s-equivalent Lagrangians in generalized mechanics

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The problem of s-equivalent Lagrangians is discussed in generalized mechanics. Some results known from ordinary (nongeneralized) mechanics are extended to the generalized case. A theorem allowing the reduction of the higher-order Lagrangian description to the usual order is derived. The theorem is found to be useful in the analysis of generalized mechanical systems, and it leads to a new type of equivalence between Lagrangian functions. Some new perspectives are pointed out.

INTRODUCTION

There has been considerable activity on the inverse problem of Lagrangian mechanics, that is, on the problem of how to find the Lagrangian function given the equations of motion (see Ref. 1 and references therein). From the physical point of view the interest in this problem arises from the question of how to find all nontrivially related s-equivalent Lagrangians, i.e., Lagrangians that lead to the same orbits in configuration space but do not merely differ from each other by the addition of a total time derivative of a suitable function or by the multiplication with a numerical factor.[1,2] The activities have led to an enrichment in the classical description of mechanical systems, and have also revealed some interesting problems concerning the quantum aspect, as it has been found that the Hamiltonians which correspond to s-equivalent Lagrangians do not necessarily lead to the same orbits in phase space.[3] However, the general problem of s-equivalent Lagrangians has not been solved.

Some useful partial results exist. If we are given a Lagrangian function $L(t,q_1,\ldots,q_n;\dot q_1,\ldots,\dot q_n)\equiv L(t,q,\dot q)$ with

$$\Lambda_i\equiv\frac{\partial L}{\partial q^i}-\frac{d}{dt}\frac{\partial L}{\partial \dot q^i}=0\ , \tag{1.1a}$$

then the problem of s equivalence is the problem of finding all Lagrangians $\bar L(t,q,\dot q)$ with

$$\bar\Lambda_i\equiv\frac{\partial \bar L}{\partial q^i}-\frac{d}{dt}\frac{\partial \bar L}{\partial \dot q^i}=0 \tag{1.1b}$$

such that Eq. (1.1a) implies Eq. (1.1b) and vice versa:

$$\Lambda_i=0\Longleftrightarrow\bar\Lambda_i=0\ . \tag{1.2}$$

One way of achieving this is to find all dynamical variables $a_j{}^k(t,q,\dot q)$, $j,k=1,\ldots,n$, such that (summation convention always understood)

$$\bar\Lambda_j=a_j{}^k\Lambda_k\ , \tag{1.3}$$

$$a=||a_j{}^k||,\ \ \det a\neq 0\ . \tag{1.3a}$$

Even this presents considerable difficulties. However, it can be shown that

$$\frac{d}{dt}(\mathrm{tr}a)=0 \tag{1.4}$$

represents a necessary condition.[4] This can be generalized to the trace of all integral powers of a.[5] In one dimension the problem is simpler because here the elements $a_j{}^k$ reduce to a single function which is a constant of the motion, as follows from (1.4). The lack of more general results perhaps explains why the problem of s equivalence has not been considered so far in generalized mechanics where the Lagrangian function contains higher-order time derivatives of the generalized coordinates (but see Ref. 6).

The main objective of the present work is to discuss the problem of s equivalence in generalized mechanics. In Sec. II we shall discuss the possibility of finding s-equivalent Lagrangians in generalized mechanics with the help of a (new) higher-order reduction theorem. From this theorem a weaker class of equivalence follows, which we have called semiequivalence. In Sec. III we approach the s-equivalence problem via Boehm's conditions and in Sec. IV we summarize and discuss our results.

II. THE ORDER-REDUCTION APPROACH

Let $L(t,q,\dot q,\ldots,q^{(N)})$ be a Lagrangian function for some system with $M>1$ degrees of freedom. Here, and throughout this paper, we use the notation $q=(q^1,\ldots,q^M)$, $\dot q=(\dot q^1,\ldots,\dot q^M)$; latin indices $i,j,k,\ldots$, range from 0 to N while greek indices indicate the values $1,2,\ldots,M$. The Euler-Lagrange equations which follow from Hamilton's principle are[7]

$$P_\alpha=\sum_k(-1)^kD_k\frac{\partial L}{\partial q_\alpha^{(k)}}=0\ , \tag{2.1}$$

where $D_k=d/dt^k$.

