Gauge field theory of gravity

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We propose a Yang-Mills field theory of gravity based on a unitary phase gauge invariance of the Lagrangian where the gauge transformations are those of the $SU(2)\times U(1)$ symmetry of the two spinors. In a classical limit this microscopic theory results in Einstein's metrical theory of gravity, where we restrict ourselves in a first step to its linearized version. Furthermore, only the case of massless particles is treated explicitly.

I. INTRODUCTION

In Einstein's general relativity the gravitational interaction is described by the metric field of space-time based on the equivalence of inertial and (passive) gravitational mass (equivalence principle). According to Einstein's field equations the source of the metric field (gravitational field) is the energy momentum stress tensor of matter, which is given ultimately because of the quantum structure of matter by the expectation values of the energy-momentum stress operators. Thus Einstein's description of gravity is a purely *macroscopic* one and on this level in best accordance with the known experimental data at least within the first-order approximation.[1] Accordingly, the classical metrical gravitational field represents the background of space-time, on which all other physics happens, which reacts back on the metrical field according to Einstein's field equations.

But just here the first problem arises: The back reaction of a single quantum process of matter on the classical metrical field cannot be considered consistently in Einstein's theory.[2] Furthermore, on the purely microscopic level neither a theory nor experiments of gravitational interactions exist today. However, there is some evidence that, for the gravitational interaction of elementary particles, the metrical description of gravity may not be the suitable one.

First of all, all attempts of quantization of the metric field have been without success. Second, only in the case of a *classical* particle does the mass drop out of the equation of motion of gravitation as a consequence of the equivalence principle, so that all masses are moved in the same way by gravity and the gravitational field can be described by a geometrical "guiding field." But in contrast with this, the motion of a *quantum* particle state is *not* independent of the particle mass even in the case of the presence of a *classical* gravitational field:[3] In spite of the validity of the equivalence principle the particle mass does not drop out of the Schrödinger equation. Thus no universal guidance of the quantum processes by gravity[4] exists and consequently there is no hint of the necessity for geometrization of the gravitational interaction on the quantum level.

Finally, Einstein's metrical theory of gravitation has a completely different structure than the successful gauge theories of electroweak and strong interactions. Therefore a unification of all interactions needs a modification of the standard description of gravity. The possibilities discussed until now mainly consist of supergravity[5] and modern Kaluza-Klein theories,[6] where the metric is still retained as a fundamental structure of gravitation on the microscopic level in both cases. But these models have other problems, especially, for instance, the necessity of the existence of too many exotic elementary particles.

For this reason we want to point in this paper to the possibility of the construction of a *unitary gauge theory of gravity* on the microscopic level in accordance with the gauge theories of electroweak and strong interactions. Then also the gravitational interaction would be produced by the exchange of usual vector gauge bosons on the background of the Minkowski space-time of special relativity. According to this, on the quantum level, gravitation would not be described by a metric tensor field, but by gauge vector fields.[7] Only in a certain classical limit, i.e., by cooperation of many vector bosons, must Einstein's metrical description of classical gravity result at least in its tested linearized version. According to this line the non-Euclidean metric of space-time is not a fundamental, but more an effective field produced by a combination of gauge vector fields like a condensation phenomenon, and has only a *classical* meaning describing the *classical* gravitational field alone. We emphasize that we pursue here the idea that the quantum or microscopic physics possesses the priority and that all macroscopic physics must be deduced as its classical limit, whereas the microphysics itself follows directly from few very general first principles.

II. THE GROUP-THEORETICAL CONCEPT

Following the mentioned line, the first aim of this paper is a quantum-theoretical description of the gravitational interaction between elementary particles by a usual gauge field theory based on a phase gauge invariance of the Lagrangian, where with respect to the interpretation of the quantum theory—as is well known—only unitary gauge transformations are allowed. Furthermore, because gravity couples to *all* particles, a gauge transformation is needed, which arises as the transformation of the intrinsic spinor structure of each particle and not as a transformation between different particles of any multiplet. Finally, gravity is connected with the concept of "mass." However this concept is still a classical one and should therefore be avoided in a quantum approach of gravity. In accordance with the modern gauge-theoretical treatment of interactions one has to start with massless particles and the mass appears subsequently by a dynamical process, namely, as a consequence of spontaneous symmetry breaking. As we shall see in Sec. IV, this is also the only passable way for obtaining Einstein's tensor theory in a classical limit starting from a microscopic vector gauge theory of gravity. Doing so, it may even be expected that the empirical fact of the equivalence between the inertial and gravitational mass of particles, which is the basis of general relativity, can be founded theoretically on the microscopic level.

