$ if and only if $||\boldsymbol{\lambda}-\gamma\mathbf{l}||\leq 1-\gamma$. Since $||\mathbf{l}||\leq 1$ this is equivalent to $\gamma\mathbf{l}\in S(0,\gamma)\cap S(\boldsymbol{\lambda},1-\gamma)$. But this intersection of two spheres is always nonempty because $||\boldsymbol{\lambda}||\leq 1$. We collect some simple statements.

Proposition 4.3. Let $E_1=E(\boldsymbol{\lambda}_1)$, $E_2=E(\boldsymbol{\lambda}_2)$ be unsharp spin properties, $F=\gamma E(\mathbf{l})$ be a spin effect. Then (1) Any spin property $E(\boldsymbol{\lambda})$ possesses nontrivial positive lower bounds F. (2) For $E(\boldsymbol{\lambda})=T_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}=\hat{\boldsymbol{\lambda}})$ the only lower bounds are $F=\gamma T_{\boldsymbol{\lambda}},\gamma\leq 1$. (3) $E_1=E(\boldsymbol{\lambda}_1)\leq E_2=E(\boldsymbol{\lambda}_2)$ implies $\boldsymbol{\lambda}_1=\boldsymbol{\lambda}_2$, thus $E_1=E_2$. (4) E_1 and E_2 $(\boldsymbol{\lambda}_1\neq\boldsymbol{\lambda}_2)$ possess positive lower bounds if and only if $\frac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||<1$ and $\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2<1$ (which can easily be satisfied). (5) $T_{\mathbf{n}}$ and $T_{\mathbf{n}'}$ $(\mathbf{n}\neq\mathbf{n}')$ possess no positive lower bound.

Proof. (1) has just been proved. (2) For $\lambda=1$ one has $S(0,\gamma)\cap S(\boldsymbol{\lambda},1-\gamma)=\{\gamma\boldsymbol{\lambda}\}$, thus the only lower bounds are $F=\gamma E(\boldsymbol{\lambda})=\gamma T_{\boldsymbol{\lambda}}$. (3) For $\gamma=1$ one has $S(\boldsymbol{\lambda}_2,0)\cap S(0,1)=\{\boldsymbol{\lambda}_2\}$, thus $E_1=E(\boldsymbol{\lambda}_2)=E_2$ is the only lower bound for E_2 with $\gamma=1$. (4) First, let $F=\gamma E(\mathbf{l})\leq E_1,E_2$. This is equivalent to $\gamma\mathbf{l}\in S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)\cap S(0,\gamma)$; it follows that $S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)\neq\varnothing$, i.e., $||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||\leq 2(1-\gamma)$ and $0<\gamma\leq 1-\frac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||$. Now $\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2=1$ would imply $\lambda_1=1=\lambda_2$ and, by (2), $\mathbf{l}=\boldsymbol{\lambda}_1$ and $\mathbf{l}=\boldsymbol{\lambda}_2$, thus $\boldsymbol{\lambda}_1=\boldsymbol{\lambda}_2$. To show the converse, let $\frac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||<1$; then there exists $\gamma>0,\gamma\leq 1-\frac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||$. If both $\lambda_1\neq 1\neq\lambda_2$ then take $\gamma<\min\{1-\lambda_1,1-\lambda_2\}$ to obtain $S(0,\gamma)\subset S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)$. If, e.g., $\lambda_1=1\neq\lambda_2$, take $\gamma<1-\lambda_2$; then $\{\gamma\boldsymbol{\lambda}_1\}=S(0,\gamma)\cap S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)$. (5) follows from (4): $\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2=1$.

Statement (5) together with Proposition (4.2) implies the following.

Corollary 4.4. A pair of projections $\{T_{\boldsymbol{\lambda}},T_{\boldsymbol{\lambda}'}\}$ with $\boldsymbol{\lambda}'\notin\{\boldsymbol{\lambda},-\boldsymbol{\lambda}\}$ cannot be coexistent.

On the other hand, (4) shows that *unsharp* properties may well be jointly testable. We shall see that condition (4) is *not* sufficient for coexistence.

C. Coexistence criterion

Proposition 4.1 brings about the following coexistence criterion for unsharp spin properties $\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)\}$.