With some dynamical variables $\lambda_\alpha{}^\beta(t,q,\ldots,q^{(N)})$ of which we require that

$$\det||\lambda_\alpha{}^\beta||\neq 0\ , \tag{2.2}$$

we define that following set of quantities:

$$\bar P_\alpha=\lambda_\alpha{}^\beta P_\beta\ . \tag{2.3}$$

From (2.2) it is found that (2.1) and (2.3) are equivalent sets in the sense that

$$P_\beta=0 \Longleftrightarrow \bar{P}_\beta=0 . \tag{2.4}$$

Thus, if we can find some $\bar{L}(t,q,\ldots,q^{(N)})$ such that

$$\bar{P}_\beta=\sum_k(-1)^k D_k\frac{\partial\bar{L}}{\partial q^{(k)}\beta}=0 , \tag{2.5}$$

we can say that L and $\bar{L}$ are s-equivalent Lagrangians in generalized mechanics. Hence the problem of s equivalence amounts to finding suitable λ's to guarantee the existence of $\bar{L}$.

Now, as it has been pointed out in Ref. 8, the higher-order Lagrangian formalism has the peculiarity that it allows a description of a mechanical system in configuration spaces with less dimensions than the usual ones. We shall exploit this feature, and we shall show that the equivalence problem can be approached from a different side where we need not worry about the λ's: the s equivalence remains defined by (2.4) but we need not go through (2.3). As a result some interesting questions will arise. Therefore we shall postpone the search for suitable λ's to the next section.

For simplicity we shall specialize on the case $M=1$. Our approach will be based on the following reduction theorem.

Theorem I. Given a Lagrangian function $L(t,q_1,\dot{q}_1,\ldots,q_1^{(N)})$ for a system in one generalized coordinate q_1 and $N>1$, it is possible to construct another Lagrangian function $\bar{L}(t,q_1,\ldots,q_1^{(N-1)},q_2,\dot{q}_2)$ involving two generalized coordinates q_1,q_2 and a maximum order of the time derivative equal to $(N-1)$, such that L and $\bar{L}$ are equivalent in the sense that both lead to the same orbits in the original configuration space. The new Lagrangian function is obtained from

$$\bar{L}=D(q_2q_1^{(N-1)})\,|_{q_1^{(N)}\equiv Q_1^{(N)}} - L(t,q_1,\ldots,q_1^{(N-1)},q_1^{(N)}\equiv Q_1^{(N)}) , \tag{2.6}$$

where

$$Q_1^{(N)}=Q_1^{(N)}(t,q_1,\dot{q}_1,\ldots,q_1^{(N-1)},q_2) \tag{2.7}$$

must be obtained from the following definition of the new generalized coordinate q_2:

$$q_2\equiv\frac{\partial L}{\partial q_1^{(N)}} . \tag{2.8}$$

Comments on theorem I. Concerning the range of applicability of theorem I we emphasize that the existence of $\bar{L}$ as given by (2.6) strongly depends on the possibility of writing down (2.7). Thus, for instance, if L is linear in $q_1^{(N)}$ it will not be possible to write (2.7). On the other hand, in this case we do not have an effective N-order system as it was shown by Ryan[9] (for the one-dimensional case considered here), and another Lagrangian function depending on (at most) $q_1^{(N-1)}$ can be constructed to describe the system. For this true higher-order Lagrangian we may expect it to be possible to write (2.7). We will not go into details here, but rather limit the applications to systems for which (2.7) follows from (2.8).

Proof of theorem I. To prove theorem I we shall proceed in a direct way. The equations of motion for the system described by L are given by (2.1) and those corresponding to $\bar{L}$ are

$$\sum_{k=0}^{N-1}(-1)^k D_k\frac{\partial\bar{L}}{\partial q^{(k)}}=0 , \tag{2.9}$$

$$\frac{\partial\bar{L}}{\partial q_2}-D\frac{\partial\bar{L}}{\partial\dot{q}_2}=0 . \tag{2.10}$$

Now, using definitions (2.6) and (2.8) it follows that (2.10) is an identity. Hence, (2.9) is the effective equation of motion and we only need to show that this equation is the same as (2.1). For this consider $0\le k\le(N-2)$. From (2.1) and (2.8) we find

$$\frac{\partial\bar{L}}{\partial q_1^{(k)}}=-\frac{\partial L}{\partial q_1^{(k)}} . \tag{2.11}$$

For $k=(N-1)$ a similar reasoning leads to

$$\frac{\partial\bar{L}}{\partial q_1^{(N-1)}}=D\frac{\partial L}{\partial q_1^{(N)}}-\frac{\partial L}{\partial q_1^{(N-1)}} . \tag{2.12}$$

Using (2.11) and (2.12) it is easy to verify that (2.9) reduces to (2.1) and this proves theorem I.

Now, theorem I can be applied successively until one arrives at an $\bar{L}$ as prescribed in usual (nongeneralized) mechanics. Hence we have the following generalization.