However, in the following, we do not want to discuss the mass problem in detail because this would go beyond the scope of this paper. We consider only the investigation of the gravitational interaction of massless particles in a first step. Thus we start from the general transformation group GL(2,*C*) of the two-spinors as the most general symmetry group. On the other hand, all generators of this group can be constructed by complex linear combinations of the generators of SU(2)$\times$U(1), i.e., of the unit and Pauli matrices.[8] Thus instead of a gauge theory of the complete group GL(2,*C*) we suppose that it is sufficient to gauge only the SU(2)$\times$U(1) symmetry by which automatically the basic quantum-theoretical requirement of the unitarity of the phase gauge transformations is fulfilled. This idea is supported by the mathematical fact that one can reduce the group GL(2,*C*) to its maximal compact subgroup U(2) by contraction.[9]

On the other hand, as is well known, the group SL(2,*C*) is the covering group of the Lorentz group. From this point of view we gauge indeed the Lorentz group in the sense that we gauge its "basic" transformations,[8] namely, those of SU(2). However, we consider the gauge group only as an *internal* group.

Although we start from the two-spinor calculus for massless particles we perform our investigations mainly in the four-spinor representation, that is, in the 4×4 representation of SU(2)$\times$U(1) with respect to a later consideration of particle masses. Furthermore, we restrict ourselves to a treatment of the theory on the level of the first quantization. However, in the case of the compact gauge group SU(2)$\times$U(1), no problems exist with quantization: The theory is renormalizable in the usual way and it gives no ghost fields.

According to our gauge group there exist four gauge-vector fields and it is possible to introduce two or, with respect to the reducibility of the 4×4 representation of SU(2)$\times$U(1), even three different gauge coupling constants. However, because all gauge bosons mediate the same interaction between all spin-$\frac{1}{2}$ particles, namely, gravity, we give all these coupling constants the same value.[10] Under this presupposition, in the following we first construct a gauge theory of the SU(2)$\times$U(1) symmetry for gravity on the background of the Minkowski space-time and then we show that in a classical limit a metrical description of gravity for the expectation values results, where the metric satisfies Einstein's field equations; in this connection we restrict ourselves for simplicity to the linearized theory.

III. THE LAGRANGIAN OF MICROSCOPIC GRAVITY

We define the transformation matrices of the group SU(2)$\times$U(1) in their 4×4 representation by[11]

$$U=e^{i\lambda_a(x^\mu)\tau^a} \tag{3.1}$$

with

$$\tau^a=\frac{1}{2}\begin{pmatrix}\sigma^a & 0\\ 0 & \sigma^a\end{pmatrix} \tag{3.1a}$$

(σ^0 is the unit matrix and σ^1, σ^2, σ^3 are Pauli matrices). The commutator relations for the generators τ^a are

$$[\tau^b,\tau^c]=i\epsilon_a{}^{bc}\tau^a\ . \tag{3.1b}$$

Then the spinor state function ψ and the Dirac matrices γ^μ transform as

$$\psi'=U\psi,\quad \gamma'^\mu=U\gamma^\mu U^{-1}\ , \tag{3.2}$$

with the invariant relation

$$\gamma^{(\mu}\gamma^{\nu)}=\eta^{\mu\nu} \tag{3.2a}$$

[$\eta^{\mu\nu}=\eta_{\mu\nu}=\mathrm{diag}(-1,1,1,1)$ Minkowski metric]. Accordingly the covariant spinor derivative reads

$$D_\mu\psi=(\partial_\mu+ig\omega_\mu)\psi \tag{3.3}$$

(g is the uniform gauge coupling constant) with the SU(2)$\times$U(1) gauge fields $\omega_{\mu a}$ defined by

$$\omega_\mu=\omega_{\mu a}\tau^a\ , \tag{3.3a}$$

which transform under gauge transformation as[12]

$$\omega'_\mu=U\omega_\mu U^{-1}+\frac{i}{g}U_{|\mu}U^{-1}\ . \tag{3.4}$$

Furthermore it follows from (3.2) that the γ^μ are only covariantly constant, i.e.,

$$D_\alpha\gamma^\mu\equiv\gamma^\mu{}_{|\alpha}+ig[\omega_\alpha,\gamma^\mu]=0\ , \tag{3.5}$$

which is, together with the condition (3.2a), the determination equation for γ^μ in addition to the field equations (3.11) and (3.12).

Now we define the gauge-field strength of microscopic gravity in the usual way by the commutator of the covariant derivative given in (3.3):

$$F_{\mu\nu}\equiv\frac{1}{ig}[D_\mu,D_\nu]=\omega_{\nu|\mu}-\omega_{\mu|\nu}+ig[\omega_\mu,\omega_\nu]\ . \tag{3.6}$$

For the components of $F_{\mu\nu}$ with respect to τ^a it follows immediately with the use of (3.3a) and (3.1b) that

$$F_{\mu\nu a}=\omega_{\nu a|\mu}-\omega_{\mu a|\nu}-g\epsilon_a{}^{bc}\omega_{\mu b}\omega_{\nu c} \tag{3.7}$$

satisfying the Bianchi identities

$$F_{a[\mu\nu|\lambda]}+g\epsilon_a{}^{bc}F_{b[\mu\nu}\omega_{\lambda]c}=0\ . \tag{3.8}$$

