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Cosmological perturbations in Kaluza-Klein models

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Recent works have studied the classical cosmology of the early Universe in Kaluza-Klein models considered as product spaces. In this paper, the growth of cosmological perturbations is examined as a natural extension of established works concerning three-dimensional cases. Appropriate gauge-invariant quantities are defined, and the perturbed field equations are expressed in terms of these. Leading-order solutions are derived for the region in which most of the growth occurs. Difficulties with the models and the sensitivity to model variations are discussed.

I. INTRODUCTION

This paper may be viewed as serving two purposes. First, it is a generalization of Bardeen's work,[1] which established a framework in which cosmological perturbations can be studied in a gauge-invariant manner. Here we consider Kaluza-Klein models, in which the background space naturally decomposes into a direct product $M_d \times M_D$. M_d (with $d=3$) is the three-dimensional space with which we are familiar, and M_D is a D-dimensional compact space which currently has a very small characteristic size $R_{\rm KK}$ (of order similar to the Planck length) and is essentially static. At early times, of course, the "extra" space is expected to play an important dynamical role. Second, the paper is a sequel to earlier work[2,3] studying the background classical behavior of this scenario. In the earlier papers, and in many other recent works studying Kaluza-Klein cosmology,[4] it has been found that there is a period during which the characteristic scale size of the extra dimensions collapses to its final value, forcing a substantial amount of inflation of our dimensions as it does so. The amount of inflation may be made arbitrarily large, thus solving the so-called flatness problem. The question as to whether the entropy problem (why the total entropy in the comoving volume of the currently observable Universe is so large, viz., $\sim 10^{88}$) has also been resolved is more a matter of taste. In Ref. 2 it was shown that this large number may be understood as a moderate excitation in each of a large (~ 40) number of dimensions; on the other hand, the total entropy is indeed an input parameter.

The paper is basically divided into two parts. First we derive differential equations for the linearized metric perturbations in general, i.e., without assuming any specific behavior of the background geometry. To do this we follow the methods of Ref. 1 in splitting a general perturbation into several pieces according to the spatial dependence. The problem is more complicated here because the background space is a direct product. We write

$$ds^2 = -dt^2 + r^2(t)\, {}^d g_{ij}(x^k) dx^i dx^j + R^2(t)\, {}^D g_{ab}(X^c) dX^a dX^b \,, \qquad (1.1)$$

where ${}^d g_{ij}$ and ${}^D g_{ab}$ are the metrics for M_d and M_D, respectively, assumed to describe spaces of constant curvature K_r, K_R. In later applications we will take M_d to be flat and M_D to be a D-sphere with $K_r = 0, K_R = 1.0$, but as yet we work in full generality. In the second part of the paper we find solutions to these equations assuming that the behavior of r, R is as described in Refs. 2 and 3. The dominant feature of this scenario is the extremely rapid inflation of r as R collapses to $R_{\rm KK}$. This inflation is given by a negative power of $r = t_0 - t$, where t_0 is the time at which R would reach zero were unknown (presumably quantum) effects not to halt its collapse. Thus, the inflation is different in character from the usual exponential growth, or even the power-law *Ansätze* of Ref. 5 where growth is as a positive power of t.

For reasons discussed later, this extremely rapid inflation of r leads to a reversal of the normal roles of the "growing" and "decaying" modes discussed in Ref. 1 and other papers,[6] and the singular behavior of the usual "decaying" mode results in extremely rapid growth of the perturbation amplitudes. Indeed, for scalar- and vector-type perturbations (defined more precisely later) the growth rate is such that for plausibly sized initial amplitudes, the perturbations will enter the nonlinear regime *before* the expansion terminates. This is, of course, a serious problem as the final amplitude of such perturbations when they reenter the horizon is generally expected to be $\sim 10^{-4}$, and it seems unduly optimistic to hope that nonlinearities could act in such a way as to keep the amplitudes small.

It is important to note, however, that the details of the singular behavior during collapse are probably the least reliable feature of the classical analysis. As discussed in Ref. 3, one might reasonably expect that quantum gravity effects will moderate this singular behavior, effectively slowing the expansion rate. While this issue is not essential to the overall evolution of the Universe, which is effectively controlled by global conservation considerations as discussed in Ref. 3, it is important for the present discussion. As a partial (if *ad hoc*) treatment of this problem, we will briefly consider the perturbation amplitudes, assuming that inflation behaves in an exponential rather than singular power fashion.

The organization of the paper is as follows. In Secs. II and III we define the problem to be solved and derive the relevant equations, paying particular attention to the perturbations of the metric rather than those of the energy-momentum tensor. In Sec. IV we solve the problem in the special case of Refs. 2 and 3 and discuss the results in Sec. V. Concluding remarks are presented in Sec. VI and the Appendix contains technical material necessary for a full understanding of the text. While the most technical details are reserved for the Appendix, Secs. II and III still contain considerable formalism. The reader who is primarily interested in the specific Kaluza-Klein model application may wish to first go directly to Sec. IV.