Theorem II. Given a Lagrangian function $L(t,q_1,\dot{q}_1,\ldots,q_1^{(N)})$ for a system in one generalized coordinate q_1 and $N>1$, it is possible to construct another Lagrangian function $\bar{L}(t,q_1,\dot{q}_1,P_J,\dot{P}_J)$, $J=0,1,\ldots,N-2$, involving N generalized coordinates and velocities so that L and $\bar{L}$ are equivalent in the sense that both lead to the same orbits in the original configuration space. The new Lagrangian function $\bar{L}$ is obtained from

$$\bar{L}=\sum_J(-1)^J D(P_Jq^{(N-J-1)})\,|_{q^{(K)}\equiv Q^{(K)}} - L(t,q_1,\dot{q}_1,q^{(K)}\equiv Q^{(K)}) , \tag{2.13}$$

where

$$Q^{(K)}\equiv Q^{(K)}(t,q_1,\dot{q}_1,P_J,\dot{P}_J),\quad K=2,3,\ldots,N \tag{2.13a}$$

must be obtained from the following definitions of the new generalized coordinates P_J:

$$P_0=\frac{\partial L}{\partial q_1^{(N)}},\quad J=0 ,$$

$$P_J=\dot{P}_{J-1}+(-1)^J\frac{\partial L}{\partial q^{(N-J)}},\quad J=1,\ldots,(N-2) . \tag{2.14}$$

The proof of theorem II can be obtained by a straightforward generalization of the technique used in establishing theorem I and so we shall omit it here. On the other hand, some comments are in order. First, as before, we note that the range of applicability of theorem II strongly depends on the possibility of resolving the definitions (2.14) for $Q^{(K)}$. Again we will not go into details limiting ourselves to nonexceptional cases, but now the situation is somewhat different from the aforementioned one. In the general case, definitions (2.14) can give rise to some relations involving P_J and $\dot{P}_J$. These relations could then be thought of as constraints (nonholonomics in general) and this raises some interesting questions concerning the generalized Lagrangian formalism for the description of nonholonomic systems. In a forthcoming paper we shall discuss those questions in connection with a previous work.[10] Another aspect is the similarity between theorem II and Ostrogradsky's method of obtaining canonical equations for Lagrangians depending on higher-order derivatives.[11] In fact these two approaches share the peculiarity of allowing a description of a given higher-order Lagrangian system in spaces of different dimensions. The main difference resides in the fact that in the Ostrogradsky method we change from configuration to phase space, the resulting equations being in Hamiltonian form, while in using theorem II we remain in configuration space changing only the dimensionality of the space. What we actually obtain by using theorem II are two equivalent Lagrangian descriptions in configuration spaces of different dimensions, and this is the main role of theorem II in what follows.

With theorems I and II we have established a kind of equivalence among usual Lagrangian functions (i.e., first time derivative only) and the generalized ones. The equivalence is obtained through the enlargement of the original configuration space. To show how this works we give some examples.

Consider first the Lagrangian function

$$L=\tfrac{1}{2}(q_1{}^2+\dot{q}_1{}^2+\ddot{q}_1{}^2) \tag{2.15}$$

which leads to

$$P\equiv q_1-\ddot{q}_1+q_1^{(4)}=0 . \tag{2.16}$$

From theorem I it follows that

$$\bar{L}=\tfrac{1}{2}(q_2{}^2-q_1{}^2-\dot{q}_1{}^2)+\dot{q}_1\dot{q}_2 \tag{2.17}$$

is an equivalent Lagrangian (in usual mechanics). In fact the Euler-Lagrange equations for $\bar{L}$ are

$$q_1-\ddot{q}_1+\ddot{q}_2=0 , \tag{2.18a}$$

$$q_2-\ddot{q}_1=0 , \tag{2.18b}$$

and it is easily seen, after eliminating the extra coordinate q_2, that (2.18) reduces to (2.16).

As another example, take

$$L=\tfrac{1}{2}[q_1{}^2+\dot{q}_1{}^2+\ddot{q}_1{}^2+(q_1^{(3)})^2] \tag{2.19}$$

for which the equation of motion is

$$q_1-\ddot{q}_1+q_1^{(4)}-q_1^{(6)}=0 . \tag{2.20}$$

Applying theorem I twice or applying directly theorem II we have

$$\bar{L}=\dot{q}_1\dot{q}_3+q_3\dot{q}_2+\tfrac{1}{2}(\dot{q}_1{}^2-\dot{q}_2{}^2)+\tfrac{1}{2}(q_1{}^2-q_2{}^2-q_3{}^2) \tag{2.21}$$

as an equivalent Lagrangian depending on three-generalized coordinates. Indeed, the Euler-Lagrange equations associated with $\bar{L}$ are

$$q_1-\ddot{q}_1-\ddot{q}_3=0 , \tag{2.22a}$$

$$\ddot{q}_2-\dot{q}_3-q_2=0 , \tag{2.22b}$$

$$\dot{q}_2-q_3-\ddot{q}_1=0 , \tag{2.22c}$$

and from (2.22b) and (2.22c) we have $q_2=q_1^{(3)}, q_3=q_1^{(4)}-\ddot{q}_1$ which changes (2.22b) and (2.22c) into identities and (2.22a) into (2.20).