In the case of the *pure* gauge field

$$\omega_\mu\equiv\omega_\mu^{(0)}=\frac{i}{g}U_{|\mu}U^{-1} \tag{3.9a}$$

it holds that

$$F_{\mu\nu} \equiv 0, \quad F_{\mu\nu a} \equiv 0 . \tag{3.9b}$$

With the principle of minimal coupling the gauge-invariant Lagrange density of gauge gravity for a massless particle takes the very simple form

$$L = -\frac{\hbar}{16\pi k} F_{\mu\nu a} F_b^{\mu\nu} \eta^{ab} + \frac{\hbar}{2} i \bar{\psi} \gamma^\mu D_\mu \psi + \text{H.c.} \tag{3.10}$$

Here we have employed as group-space metric the Minkowski matrix, η^{ab}, because, with respect to the classical limit of this microscopic theory, we need the U(1) gauge potential as timelike and those of SU(2) as spacelike.[13] Furthermore we have taken the opportunity to introduce, aside from the gauge coupling constant g, a second coupling constant k between the two gauge-invariant parts of the Lagrange density. Both coupling constants must be determined by experiment or by the transition to the macroscopic classical limit, where k comes out to be proportional to the Newtonian gravitational constant G, whereas g does not possess a classical limiting value and has only the meaning of a microphysical "heavy charge."[20] But both coupling constants are necessary in the following and cannot be substituted for one another.

The field equations following from the action principle belonging to (3.10) are given by

$$\partial_\mu F^{\mu\nu a} + g \epsilon^{abc} F_b^{\mu\nu} \omega_{\mu c} = 2\pi k g \bar{\psi} \{\gamma^\nu, \tau^a\} \psi \tag{3.11}$$

and, using (3.5), by

$$\gamma^\mu (\partial_\mu + i g \omega_\mu) \psi = 0 . \tag{3.12}$$

Finally the gauge-invariant canonical energy-momentum tensor of the whole system takes the form, with respect to (3.12),

$$T_\mu{}^\nu = \tfrac{1}{2} i \hbar (\bar{\psi} \gamma^\nu D_\mu \psi - \bar{D}_\mu \bar{\psi} \gamma^\nu \psi) - \frac{\hbar}{4\pi k} (F_{\mu\alpha a} F^{\nu\alpha a} - \tfrac{1}{4} F_a^{\alpha\beta} F_{\alpha\beta}^a \delta_\mu{}^\nu) . \tag{3.13}$$

It has the property with the use of the field equations

$$T_\mu{}^\nu{}_{|\nu} = 0 . \tag{3.13a}$$

Evidently in our microscopic theory of gravitation there exist four gauge-vector fields $\omega_{\mu a}$, which couple to four "heavy" gauge currents of matter (ψ field) as usual in Yang-Mills field theories:

$$j^{\mu a} = \frac{g}{2} \bar{\psi} \{\gamma^\mu, \tau^a\} \psi . \tag{3.14}$$

With respect to the fact that the U(1) gauge current $j^{\mu 0}$ is proportional to the timelike probability current density and therefore correlated with the four-momentum density of the particle field, it is expected that the field equations (3.11) and (3.12) are indeed those for gravity. This supposition will be confirmed subsequently by investigation of the classical limit of this microscopic theory.

IV. THE CLASSICAL LIMIT

We can be sure that the theory proposed in Sec. III is a theory of gravity, if in the classical limit Einstein's metric theory of gravitation results. For this it must be shown first that according to Ehrenfest's theorem the expectation values of the four-momentum of the particle field (ψ field) satisfy the usual equations of motion for classical gravity. This means in detail that from the equations of motion the classical gravitational force is to be read off, which must be reducible to geometrical connection coefficients (macroscopic field strengths), because classical gravity is a geometrical theory. Second, it must be shown that the connection coefficients are metric and third, that they satisfy Einstein's field equations with the energy-momentum tensor of the ψ field (matter field) as a source. In doing this we restrict ourselves for simplicity to the linearized Einstein theory as the only empirically confirmed part of the complete theory.

A. The equation of motion

We start from the result (3.13a). Then we have[14]

$$T_\mu{}^0{}_{|0} + T_\mu{}^j{}_{|j} = 0 . \tag{4.1}$$

After insertion of (3.13) into (4.1) and integration over the three-dimensional hypersurface $t = \text{const}$ we obtain

$$\partial_0 \int \tfrac{1}{2} i \hbar (\bar{\psi} \gamma^0 D_\mu \psi - \bar{D}_\mu \bar{\psi} \gamma^0 \psi) d^3x = \frac{\hbar}{4\pi k} \left[\int (F_{\mu\alpha a} F^{\beta\alpha a})_{|\beta} d^3x - \tfrac{1}{4} \int (F_a^{\alpha\beta} F_{\alpha\beta}^a)_{|\mu} d^3x \right] , \tag{4.2}$$

where boundary integrals concerning the pure ψ field are dropped out with respect to the normalization condition for the wave function $\int \psi^\dagger \psi \, d^3x = 1$.