Theorem 4.5. A pair $\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)\}$ of unsharp spin properties is coexistent if and only if

$$\tfrac{1}{2}||\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2||+\tfrac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||\le 1 .$$

Proof. Evaluation of criterion (4.1) gives rise to the following set of inequalities for the parameters of some $F=\gamma E(\boldsymbol{l})$:

$$||\boldsymbol{\lambda}_1-\gamma\boldsymbol{l}||\le 1-\gamma,\quad ||\boldsymbol{\lambda}_2-\gamma\boldsymbol{l}||\le 1-\gamma ,$$
$$||\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2-\gamma\boldsymbol{l}||\le\gamma$$

being equivalent to

$$\gamma\boldsymbol{l}\in S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)\cap S(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2,\gamma)\cap S(0,\gamma)\neq\varnothing .$$

However, $S(\boldsymbol{\lambda}_1,1-\gamma)\cap S(\boldsymbol{\lambda}_2,1-\gamma)\neq\varnothing$ iff $\gamma\le 1-\tfrac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||$, and $S(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2,\gamma)\cap S(0,\gamma)\neq\varnothing$ iff $\gamma\ge\tfrac{1}{2}||\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2||$. On the other hand, these last two inequalities are already sufficient for coexistence since the point $\tfrac{1}{2}(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)$ will then be contained in all four spheres. This proves the theorem.

With this result we have found both a complete generalization of Prugovecki's results [referring to $E(\lambda_1\hat{\mathbf{x}})$, $E(\lambda_2\hat{\mathbf{z}})$], and a compact characterization of coexistence in two-dimensional state spaces. Moreover, we have established an interpretation of coexistence conditions in terms of unsharpness.

Let us note some immediate implications.

Corollary 4.6. Let $\mathscr{E}_{12}=\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)\}$ denote a pair of spin properties. Then (1) for $\lambda_1{}^2+\lambda_2{}^2\le 1$, all pairs $\mathscr{E}_{12}$ are coexistent, (2) for *orthogonal* $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\mathscr{E}_{12}$ is coexistent iff $\lambda_1{}^2+\lambda_2{}^2\le 1$, (3) for $\lambda_1=\lambda_2=\lambda,\mathscr{E}_{12}$ is coexistent iff

$$\lambda\le 2(||\hat{\boldsymbol{\lambda}}_1+\hat{\boldsymbol{\lambda}}_2||+||\hat{\boldsymbol{\lambda}}_1-\hat{\boldsymbol{\lambda}}_2||)^{-1} =\{1+[1-(\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{\lambda}}_2)^2]^{1/2}\}^{-1/2} .$$

This upper bound is always less than unity, except for $\boldsymbol{\lambda}_2\in\{\boldsymbol{\lambda}_1,-\boldsymbol{\lambda}_2\}$. (4) For $\lambda_1=1,\mathscr{E}_{12}$ is coexistent iff $\hat{\boldsymbol{\lambda}}_2\in\{\boldsymbol{\lambda}_1,-\boldsymbol{\lambda}_1\}$ iff $E(\boldsymbol{\lambda}_2)=E(\pm\lambda_2\boldsymbol{\lambda}_1)$ iff $E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)$ commute.

Proof. Consider the quantity

$$\Sigma:=\tfrac{1}{2}||\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2||+\tfrac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2|| ,$$
$$\Sigma^2=\tfrac{1}{2}(\lambda_1{}^2+\lambda_2{}^2)+\tfrac{1}{2}[(\lambda_1{}^2+\lambda_2{}^2)^2-4\lambda_1{}^2\cdot\lambda_2{}^2(\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{\lambda}}_2)^2]^{1/2} .$$

Then $\mathscr{E}_{12}$ is coexistent iff $\Sigma^2\le 1$ [Theorem (4.5)]. (1) $\lambda_1{}^2+\lambda_2{}^2\le 1$ implies $\Sigma^2\le 1$. (2) $\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2=0$ gives $\Sigma^2=\lambda_1{}^2+\lambda_2{}^2$. (3) This condition can be immediately read off from Theorem (4.5). (4) We have $\Sigma^2\ge\tfrac{1}{2}(\lambda_1{}^2+\lambda_2{}^2)+\tfrac{1}{2}\,|\lambda_1{}^2-\lambda_2{}^2|$ with equality only for $(\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2)^2=1$. For $\lambda_1=1$ this reads $\Sigma^2\ge 1$; however, in the case of coexistence we have $\Sigma^2\le 1$; thus, $\Sigma^2=1$. For $\lambda_1=1$, $\Sigma^2=1$ implies $\lambda_2=0$ or $(\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{\lambda}}_2)^2=1$ which, in turn, is equivalent to commutativity. Finally, commutativity implies coexistence.