Now, our objective is to find s-equivalent generalized Lagrangians. For this aim we can look at theorem II as a bridge from the generalized to the usual Lagrangian formalism in such a way that the known results for the last case may be used to obtain s-equivalent Lagrangians in the generalized case. To be explicit, let us consider a given $L(t,q_1,\ldots,q_1^{(N)})$. From theorem II we pass to $\bar{L}(t,q_1,\dot{q}_1,P_J,\dot{P}_J)$. The next step consists in finding a family $\epsilon\bar{L}_1$ of s-equivalent Lagrangians to $\bar{L}$. Now, for each member of this family we eliminate the extra coordinates, so coming back to higher-order Lagrangians which we could expect to be s equivalent to L. Unfortunately, as we shall see, this is not so. What we effectively obtain is a weaker class of equivalence, which we call semi-equivalence following a recent proposal in field theory.[12] Indeed the Lagrangians so obtained are equivalent to L in the sense that they lead to equations of motion $\bar{P}=0$ in such a way that

$$P=0\Longrightarrow\bar{P}=0 , \tag{2.23}$$

but not vice versa, as required by (2.4).

As an example, consider L given by (2.15). We found $\bar{L}$ in (2.17) after using theorem I. An s-equivalent Lagrangian to $\bar{L}$ is

$$\bar{L}_1=\dot{q}_1\dot{q}_2-\tfrac{1}{2}\dot{q}_2{}^2-\tfrac{1}{2}q_1{}^2+q_1q_2 . \tag{2.24}$$

Going back to higher-order dependence we arrive at

$$\bar{L}_2=-[\ddot{q}_1{}^2+\tfrac{1}{2}(q_1^{(3)})^2+\tfrac{1}{2}q_1{}^2+\dot{q}_1{}^2] \tag{2.25}$$

and the corresponding equation of motion is

$$\bar{P}=\ddot{P}-P=0 , \tag{2.26}$$

where P is given by (2.16). Thus we have (2.23) but not (2.4), and for this reason we say that $\bar{L}_2$ and L are semi-equivalent.

We think that semiequivalence is a question that merits deeper investigation. However, we shall leave this, as well as the question of generalizing the previous results to more than one degree of freedom, for another publication.

III. EQUIVALENT LAGRANGIANS IN GENERALIZED MECHANICS: A THEOREM

Now consider the equivalence problem as defined by Eqs. (2.1)–(2.5) in full generality, i.e., N and $M>1$. Our task is to find suitable λ's in such a way as to guarantee the existence of $\bar{L}$ given L. For this we shall make use of the conditions for the existence of a higher-order formalism corresponding to a given set of equations.

Let $P_\alpha(t,q,\ldots,q^{(2N)})=0$ be a set of differential equations. In order that there exists some function $L(t,q,\ldots,q^{(N)})$ such that

$$P_\alpha=\sum_k(-1)^k D_k\frac{\partial L}{\partial q^{(k)_\alpha}}=0\ , \tag{3.1}$$

the following set of Boehm conditions[13] must be satisfied:

$$\frac{\partial P_\alpha}{\partial q^{(A)_\beta}}-\begin{bmatrix}A+1\\1\end{bmatrix}D\frac{\partial P_\alpha}{\partial q^{(A+1)_\beta}}+\cdots+(-1)^{2N-A}\begin{bmatrix}2N\\2N-A\end{bmatrix}D_{2N-A}\frac{\partial P_\alpha}{\partial q^{(2N)_\beta}}=(-1)^A\frac{\partial P_\beta}{\partial q^{(A)_\alpha}}\ , \tag{3.2}$$

where $A=0,1,2,\ldots,2N$, $\alpha,\beta=1,2,\ldots,M$.

Now, it is possible to rewrite (3.1) in the form

$$P_\alpha=Q_\alpha+l_{\alpha\beta}q^{(2N)_\beta}\ , \tag{3.3}$$

where $Q_\alpha=Q_\alpha(t,q,\ldots,q^{(2N-1)})$ is known from L and

$$l_{\alpha\beta}\equiv\frac{\partial^2L}{\partial q^{(N)_\alpha}\partial q^{(N)_\beta}}=l_{\alpha\beta}(t,q,\ldots,q^{(N)}) \tag{3.4}$$

is assumed to satisfy the following condition:

$$\det||l_{\alpha\beta}||\neq 0\ . \tag{3.5}$$

With (3.3) the Boehm conditions (3.2) become relations involving Q_α and $l_{\alpha\beta}$, but we shall not need the whole set. We consider (3.2) for the cases $A=2N$ and $2N-1$. Then we have

$$\frac{\partial P_\alpha}{\partial q^{(2N)_\beta}}=\frac{\partial P_\beta}{\partial q^{(2N)_\alpha}} \tag{3.6}$$

for $A=2N$ and

$$\frac{\partial P_\alpha}{\partial q^{(2N-1)_\beta}}+\frac{\partial P_\beta}{\partial q^{(2N-1)_\alpha}}=2ND\frac{\partial P_\alpha}{\partial q^{(2N)_\beta}} \tag{3.7}$$

for $A=2N-1$. These two subsets of conditions suffice for our purpose and shall be exploited in the following.