Considering (3.13) the *usual* canonical energy-momentum tensor of the matter field is given by

$$T_\mu{}^\nu(\psi) = \tfrac{1}{2} i \hbar (\bar{\psi} \gamma^\nu \psi_{|\mu} - \bar{\psi}_{|\mu} \gamma^\nu \psi) . \tag{4.3}$$

Herewith Eq. (4.2) takes the form

$$\partial_0 \int \left[T_\mu{}^0(\psi) - \frac{\hbar}{2} g \bar{\psi} (\gamma^0 \omega_\mu + \omega_\mu \gamma^0) \psi \right] d^3x = \frac{\hbar}{4\pi k} \int [(F_{\mu\alpha a} F^{\beta\alpha a})_{|\beta} - \tfrac{1}{4} (F_a^{\alpha\beta} F_{\alpha\beta}^a)_{|\mu}] d^3x . \tag{4.4}$$

Here the large parentheses on the left-hand side represent the gauge-invariant canonical four-momentum density of the particle field following from (4.3) by replacing the ordinary derivatives through the covariant ones. However, the second term on the left-hand side of (4.4) is of the order of the coupling constant g compared with the first term and therefore can be neglected within a first-order approximation with respect to gravitational interaction. Then one obtains with the help of the Bianchi identities (3.8) and the field equations (3.11) the following equation of motion for the expectation value of the four-momentum of the ψ field:

$$\partial_0 \int T_\mu{}^0(\psi) d^3x = \hbar \int F_{\mu\alpha a} j^{\alpha a} d^3x . \tag{4.5}$$

Here on the right-hand side

$$K_\mu \equiv \hbar \int F_{\mu\alpha a} j^{\alpha a} d^3x = \int F_{\mu\alpha a} \frac{\hbar g}{2} \bar{\psi}\{\gamma^\alpha, \tau^a\}\psi\, d^3x \tag{4.6}$$

represents the classical gravitational four-force as a product between the gauge-field strengths and the gauge currents of matter.

In view of a classical *geometrical* interpretation the integrand in (4.6) must be reduced to a product between classical nonalgebra-valued connection coefficients and the energy-momentum tensor of the matter field. For this reason we write

$$K_\mu = \int F_{\mu\alpha b} \delta_a{}^b \frac{\hbar g}{2} \bar{\psi}\{\gamma^\alpha, \tau^a\}\psi\, d^3x \tag{4.6a}$$

and introduce as a "correspondence condition" between the quantum and the classical theory of gravity the following normalization of the gauge-vector fields:

$$\omega_a^\nu \omega_\nu^b = \delta_a{}^b + \epsilon_a{}^b(x^\mu) , \tag{4.7}$$

where we choose

$$|\epsilon_a{}^b| \ll 1 \tag{4.7a}$$

in view of a weak-field approximation of the theory.[15] According to this, in the case of the *pure* gauge fields (3.9a) where $F_{\mu\nu a} \equiv 0$, one has to take in (4.7) $\epsilon_a{}^b \equiv 0$ which is always possible. Then (small) deviations from the *pure* gauge-field case produce (small) values for $F_{\mu\nu a}$ and $\epsilon_a{}^b$ different from zero, so that the normalization condition (4.7) has no restrictive meaning. Accordingly, the U(1) gauge field $\omega_{\mu 0}$ is timelike, whereas the three SU(2) gauge fields are spacelike with respect to the group-space metric introduced in (3.10). Only in this way is the pseudo-Riemannian structure of the metric guaranteed later.

Consequently, in the case of the *linearized* theory the Kronecker symbol in (4.6a) can be substituted by the left-hand side of (4.7). One obtains immediately with the use of (3.3a):

$$K_\mu = \int F_{\mu\alpha b} \omega_\nu^b \frac{\hbar g}{2} \bar{\psi}\{\gamma^\alpha, \omega^\nu\}\psi\, d^3x . \tag{4.8}$$

By this procedure we have found that the product in (4.6a) is separated into two factors, both of which are (pseudo)scalars with respect to the gauge transformations and therefore can be interpreted classically: The first scalar in (4.8) corresponds to the connection coefficients, the second one should be related with the energy-momentum tensor of the matter field.

For investigation of this last relation we eliminate in the anticommutator of (4.8) the ω^ν field with the help of the field equation (3.12). Then we get, using (3.2a)

$$\frac{\hbar g}{2} \bar{\psi}\{\gamma^\alpha, \omega^\nu\}\psi = \tfrac{1}{2} i\hbar (\bar{\psi}\gamma^\alpha \psi^{|\nu} - \bar{\psi}^{|\nu}\gamma^\alpha \psi) , \tag{4.9}$$

where the terms proportional to the spin operator $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, interacting with the gauge-field strength [in (4.8)], are neglected with regard to the classical limit. The comparison with (4.3) shows that the right-hand side of (4.9) is identical with the usual canonical energy-momentum tensor $T^{\nu\alpha}(\psi)$ of the matter field. We mention explicitly, that this essential result, which connects the gauge currents with the energy-momentum tensor and which makes possible lastly the transition to Einstein's theory, is valid only in the absence of the mass term in the field equation (3.12).

Hence with the use of (4.9) Eq. (4.8) takes the form

$$K_\mu = \int F_{\mu\alpha a} \omega_\nu^a T^{\nu\alpha}(\psi) d^3x , \tag{4.10}$$

which has now the structure of the usual gravitational force. We note that as a consequence of relation (4.9) the gauge coupling constant g no longer appears explicitly in (4.10).