Statements (1)—(3) tell that arbitrary spin components $\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)\}$ can be made coexistent by taking sufficiently small λ_1,λ_2, that is, large unsharpnesses $u_1=\tfrac{1}{2}(1-\lambda_1)$, $u_2=\tfrac{1}{2}(1-\lambda_2)$. Especially for orthogonal $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2$ the coexistence criterion takes a particularly simple form. $\mathscr{E}_{12}$ can be coexistent without commuting:

$$E(\boldsymbol{\lambda}_1)\cdot E(\boldsymbol{\lambda}_2)-E(\boldsymbol{\lambda}_2)\cdot E(\boldsymbol{\lambda}_1)=\frac{i}{2}(\boldsymbol{\lambda}_1\times\boldsymbol{\lambda}_2)\cdot\boldsymbol{\sigma} .$$

This is zero if and only if $(\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{\lambda}}_2)^2=1$. We see that coexistence coincides with commutativity if at least one of both effects is a projection; for that reason, the statement of Corollary (4.4) is an immediate consequence of the above statement (4).

Once one has chosen an effect $E=E_{12}$ satisfying the condition of Proposition (4.1) for coexistent $\mathscr{E}_{12}=\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)\}$ it is straightforward to write down a joint observable. A simple example is induced by

$$E_{12}=\tfrac{1}{2}E(\boldsymbol{\lambda}_1)\cdot E(\boldsymbol{\lambda}_2)+E(\boldsymbol{\lambda}_2)\cdot E(\boldsymbol{\lambda}_1) .$$

Let $\boldsymbol{\lambda}_{\bar{1}}=-\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_{\bar{2}}=-\boldsymbol{\lambda}_2$, $\boldsymbol{\lambda}_i\in\{\boldsymbol{\lambda}_1,-\boldsymbol{\lambda}_1\}$, $\boldsymbol{\lambda}_j\in\{\boldsymbol{\lambda}_2,-\boldsymbol{\lambda}_2\}$, $\alpha_{ij}=\tfrac{1}{2}(1+\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j)$. Then the joint observable is completely defined by

$$E_{ij}=\tfrac{1}{2}\alpha_{ij}I+\tfrac{1}{4}(\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j)\cdot\boldsymbol{\sigma}=\alpha_{ij}E\left[\frac{1}{\alpha_{ij}}\tfrac{1}{2}(\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j)\right] .$$

Marginality conditions are trivially satisfied, e.g., $E_{12}+E_{1\bar{2}}=E(\boldsymbol{\lambda}_1)$, etc. Furthermore, positivity of all E_{ij} is given if and only if $\mathscr{E}_{12}$ is coexistent: one readily calculates

$$\alpha_{ij}\le 1-\tfrac{1}{2}||\boldsymbol{\lambda}_i-\boldsymbol{\lambda}_j||\ \text{ iff }\ \alpha_{ij}\ge\tfrac{1}{2}||\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j||\ \text{ iff }\ \tfrac{1}{2}||\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2||+\tfrac{1}{2}||\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2||\le 1 .$$

For the discussion of the EPR experiment a consideration of coexistence of a triple $\mathscr{E}_{123}=\{E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2),E(\boldsymbol{\lambda}_3)\}$ of unsharp spin properties turns out to be desirable. It appears difficult to give a general treatment since eight E_{ijk} are involved in a set of inequalities. Geometrically one has to investigate the intersections of (at least) four spheres; this does not seem to lead to simple solutions. Therefore I shall only present a simple *sufficient* condition. The operators

$$E_{ijk}=\tfrac{1}{2}\alpha_{ijk}I+\tfrac{1}{8}\boldsymbol{\lambda}_{ijk} ,$$
$$\alpha_{ijk}=\tfrac{1}{4}(1+\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j+\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_k+\boldsymbol{\lambda}_j\cdot\boldsymbol{\lambda}_k) ,$$
$$\boldsymbol{\lambda}_{ijk}=\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j+\boldsymbol{\lambda}_k ,$$

satisfy the marginality conditions $E_{ijk}+E_{ij\bar{k}}=E_{ij}$, etc., with E_{ij},E_{ik},E_{jk} being the above pair observables. Therefore the E_{ijk} define a joint observable if and only if they are positive, i.e.,

$$\tfrac{1}{4}\|\boldsymbol{\lambda}_{ijk}\| \le \alpha_{ijk} = \tfrac{1}{8}(1-\lambda_1{}^2-\lambda_2{}^2-\lambda_3{}^2+1+\|\boldsymbol{\lambda}_{ijk}\|^2) ,$$

or

$$0 \le (\|\boldsymbol{\lambda}_{ijk}\|-1)^2+(1-\lambda_1{}^2-\lambda_2{}^2-\lambda_3{}^2) .$$

The last inequality is satisfied for all directions $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_3$ if $\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2\le 1$. This proves the statements made in Ref. 25 with respect to joint triple observables.