We now consider the following set of equations

$$\bar{P}_\alpha=\lambda_\alpha{}^\beta P_\beta\ , \tag{3.8}$$

where $\lambda_\alpha{}^\beta$ are some dynamical functions for which we require

$$\det||\lambda_\alpha{}^\beta||\neq 0 \tag{3.9}$$

in order that conditions (2.4) are met. The problem then is to find $\lambda_\alpha{}^\beta$ such that there exists some $\bar{L}(t,q,\ldots,q^{(N)})$ which satisfies (2.5). Using (3.3) and with the identifications

$$\bar{Q}_\alpha=\lambda_\alpha{}^\beta Q_\beta\ , \tag{3.10}$$

$$\bar{l}_{\alpha\nu}=\lambda_\alpha{}^\beta l_{\beta\nu}\ , \tag{3.11}$$

we can write Eq. (3.8) as

$$\bar{P}_\alpha=\bar{Q}_\alpha+\bar{l}_{\alpha\nu}q^{(2N)_\nu}\ . \tag{3.12}$$

The existence of $\bar{L}$ is assured if we impose the Boehm conditions to $\bar{P}_\alpha$ as given by (3.12). This will result in conditions to be obeyed by the λ's if we assume that (3.2) is valid. Because of the existence of higher-order time derivatives this is not a simple problem. Nevertheless it is possible to handle it. Let us write the Boehm conditions corresponding to $A=2N,2N-1$, for $\bar{P}_\alpha$. We have

$$\frac{\partial \bar{P}_\alpha}{\partial q^{(2N)_\beta}}=\frac{\partial \bar{P}_\beta}{\partial q^{(2N)_\alpha}}\ , \tag{3.6a}$$

$$\frac{\partial \bar{P}_\alpha}{\partial q^{(2N-1)_\beta}}+\frac{\partial \bar{P}_\beta}{\partial q^{(2N-1)_\alpha}}=2nD\frac{\partial \bar{P}_\alpha}{\partial q^{(2N)_\beta}}\ . \tag{3.7a}$$

Consider first the set (3.6a). From (3.12) it follows that

$$\frac{\partial\lambda_\alpha{}^\nu}{\partial q^{(2N)_\beta}}(Q_\nu+l_{\nu\mu}q^{(2N)_\mu})+\lambda_\alpha{}^\nu l_{\nu\beta}=\frac{\partial\lambda_\beta{}^\nu}{\partial q^{(2N)_\alpha}}(Q_\nu+l_{\nu\mu}q^{(2N)_\mu})+\lambda_\beta{}^\nu l_{\nu\alpha}\ , \tag{3.13}$$

where we have also used (3.10) and (3.11). Until now we have not specified the dependence of $\lambda_{\alpha\beta}$ on $q^{(k)_\alpha}$. From (3.8) one might expect to obtain this dependence up to the $2N$-order time derivative. On the other hand, the effective dependence will be dictated by the conditions we must satisfy for solving the equivalence problem. This requires that both $P_\alpha=0$ and $\bar{P}_\alpha=0$ give the same orbits in configuration space, as follows from (2.4). For these orbits (3.13) reduces to

$$\lambda_\alpha{}^\nu l_{\nu\beta}=\lambda_\beta{}^\nu l_{\nu\alpha}\ . \tag{3.14}$$

Equation (3.14) tells us nothing more than the symmetry of the $\bar{l}_{\alpha\beta}$ term, as it is seen from the definitions (3.11). Now, from the structure of Eq. (3.12) we expect $\bar{l}_{\alpha\beta}$ to be related to the unknown $\bar{L}$ as in (3.4):

$$\bar{l}_{\alpha\beta}=\frac{\partial^2\bar{L}}{\partial q^{(N)_\alpha}\partial q^{(N)_\beta}}\ . \tag{3.15}$$

Thus two restrictions emerge quite naturally:

$$\det||\bar{l}_{\alpha\beta}||\neq 0\ , \tag{3.16}$$

$$\bar{l}_{\alpha\beta}=\bar{l}_{\alpha\beta}(t,q,\ldots,q^{(N)})\ . \tag{3.17}$$