For the full linearized classical limit of the equation of motion one has to use in (4.10) the energy-momentum tensor $T^{\mu\nu}(\psi)$ in the lowest WKB approximation of the free particle field. One finds from (4.3) (see the Appendix)

$$T^{[\mu\nu]}(\psi) = 0, \quad T_\mu{}^\mu(\psi) = 0 , \tag{4.11}$$

according to which $T^{\mu\nu}(\psi)$ is *symmetric* and *traceless*. The first property results from neglecting the spin of the particles and the last one is a direct consequence of the masslessness of the particles and valid only in that case.

Applying this result to relation (4.10) we find for the equation of motion (4.5) (Ref. 16)

$$\partial_0 \int T^{\mu 0}(\psi) d^3x = \int F^\mu_{a(\alpha} \omega_{\nu)}^a T^{\nu\alpha}(\psi) d^3x , \tag{4.12}$$

which we have now to compare with the corresponding one according to general relativity:

$$\partial_0 \int T^{\mu 0}(\psi) d^3x = -\int \Gamma^\mu{}_{\alpha\nu} T^{\nu\alpha}(\psi) d^3x - \int \Gamma^\nu{}_{\nu\alpha} T^{\mu\alpha}(\psi) d^3x . \tag{4.13}$$

The last term on the right-hand side of (4.13) vanishes because of $\Gamma^\nu{}_{\nu\alpha} = 0$ in the case of a traceless energy-momentum tensor [see (4.11)] as a source of the classical gravitational field [cf. Eq. (4.32)]. Then we find by comparison of (4.12) and (4.13) the following correlation between the general geometrical connection coefficients $\tilde{\Gamma}^\mu{}_{\alpha\nu}$ and the microscopic gauge-field strengths $F_{\mu\nu a}$:

$$\tilde{\Gamma}^\mu{}_{\alpha\nu} = -F^\mu_{a(\alpha} \omega_{\nu)}^a . \tag{4.14}$$

Herewith we have arrived at a *geometrical* description of the gravitational force in the classical limit of our microscopic gauge theory without using any principle like the equivalence principle.

B. The metric.

In order to find out the metrical structure of our connection coefficients we reform the right-hand side of Eq. (4.14) by inserting the definition (3.7) and obtain

$$\tilde{\Gamma}^\mu{}_{\alpha\nu} = \tfrac{1}{2}\eta^{\mu\lambda}[(\omega_{\lambda a}\omega_\nu^a)_{|\alpha} + (\omega_{\lambda a}\omega_\alpha^a)_{|\nu} - (\omega_{\alpha a}\omega_\nu^a)_{|\lambda}] - \tfrac{1}{2}(\omega^a_{\nu|\alpha} + \omega^a_{\alpha|\nu})\omega_a^\mu . \tag{4.15}$$

Herein the bracket has indeed the structure of a *linearized* Christoffel symbol $\Gamma^\mu{}_{\alpha\nu}$ with the metric defined by

$$g_{\mu\nu}\equiv\omega_{\mu a}\omega^a_\nu\ . \tag{4.16}$$

Then the connection (4.15) is a *metrical* one after choosing the "gauge-fixing condition"

$$\omega^a_{(\mu\,|\,\nu)}=0 \tag{4.17}$$

as a constraint for the classical metrical limit of our theory, which is realizable in the lowest WKB limit of Eq. (3.11).

However, we are aware of the fact that the metrical condition (4.17) is strong, because it cannot be fulfilled in general by a gauging alone; in this way the following two consequences of (4.17) are achievable: The generalized Lorentz gauge

$$\omega^{\mu a}{}_{|\mu}=0 \tag{4.17a}$$

and

$$\omega^a{}_{(\mu\,|\,\nu)}\omega^\nu_a=0\ . \tag{4.17b}$$

Consequently the full condition (4.17) restricts the set of admissible solutions for construction of the macroscopic classical Einstein limit to those gauge fields, which are at least in the WKB-limit Killing vector fields with respect to the Minkowski metric. On the other hand, we remember that our theory is still incomplete because of the masslessness of the particles, in consequence of which the set of solutions is also restricted already on the microscopic level. There are some hints that these two things are correlated and that therefore a final judgment of the condition (4.17) will be possible only in connection with solving the mass problem, for instance, with the help of a Higgs field. Thus in the present form of the theory even an approximate fulfillment of the condition (4.17) satisfying (4.17a) and (4.17b) will be sufficient. Then $\tilde{\Gamma}^\mu{}_{\alpha\nu}\simeq\Gamma^\mu{}_{\alpha\nu}$ is valid and the covariant derivative of the metric (4.16) vanishes:

$$g_{\mu\nu\,|\,\lambda}-\tilde{\Gamma}^\alpha{}_{\mu\lambda}g_{\alpha\nu}-\tilde{\Gamma}^\alpha{}_{\nu\lambda}g_{\mu\alpha}=0\ . \tag{4.18}$$

In the case of *pure* gauge fields (3.9a) with $F_{\mu\nu a}\equiv 0$ the connection coefficients (4.14) and (4.15) vanish identically and the metric (4.16) is reduced to the Minkowski metric $\eta_{\mu\nu}$ where in the normalization condition (4.7) $\epsilon_a{}^b\equiv 0$ is valid. In general by decomposition of the metric (4.16) according to