The existence of joint lower bounds and of joint observables implies the existence of joint test operations and of "joint" instruments, respectively. Therefore we conclude that noncommuting *unsharp* spin properties can indeed be simultaneous elements of reality of a system in an operational sense. It remains to note that arbitrary pairs $E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)$ with $\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2\ge 0$ can be simultaneously ϵ actual: take the pure state $T_{\mathbf{n}}$ with $\mathbf{n}=(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)/\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|$, then

$$\mathrm{tr}[T_{\mathbf{n}}\cdot E(\boldsymbol{\lambda}_1)] = \frac{1}{2}\left[1+\frac{\lambda_1{}^2+\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2}{\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|}\right] > \frac{1}{2} ,$$

$$\mathrm{tr}[T_{\mathbf{n}}E(\boldsymbol{\lambda}_2)] = \frac{1}{2}\left[1+\frac{\lambda_2{}^2+\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2}{\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|}\right] > \frac{1}{2} .$$

In that state the sum of probabilities is even maximal and equal to $1+\frac{1}{2}\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|$.

D. Informational completeness

Until now we have been mainly concerned with the capability of measurements to detect *properties* of systems; next we turn to the determinative features, i.e., the ability of measurements to determine the *states* of systems. As is well known, the statistics of *all* three spin components s_x,s_y,s_z is needed in order to uniquely determine the premeasurement state. An appreciable improvement can be achieved by means of unsharp observables.

Theorem 4.7. (1) A joint spin observable (Σ_{12},A_{12}) for unsharp spin properties $E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)$ is informationally complete if and only if A_{12} is generated by an effect $E_{12}=\gamma_{12}E(\boldsymbol{\lambda}_{12})$ such that the set of vectors $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_{12}$ is linearly independent. (2) There exist informationally complete observables for $E(\boldsymbol{\lambda}_1),E(\boldsymbol{\lambda}_2)$ if and only if

$$\tfrac{1}{2}\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|+\tfrac{1}{2}\|\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2\|<1$$

and $\{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2\}$ linearly independent .

Proof. (1) State operators α_i can be represented as $\alpha_i=E(\mathbf{a}_i)=\frac{1}{2}(I+\mathbf{a}_i\cdot\boldsymbol{\sigma}),\|\mathbf{a}_i\|\le 1$. Then

$$0=\mathrm{tr}[(\alpha_1-\alpha_2)\cdot E_{ij}] \text{ for all } (i,j)\in\{1,\bar{1}\}\times\{2,\bar{2}\}$$

iff

$$0=(\mathbf{a}_1-\mathbf{a}_2)\cdot\boldsymbol{\lambda}_{12}=(\mathbf{a}_1-\mathbf{a}_2)\cdot(\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_{12})$$
$$=(\mathbf{a}_1-\mathbf{a}_2)(\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_{12}) .$$

One can infer $\mathbf{a}_1=\mathbf{a}_2$, i.e., $\alpha_1=\alpha_2$ if and only if the set $\{\boldsymbol{\lambda}_{12},\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_{12},\boldsymbol{\lambda}_2-\boldsymbol{\lambda}_{12}\}$ is linearly independent if and only if $\{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_{12}\}$ is linearly independent.

(2) If there exists an informationally complete joint observable defined by $E_{12}=\gamma_{12}E(\boldsymbol{\lambda}_{12})$ then, according to (1), $\boldsymbol{\lambda}_1\ne 0\ne\boldsymbol{\lambda}_2,\boldsymbol{\lambda}_1\ne\boldsymbol{\lambda}_2$. Were $\Sigma=\frac{1}{2}\|\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2\|+\frac{1}{2}\|\boldsymbol{\lambda}_1-\boldsymbol{\lambda}_2\|=1$ then the set $S(\boldsymbol{\lambda}_1,1-\gamma_{12})\cap S(\boldsymbol{\lambda}_2,1-\gamma_{12})\cap S(\mathbf{0},\gamma_{12})=\{\frac{1}{2}(\boldsymbol{\lambda}_1+\boldsymbol{\lambda}_2)\}$ contains only the vector $(\gamma_{12}\equiv 1)$ $\gamma_{12}\boldsymbol{\lambda}_{12}$, in contradiction to informational completeness. On the other hand, if $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2$ are linearly independent and $\Sigma<1$ then the intersection of the above three spheres contains more than one element, and $\boldsymbol{\lambda}_{12}$ can be chosen to be noncollinear with $\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2$: according to (1) it gives rise to an informationally complete observable. This completes the proof.