Condition (3.16) expresses the nonsingular character of $\bar{L}$, and due to (3.5) and (3.14) leads to (3.9). From (3.17) and (3.11) we have

$$\frac{\partial \bar{l}_{\alpha\beta}}{\partial q^{(B)}{}_\mu}=\frac{\partial \lambda_\alpha{}^\nu}{\partial q^{(B)}{}_\mu} l_{\nu\beta}=0 \tag{3.18}$$

for $B=N+1,N+2,\ldots,2N$. Multiplying these equations by the inverse matrix elements $(l^{-1})^\beta{}_\gamma$ and summing over β we arrive at the effective maximum dependence of $\lambda_\alpha{}^\beta$ on $q^{(k)}{}_\alpha$:

$$\frac{\partial \lambda_\alpha{}^\gamma}{\partial q^{(B)}{}_\mu}=0 \rightarrow \lambda_\alpha{}^\beta=\lambda_\alpha{}^\beta(t,q,\ldots,q^{(N)}) . \tag{3.19}$$

From the foregoing results condition (3.7a) can be written as

$$\lambda_\alpha{}^\nu \frac{\partial Q_\nu}{\partial q^{(2N-1)}{}_\beta}+\lambda_\beta{}^\nu \frac{\partial Q_\nu}{\partial q^{(2N-1)}{}_\alpha}=2N(D\lambda_\alpha{}^\nu) l_{\nu\beta} + 2N\lambda_\alpha{}^\nu (D l_{\nu\beta}) . \tag{3.20}$$

Condition (3.7) relates Q_ν and $l_{\alpha\beta}$ such that

$$\frac{\partial Q_\alpha}{\partial q^{(2N-1)}{}_\beta}+\frac{\partial Q_\beta}{\partial q^{(2N-1)}{}_\alpha}=2N(D l_{\alpha\beta}) . \tag{3.21}$$

Then, inserting (3.21) into (3.20), multiplying the resulting equation by $(l^{-1})^\beta{}_\rho$, and summing over β we find

$$(l^{-1})^\beta{}_\rho \lambda_\beta{}^\nu \frac{\partial Q_\nu}{\partial q^{(2N-1)}{}_\alpha}-(l^{-1})^\beta{}_\rho \lambda_\alpha{}^\nu \frac{\partial Q_\beta}{\partial q^{(2N-1)}{}_\nu} =2N(D\lambda_{\alpha\rho}) . \tag{3.22}$$

From (3.14) it is easily seen that $\lambda_\alpha{}^\rho=\lambda_\beta{}^\gamma l_{\nu\alpha}(l^{-1})^{\beta\rho}$. Using this in Eq. (3.22) we have

$$2N(D\lambda_\alpha{}^\rho)=\lambda_\mu{}^\rho (l^{-1})^{\mu\nu}\frac{\partial Q_\nu}{\partial q^{(2N-1)}{}_\alpha} -\lambda_\alpha{}^\nu (l^{-1})^{\beta\rho}\frac{\partial Q_\beta}{\partial q^{(2N-1)}{}_\nu} . \tag{3.23}$$

Finally for $\alpha=\rho=\gamma$ and summing over γ we arrive at

$$\frac{d}{dt}(\mathrm{tr}\,||\lambda_\alpha{}^\beta||)=0 . \tag{3.24}$$

This result can easily be generalized for the trace of integral powers of the matrix $\Lambda=||\lambda_\alpha{}^\beta||$. In fact, using (3.23) in the relation $D[\mathrm{tr}(\Lambda^m)]=m(\Lambda^{m-1})_\rho{}^\alpha \dot{\lambda}_\alpha{}^\rho$ for any integer m, and taking into account that $(\Lambda^{m-1})_\rho{}^\alpha \lambda_\mu^\rho=(\Lambda^{m-1})_\mu{}^\rho \lambda_\rho{}^\alpha$, as can easily be established, it is a straightforward matter to show that

$$\frac{d}{dt}[\mathrm{tr}(\Lambda^m)]=0, \quad m=1,2,\ldots . \tag{3.25}$$

Condition (3.25) is necessary in order to get equivalent higher-order Lagrangians and it is worth noting the general validity of this result in the various forms of the Lagrangian formalism. We have established it here for generalized mechanics. Its validity for usual mechanics was already mentioned in Sec. I. For field theory it was proved by Farias and Teixeira,[14] and in Ref. 15 it was proved for the first-order Lagrangian formalism.