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu},\ |\,h_{\mu\nu}\,|\ll 1 \tag{4.19}$$

one finds with the use of (4.7) the following relations within the linearized theory:

$$\begin{aligned} h_{\mu\nu}&=\epsilon_a{}^b\omega_{\mu b}\omega^a_\nu\ ,\\ \epsilon_a{}^b&=\omega^{\mu b}\omega^\nu_a h_{\mu\nu}\ . \end{aligned} \tag{4.20}$$

Herewith the normalization condition (4.7) takes the form of the usual tetrad condition in general relativity

$$\omega^b_\mu\omega_{\nu a}g^{\mu\nu}=\delta^b{}_a\ , \tag{4.20a}$$

where $g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}$ with $g^{\mu\nu}g_{\nu\lambda}=\delta^\mu{}_\lambda$ (Ref. 17).

Of course, the metric (4.16) is not a gauge-invariant quantity (pseudoscalar). From this it follows that by a gauge transformation of the gauge fields a coordinate transformation of the metric field is induced. Here we restrict ourselves within the linearized theory to infinitesimal transformations. Then from (3.1), (3.3a), and (3.4) the gauge-field transformation follows ($|\,\lambda_a\,|\ll 1$):

$$\omega'_{\mu a}=\omega_{\mu a}-\frac{1}{g}\lambda_{a\,|\,\mu}-\epsilon_a{}^{bc}\lambda_b\omega_{\mu c}\ . \tag{4.21}$$

Herewith we obtain according to (4.16) the "new" metric,

$$g'_{\mu\nu}=\eta_{\mu\nu}+h'_{\mu\nu}\equiv\omega'_{\mu a}\omega'^a_\nu\ , \tag{4.22}$$

where in view of (4.19)

$$h'_{\mu\nu}=h_{\mu\nu}-\frac{1}{g}\omega^a_\mu\lambda_{a\,|\,\nu}-\frac{1}{g}\omega^a_\nu\lambda_{a\,|\,\mu}\ . \tag{4.22a}$$

On the other hand, in the case of a transition from $g_{\mu\nu}$ to $g'_{\mu\nu}$ by the infinitesimal coordinate transformation $x'^\mu=x^\mu+\xi^\mu(x^\nu)$ (with $|\,\xi^\mu\,|\ll|\,x^\mu\,|$) there exists the well-known transformation law for the deviations from the Minkowski metric:

$$h'_{\mu\nu}=h_{\mu\nu}-\xi_{\mu\,|\,\nu}-\xi_{\nu\,|\,\mu}\ . \tag{4.23}$$

The comparison of (4.22a) and (4.23) yields immediately the following connection between infinitesimal gauge and coordinate transformations:

$$\xi^\mu=\frac{1}{g}\omega^{\mu a}\lambda_a\ . \tag{4.24}$$

However, as gauge transformations are only those allowed, which do not disturb the gauge-fixing condition (4.17), see (4.17a) and (4.17b). Accordingly, also the (infinitesimal) coordinate transformations are limited. As we see later, the gauge-fixing condition (4.17) implies on the level of the field equations the de Donder gauge for the metric, to which obviously the coordinate transformations are bounded, see (4.28b) and (4.29).

C. The field equations

After definition of the metric (4.16) through the gauge potentials it remains to be shown that the metric or its connection coefficients satisfy Einstein's linearized field equations. These must follow as a classical limit of the gauge-field equations (3.11). Therefore one has to remove at first the algebra valuedness of them. For this we multiply Eqs. (3.11) by the gauge fields $\omega_{\sigma a}$ and obtain, with the use of (3.3a) and (3.7),

$$\partial_\mu(F^\mu_{a\nu}\omega^a_\sigma)-F^\mu_{a\nu}F^a_{\mu\sigma}-F^\mu_{a\nu}\omega^a_{\mu\,|\,\sigma}=2\pi kg\bar{\psi}\{\gamma_\nu,\omega_\sigma\}\psi\ . \tag{4.25}$$

The right-hand side is with respect to the basic relation (4.9) already proportional to the energy-momentum tensor of the matter field, where terms containing the spin tensor $\sigma^{\mu\nu}$ are neglected with regard to the classical limit. On the left-hand side of (4.25) the second and the third terms can be neglected in view of the linearization. Thus we get

$$\partial_\mu(F^\mu_{a\nu}\omega^a_\sigma)=4\pi\frac{k}{\hbar}T_{\sigma\nu}(\psi)\ , \tag{4.26}$$

which is reduced with respect to (4.11) to its symmetric and traceless part:[16]

$$\partial_\mu(F^\mu_{a(\nu}\omega^a_{\sigma)})=4\pi\frac{k}{\hbar}T_{\nu\sigma}(\psi) \tag{4.27}$$

with

$$\partial_\mu(F^\mu_{a\nu}\omega^{\nu a})=0 . \tag{4.28}$$

Obviously, also in the classical limit of the field equations—as in the classical limit of the equations of motion—the gauge coupling constant g no longer appears explicitly, so that it cannot be determined by comparison with the classical Einstein theory.