The above results may be interpreted in the following way. Introduction of unsharpness on one hand leads to a decrease of the ideally available information with respect to measuring values ("smearing of probability distributions increases entropy"[26]). On the other hand, unsharpness is necessary for the possibility of joint measurements and for informational completeness of a single observable.

V. CONCLUSION: THE EPR EXPERIMENT

In Bohm's version of the EPR experiment a system consisting of two spin-$\frac{1}{2}$ particles is prepared in the singlet state

$$\Psi=\frac{1}{\sqrt{2}}[\phi_+(\mathbf{n})\otimes\phi_-(\mathbf{n})-\phi_-(\mathbf{n})\otimes\phi_+(\mathbf{n})] .$$

The argument deals with spin measurements on one of the subsystems performed in a stage when both subsystems are spatially separated so that they cannot interact with each other during the measurement period. Then, according to the EPR *locality assumption,* a measurement of any spin property $E(\boldsymbol{\lambda})=E^{\mathrm{I}}$ on subsystem I determines the value of the correlated property $E(-\boldsymbol{\lambda})=E^{\mathrm{II}}$ of the second subsystem without disturbing it. Therefore one can apply the EPR reality criterion to conclude that E^{II} must be an element of reality. [For a detailed presentation of the argument in abstract quantum language see Ref. 27; the translation into the language of unsharp properties will be found in Ref. 25 where the relevance of the Lüders operations ϕ_L (cf. Sec. III) is evaluated and a quantitative treatment of "unsharp correlation" is given.] Since $\boldsymbol{\lambda}$ was arbitrary this conclusion holds for *all* spin properties. This leads in a well-known way to Bell's inequality:

$$p(E_1^{\mathrm{I}},E_2^{\mathrm{II}})\le p(E_1^{\mathrm{I}},E_3^{\mathrm{II}})+p(E_3^{\mathrm{I}},E_2^{\mathrm{II}}) .$$

Inserting the quantum probabilities

$$p(E_i^{\mathrm{I}},E_k^{\mathrm{II}})=(\Psi,E_i^{\mathrm{I}}\otimes E_j^{\mathrm{II}}\Psi)$$
$$=\tfrac{1}{4}(1-\boldsymbol{\lambda}_i\cdot\boldsymbol{\lambda}_j)$$
$$=(1-2\epsilon)\tfrac{1}{2}\sin^2(\theta_{12}/2)+\epsilon/2 ,$$

$$\epsilon=\tfrac{1}{2}(1-\lambda^2),\ \ \lambda=\lambda_1=\lambda_2$$

gives

$$\tfrac{1}{2}\sin^2\left[\frac{\theta_{12}}{2}\right]\le\tfrac{1}{2}\sin^2\left[\frac{\theta_{13}}{2}\right]+\tfrac{1}{2}\sin^2\left[\frac{\theta_{32}}{2}\right]+\frac{1}{2}\frac{\epsilon}{1-2\epsilon} .$$

It follows that Bell's inequality can be violated as long as $\epsilon < \frac{1}{6}$, i.e., $\lambda > (\frac{2}{3})^{1/2}$. Therefore, for reasonably small unsharpnesses—as they occur in the present experiments—the EPR-Bell argument against *local quantum reality* remains valid: small deviations from the idealizing assumption of sharp properties do not disturb the argument. On the other hand, in principle one might construct measuring devices with $\epsilon \geq \frac{1}{6}$. For such experiments the contradiction between quantum mechanics and locality will not occur: Bell's inequality are no longer violated. [Note: the analysis of Sec. II shows that quantum theory is compatible with (unsharp) reality; thus a possible contradiction must be rooted in the additional assumption of locality.] In other words, objectivity of sufficiently unsharp properties is an admissible assumption. As shown in Ref. 25, this fact can be explained by the observation that (a set of 4) Bell inequalities represent necessary *and sufficient* conditions for the existence of *state-dependent* joint probability distributions for triples $\{E(\lambda_1), E(\lambda_2), E(\lambda_3)\}$ of unsharp spin properties. In fact, the triple observable constructed in Sec. IV entails such joint distribution for the subsystem state $\alpha = \frac{1}{2}I$ corresponding to the singlet state Ψ:

$$p_\alpha(E_i, E_j, E_k) = \tfrac{1}{2}\alpha_{ijk} = \mathrm{tr}(\alpha \cdot F_{ijk}) \ .$$

To conclude, one may say that introduction of unsharp POV observables into quantum theory provides a step from quantum no-go statements (incompatibility, nonlocality) "back" to more "classical" possibilities (joint distributions, simultaneous reality in EPR-like situations).

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