With (3.25) we have not completely solved the equivalent problem in generalized mechanics. In fact this result is a consequence of the analysis of only one subset, namely, conditions (3.6a) and (3.7a), of the whole set of Boehm's conditions to be imposed on $\bar{P}_\alpha$ given by (3.12). Thus, condition (3.25) is not enough to guarantee the existence of $\bar{L}$. In the absence of the general solution we still can treat each problem individually, with (3.25) as a guide. For the simplest case, $N=2, M=1$, of an acceleration-dependent Lagrangian function the $\lambda_{\alpha\beta}$ reduce to a single function $\lambda(t,q,\dot{q},\ddot{q})$ which must be such that $\dot{\lambda}=0$. However, one cannot conclude that λ is a constant of motion in the usual sense. Indeed, we usually say that some dynamical function R is a constant of motion if $\dot{R}=0$ with the equations of motion being taken into account. Here $\lambda=\lambda(t,\ldots,\ddot{q})$ is of second order while the equation of motion is of fourth order. Thus $\lambda=c$, where c is a number. This appears as a surprising result, characteristic of the higher-order Lagrangian formalism for $M=1$. In this case L and $\bar{L}$ will result, trivially related by a constant [it being also permitted that $\bar{L}-L=D(\psi)$, for a suitable function ψ].

We conclude this section by considering the Lagrangian function[8]

$$L=\frac{1}{2}m\left[\mathbf{X}^2-\frac{1}{\omega^2}\ddot{\mathbf{X}}^2\right] , \tag{3.26}$$

where $\mathbf{X}=(q_1,q_2,q_3)$ in our notation. It is not difficult to find

$$\Lambda=\left|\left|\begin{matrix}0&0&1\\0&1&0\\1&0&0\end{matrix}\right|\right|$$

as a suitable matrix. In this case the set $\bar{P}_\alpha$ corresponding to the unknown $\bar{L}$ is seen from (3.8) to be $\bar{P}_1=P_3$, $\bar{P}_2=P_2$, $\bar{P}_3=P_1$, where

$$P_\alpha=-m\ddot{q}_\alpha-\frac{m}{\omega^2}q_\alpha^{(4)}=0, \quad \alpha=1,2,3 \tag{3.27}$$

are the Euler-Lagrange equations associated with L. Then $\bar{L}$ can be constructed from $\bar{P}_\alpha$ using, for instance, Ref. 16. It follows that

$$\bar{L}=m\dot{q}_1\dot{q}_3+\frac{m}{2}\dot{q}_2{}^2-\frac{m}{\omega^2}\ddot{q}_1\ddot{q}_3-\frac{m}{2\omega^2}\ddot{q}_2{}^2$$

is s equivalent to L.

IV. CONCLUSIONS

We have considered the equivalence problem for generalized mechanics from two different approaches, in one of which we were able to state a basic result, condition (3.25), in the search for s-equivalent generalized Lagrangians. In the other approach we derived a theorem, theorem II, that allows the setting up of s-equivalent Lagrangians in configuration spaces of different dimensions. Then using this theorem in the search of s equivalence in higher-order mechanics we found a weaker class of equivalence which we have called semiequivalence.

Theorem II seems to be an interesting result for the analysis of higher-order systems. We have also pointed out the possibility of discussing constrained systems with the help of theorem II. Another interesting question is the problem of finding constants of motion associated with a generalized Lagrangian. Nowadays the extensions of Noether's theorem for this case is well known,[8,17] but theorem II permits a new insight into this question. In fact, given a Lagrangian function $L(t,q_1,\dots,q_1^{(N)})$ one can find $\bar{L}(t,q_1,\dot{q}_1,P_J,\dot{P}_J)$ when applying theorem II. Now Noether's theorem may be used to find constants of motion associated with symmetries of $\bar{L}$. These constants are written in terms of the new coordinates and momenta $P_J,\dot{P}_J$. Then, what we must do in order to come back to the effective higher-order dependence is to eliminate the p's with the help of their definitions, (2.14). After this is done we obtain the constants of motion assigned to the given L. Now, there will result $2N$ independent constants of motion corresponding to $\bar{L}$, as it is known from elementary mechanics, and this will also be the number of independent constants corresponding to L as it is also easily deduced from the fact that the corresponding Euler-Lagrange equation is of order $2N$.