Using (3.7) Eq. (4.28) implies

$$(\omega_{\mu a\,|\,\nu}-\omega_{\nu a\,|\,\mu})\omega^{\nu a}=0 , \tag{4.28a}$$

from which immediately with the help of (4.17a) the usual de Donder gauge of the metric follows [cf. (4.19)]:[18]

$$h_\mu{}^\nu{}_{|\nu}-\tfrac{1}{2}h_{|\mu}=0 \quad (h=h_\mu{}^\mu) . \tag{4.28b}$$

This relation will be finally reduced by the condition (4.17b) to the *special* de Donder gauge:

$$h_\mu{}^\nu{}_{|\nu}=0, \quad h_{|\mu}=0 . \tag{4.29}$$

Consequently the gauge-fixing conditions (4.17a) and (4.17b) induce the special de Donder gauge (4.29) of the metric for the case of traceless field equations. The remaining equation (4.27) results, with the use of (4.14) and remembering also (4.16) and (4.17), in

$$\partial_\mu\Gamma^\mu{}_{\nu\sigma}=-4\pi\frac{k}{\hbar}T_{\nu\sigma}(\psi) . \tag{4.30}$$

These results are now to be compared with Einstein's field equation for a traceless energy-momentum tensor in its linearized version ($c=1$):

$$\partial_\mu\Gamma^\mu{}_{\nu\sigma}-\partial_\sigma\Gamma^\mu{}_{\mu\nu}=-8\pi G T_{\nu\sigma} , \tag{4.31}$$

where with regard to (4.19)

$$\Gamma^\mu{}_{\mu\nu}=\tfrac{1}{2}h_{|\nu} . \tag{4.31a}$$

Taking into account explicitly the tracelessness of the Ricci tensor $R_{\mu\nu}$ (as a consequence of the tracelessness of the energy-momentum tensor) in the linearized form considering additionally the usual de Donder gauge (4.28b) we obtain, in view of (4.31a) and in agreement with (4.29),

$$h^{\mu\nu}{}_{|\mu|\nu}=0, \quad h^{|\mu}{}_{|\mu}=0 \Longrightarrow h_{|\mu}=0 \Longrightarrow \Gamma^\mu{}_{\mu\nu}=0 . \tag{4.32}$$

Then Eqs. (4.30) and (4.31) are identical by choosing the coupling constant k as follows:

$$k=2\hbar G . \tag{4.33}$$

Herewith Einstein's linearized field equations are arrived at as a classical limit of our microscopic gauge-field equations. Finally we note that, if the gauge fields $\omega_{\mu a}$ would become massive as a consequence of spontaneous symmetry breaking, the transition to Einstein's theory performed above would lead immediately to a cosmological constant Λ proportional to the square of the gauge-field masses.

V. CONCLUSIONS

As already mentioned in Sec. II the second quantization of our theory in its presented form is straightforward. However, there are some other points for discussion and remarks concerning an extension of the theory.

At first we note that according to condition (4.7) and the definition (4.16) the gauge fields $\omega_{\mu a}$ are dimensionless. Then in view of (3.3), the gauge coupling constant g, which has the meaning of a microscopic heavy charge [see (3.14)] and which disappears in the classical limit, has the dimension of a reciprocal length. This one may be the Planck length $\sqrt{\hbar G}$ and consequently g should be proportional to the Planck mass $\sqrt{\hbar/G}$ in accordance with the accepted strength of the gravitational interaction[19] (10^{19} GeV), rather than a new universal constant. However its true value remains undetermined within our present theoretical approach.[20]

But here exists the difficulty of any microscopic theory of gravity, namely, the impossibility of a direct confrontation with experiments in the near future. Thus for constructing such theories there remains only a foundation on most general and well-established first principles, such as the postulate of unitary compact gauge symmetries, and as indirect proof the transition to the tested macroscopic limit. In this respect we want to point to the fact that the transition from our microscopic vector field theory of gravity to Einstein's macroscopic tensor field theory with the energy-momentum tensor of matter as the source is not trivial, because the energy momentum tensor must be constructed from the currents of matter according to the vector field theory. But this procedure is possible, as we have seen in Eq. (4.9), only in the case of massless particles. Therefore, masslessness is a very essential presupposition of our model and mass can be introduced alone dynamically by spontaneous symmetry breaking. It is hoped that in this way—as already suggested in Sec. II—the equivalence principle, which has not been used within our approach, can be deduced. By all means, the classical concept of the geometrization of gravity as a consequence of the equivalence principle follows in our theory from the microscopic "minimal gauge principle."

In the present paper the gravitational interaction has been investigated only between (massless) fermions. On the other hand, as already stated in Sec. II, gravity coupled to *all* fields, i.e., also to bosonic ones. The construction of the gravitational interaction concerning bosons should be possible following the line of this paper by a grand unification of all interactions using higher U(N) symmetry groups acting on a high-dimensional spin-isospin space.

Finally the question arises whether it is possible to reach even Einstein's *nonlinear* theory. So far as we see, the answer to this question can be given only by the application of an iterative approximation method with all its difficulties. However it cannot be excluded that the limit in question can only be arrived at after a spontaneous symmetry breaking solving the mass problem simultaneously.