We conclude presenting an example. The Lagrangian function (3.26) was considered in a paper[8] where Noether's theorem was extended to generalized mechanics. After using theorem II we find

$$\bar{L}=\dot{\mathbf{Y}}\cdot\dot{\mathbf{X}}-\frac{1}{2}\frac{\omega^2}{m}\mathbf{Y}^2-\tfrac{1}{2}m\dot{\mathbf{X}}^2 , \tag{4.1}$$

where

$$\mathbf{Y}\equiv(Q_1,Q_2,Q_3)=-\frac{m}{\omega^2}\ddot{\mathbf{X}}=-\frac{m}{\omega^2}(\ddot{q}_1,\ddot{q}_2,\ddot{q}_3) . \tag{4.2}$$

From (4.1) one sees that the $\mathbf{X}=(q_1,q_2,q_3)$ are cyclic coordinates. Hence the corresponding momenta are constants of motion:

$$\bar{\mathbf{p}}_1=\dot{\mathbf{Y}}-m\dot{\mathbf{X}}=\text{const} . \tag{4.3}$$

Using (4.2) we find the corresponding constant in higher-order mechanics:

$$\mathbf{p}_1=-\frac{m}{\omega^2}\mathbf{X}^{(3)}-m\dot{\mathbf{X}} . \tag{4.4}$$

This result, except for the sign, corresponds to the same constant as found in Ref. 8 associated with the invariance under the groups of space translations. Another constant which can easily be obtained from $\bar{L}$ is the "energy"

$$H=\mathbf{p}_Y\cdot\mathbf{p}_X+\frac{\omega^2}{2m}\mathbf{Y}^2+\frac{m}{2}\mathbf{p}_Y^2 , \tag{4.5}$$

where $\mathbf{p}_Y=\dot{\mathbf{X}}$, $\mathbf{p}_X=\dot{\mathbf{Y}}-m\dot{\mathbf{X}}$. The corresponding constant in higher-order mechanics is

$$H=\dot{\mathbf{X}}\cdot\mathbf{p}_1+\frac{m}{2}\dot{\mathbf{X}}^2+\frac{m}{2\omega^2}\ddot{\mathbf{X}}^2 . \tag{4.6}$$

Again this is the same result as in Ref. 8 because (4.6) is, except for the sign, the generator of the group of time translations corresponding to (3.26).

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[1]R. M. Santilli, Ann. Phys. (N.Y.) **103**, 354 (1977); **103**, 409 (1977); **105**, 227 (1977); *Foundations of Theoretical Mechanics I, II* (Springer, New York, 1977).

[2]D. Currie and E. Saletan, J. Math. Phys. **9**, 967 (1966); Y. Gelmann and E. Saletan, Nuovo Cimento **18B**, 63 (1973); P. Havas, Nuovo Cimento **5**, 363 (1957); W. Sarlet and F. Cantrijn, Hadronic J. **1**, 101 (1978); J. Phys. A **16**, 1383 (1983); J. A. Kobussen, Acta. Phys. Austriaca **51**, 293 (1979); E. Engels, Nuovo Cimento **26B**, 481 (1975); M. Henneaux, J. Phys. A **1**, L93 (1982); R. W. Atherton and G. M. Homsy, Stud. Appl. Math. **54**, 31 (1975).

[3]M. Razavy, Can. J. Phys. **50**, 2037 (1972); F. J. Kennedy and E. H. Kerner, Am. J. Phys. **33**, 463 (1965); S. Okubo, Phys. Rev. D **22**, 919 (1980); S. Hojman and R. Montemayor, Hadronic J. **3**, 1644 (1980).

[4]J. R. Farias and L. J. Negri, J. Phys. A **16**, 707 (1983).

[5]S. Hojman and H. Harleston, J. Math. Phys. **22**, 1414 (1981).

[6]D. Anderson, J. Math. Phys. **14**, 934 (1973); M. Sz. Kirkovits, Acta. Math. Hung. **43**, 341 (1984).

[7]M. Borneas, Phys. Rev. **186**, 1299 (1969); Am. J. Phys. **27**, 265 (1959); **40**, 248 (1972); J. Koestler and J. Smith, *ibid.* **33**, 140 (1965); J. Kruger and D. Callebaut, *ibid.* **36**, 557 (1968); L. M. C. de Souza and P. R. Rodrigues, J. Phys. A **2**, 304 (1969).

[8]G. C. Constantelos, Nuovo Cimento **21B**, 279 (1974).

[9]C. Ryan, J. Math. Phys. **13**, 3 (1972); **13**, 283 (1972).

[10]C. A. P. Galvão and L.J. Negri, J. Phys. A **16**, 4183 (1983).

[11]E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* 4th ed. (Cambridge University Press, Cambridge, England, 1959), p. 266.

[12]J. R. Farias, Dr. Sc. thesis, 1983.

[13]K. Boehm, J. Reine Angew. Math. **121**, 124 (1900).

[14]J. R. Farias and N. L. Teixeira, J. Phys. A **16**, 1517 (1983).

[15]S. Hojman and J. Gomes, J. Math. Phys. **25**, 1776 (1984).

[16]A. L. Vanderbauwhede, Hadronic J. **1**, 1177 (1978).

[17]J. D. Logan and J. S. Blakeslee, J. Math. Phys. **16**, 1374 (1975); Dj. S. Djukic and A. M. Strauss, J. Phys. A **13**, 431 (1980).