Note added in proof. In the static case the force (4.6) has, on the Yang-Mills level with the use of (4.7), a Coulomb form belonging to the U(1) part and a Yukawa-type form with a range of g^{-1} coming from the SU(2) part. This result may be of interest in view of the paper of E. Fischbach *et al.*, Phys. Rev. Lett. **56**, 3 (1986).

APPENDIX

With the WKB ansatz

$$\psi = \sum_{n=0}^{\infty} a_n(x^\mu)(-i\hbar)^n e^{(i/\hbar)S(x^\mu)} \tag{A1}$$

one gets in the limiting case $\hbar \to 0$ the leading expressions for the canonical energy-momentum tensor (4.3),

$$T^{\mu\nu}(\psi) = -\bar{a}_0 \gamma^\nu a_0 S^{|\mu} , \tag{A2}$$

and for the field equation (3.12) in the case of free particles,

$$\gamma^\mu a_0 S_{|\mu} = 0 . \tag{A3}$$

The only nontrivial solution of (A3) is, disregarding an irrelevant factor,

$$\bar{a}_0 \gamma^\mu a_0 \sim S^{|\mu}, \quad S^{|\mu} S_{|\mu} = 0 , \tag{A4}$$

which means that the probability current is lightlike in the lowest WKB limit, as it was to be expected in the case of massless free particles. Then the energy-momentum tensor (A2) for the free matter field reads, in its classical limit,

$$T^{\mu\nu}(\psi) \sim S^{|\mu} S^{|\nu}, \quad T_\mu{}^\mu(\psi) = 0 . \tag{A5}$$

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[1]As long as the solar mass quadrupole moment is unknown, the perihelion shift of Mercury is not a precise test with regard to the second-order approximation.

[2]Don N. Page and C. D. Geilker, Phys. Rev. Lett. **47**, 979 (1981).

[3]A. W. Overhauser and R. Collela, Phys. Rev. Lett. **32**, 1237 (1974); R. Collela, A. W. Overhauser, and S. A. Werner, *ibid.* **34**, 1472 (1975).

[4]L. Rosenfeld, in *Entstehung, Entwicklung und Perspektiven der Einsteinschen Gravitationstheorie,* edited by J. Treder (Akademie, Berlin, 1966); H. Hönl, Phys. Bl. **37**, 26 (1981).

[5]For a review, see P. Van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981).

[6]For a review see A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982); E. Witten, Nucl. Phys. **B186**, 412 (1981).

[7]After preparing this paper we have been informed that another attempt in this direction exists however with fully different assumptions, e.g., a Euclidean metric and a non-Yang-Mills Lagrangian: H. R. Pagels, Phys. Rev. D **29**, 1960 (1984).

[8]B. G. Wybourne, *Classical Groups for Physicists* (Wiley, New York, 1974); E. Merzbacher, *Quantum Mechanics,* 2nd ed. (Wiley, New York, 1970), p. 271.

[9]Ch. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic, London, 1983), Chap. 7.6.

[10]A deeper understanding of this ansatz can be expected from a later unification of all interactions.

[11]In the following Einstein's sum convention is used. Latin indices a, b, etc., run from 0 to 3 and are related to the basis of the group algebra. Greek indices run from 0 to 3 and are related to the space-time coordinates.

[12]The symbol $|\mu$ means the usual partial derivative with respect to the coordinate x^μ.

[13]Evidently the action corresponding to (3.10) is bounded from below after a Wick rotation of time, according to which not only the space-time metric but also the group-space metric goes over from a pseudo-Euclidean to a Euclidean structure.

[14]Latin indices j,k, etc., run from 1 to 3 and are related to the spacelike coordinates only.

[15]If we avoid in (3.3) the explicit appearance of a gauge coupling constant, this must be introduced at the latest in (4.7), see also (4.16).

[16]Of course, there exists also torsion represented by the antisymmetric part $F^\mu{}_{a[\nu}\omega_{\sigma]}{}^a$ of the connections. Then the antisymmetric part of the field equations (4.25) and (4.26) is the field equation for the torsion. However, we do not investigate it in this paper explicitly.

[17]Note, that in contrast with the remaining designation in this paper here $g^{\mu\nu} \neq \eta^{\mu\sigma}\eta^{\nu\rho}g_{\sigma\rho}$, but $g^{\mu\nu}$ is inverse to $g_{\mu\nu}$.

[18]See, e.g., L. D. Landau and E. M. Lifshitz, *Klassische Feldtheorie* (Akademie, Berlin, 1964), Sec. 101.

[19]Note that in this case the total coupling constant in (3.11) would be, with respect to (4.37), the square root of the gravitational constant, namely, $kg = \sqrt{\hbar G}$, in accordance with the situation in the electrodynamics.

[20]In this connection we point to the fact that our theory can be already considered as a unified theory of gravity and electromagnetism on the microscopic level in the sense that the U(1) part of the theory can be identified with the electrodynamics after choosing $g = (e^2/\hbar)^{1/2}/(2\hbar G)^{1/2}$ and $kg = e\sqrt{2G}$.