II. THE METRIC AND ENERGY-MOMENTUM TENSORS

We begin by defining the spatial dependence of the perturbations, as this will be used to split the calculations into several disjoint parts. In Ref. 1 this was done by classifying perturbations as scalar (S), vector (V), or tensor (T) according to whether the spatial dependence was defined in terms of solutions to a scalar, vector, or tensor Helmholtz equation. It should be noted that while, e.g., tensors may be constructed from a solution to a scalar Helmholtz equation, the reverse is not true—vector or tensor perturbations are intrinsically one- or two-index objects, and by construction, nontrivial tensorial objects with fewer indices cannot be derived from them. This section follows closely the treatment given in Ref. 1, so familiarity with that approach would be helpful. There are, however, certain notational differences.

The complication here is, of course, the fact that the background space is a direct product of the two spaces M_d and M_D, possibly with different curvatures and so without necessarily any overall $(d+D)$-dimensional rotational symmetry. Hence, within our framework there are no $(d+D)$-dimensional tensors, only D-dimensional or d-dimensional ones. Thus, a given perturbation has spatial dependence S, V, or T in each space separately, and an overall spatial dependence given by the product of these, so that there are more than three possible overall dependences. Some possibilities are excluded as we wish only to construct perturbations to the metric and energy-momentum tensor, both symmetrical two-index objects. Let us begin with intrinsically scalar quantities. These are derived from a scalar Helmholtz equation

$$q^{(0)|i}{}_{|i}+k_r^{(0)2}q^{(0)}=0\ ,\qquad Q^{(0)|a}{}_{|a}+k_R^{(0)2}Q^{(0)}=0\ , \tag{2.1}$$

where $q^{(0)}=q^{(0)}(x^i)$, $Q^{(0)}=Q^{(0)}(X^a)$, $i=1,\ldots,d$, $a=d+1,\ldots,d+D$. The bar denotes a covariant derivative with respect to indices in only one or the other space; covariant derivatives in the full space-time will be denoted by a semicolon. We may construct vector and tensor quantities from $q^{(0)}$ and $Q^{(0)}$, where the tensorial character refers to rotations within just one of the spaces. Note that indices are raised and lowered on q (Q) by ${}^dg_{ij}$ (${}^Dg_{ab}$) and *not*

$$g_{ij}=r^2\,{}^dg_{ij}(g_{ab}=R^2\,{}^Dg_{ab})\ .$$

So, following Ref. 1 we have

$$q_i^{(0)}=-\frac{1}{k_r^{(0)}}q^{(0)}_{|i},\quad q_{ij}^{(0)}=\frac{1}{k_r^{(0)2}}q^{(0)}_{|ij}+\frac{1}{d}\,{}^dg_{ij}q^{(0)}\ ,$$
$$Q_a^{(0)}=-\frac{1}{k_R^{(0)}}Q^{(0)}_{|a},\quad Q_{ab}^{(0)}=\frac{1}{k_R^{(0)2}}Q^{(0)}_{|ab}+\frac{1}{D}\,{}^Dg_{ab}Q^{(0)}\ , \tag{2.2}$$

which do not inherently solve a Helmholtz equation and may be reduced to scalar $q^{(0)}$'s or $Q^{(0)}$'s.

Intrinsically vector quantities are derived from solutions of

$$q^{(1)i|j}{}_{|j}+k_r^{(1)2}q^{(1)i}=0\ ,\qquad Q^{(1)a|b}{}_{|b}+k_R^{(1)2}Q^{(1)a}=0\ , \tag{2.3}$$

where $q^{(1)i}$ and $Q^{(1)a}$ are divergenceless and so cannot be reduced to scalars. From these we may construct tensors

$$q_{ij}^{(1)}=-\frac{1}{2k_r^{(1)}}(q^{(1)}_{i|j}+q^{(1)}_{j|i})\ ,\qquad Q_{ab}^{(1)}=-\frac{1}{2k_R^{(1)}}(Q^{(1)}_{a|b}+Q^{(1)}_{b|a})\ . \tag{2.4}$$

Intrinsically tensor quantities are solutions of

$$q^{(2)ij|k}{}_{|k}+k_r^{(2)2}q^{(2)ij}=0\ ,\qquad Q^{(2)ab|c}{}_{|c}+k_R^{(2)2}Q^{(2)ab}=0\ , \tag{2.5}$$

which are again divergenceless and are also traceless.

We are now in a position to see which types of perturbation can contribute to an arbitrary symmetric two-index tensor $A_{\mu\nu}$. We have

$$A_{\mu\nu}=\begin{bmatrix} A_{00} & A_{0j} & A_{0b}\\ A_{i0} & A_{ij} & A_{ib}\\ A_{a0} & A_{ai} & A_{ab}\end{bmatrix} \tag{2.6a}$$

and it is easy to see that, for example, $q_{ij}^{(2)}$ can contribute only to A_{ij}. We may illustrate which spatial dependences contribute where in the following way. Denote by, for example, SV a quantity which derives from $q^{(0)}$ for its x dependence and $Q^{(1)a}$ for its X dependence (regardless of what indices the quantity actually carries). Then possible contributions are

$$\begin{bmatrix} SS & SS,VS & SS,SV\\ SS,VS & SS,VS,TS & SS,VS,SV,VV\\ SS,SV & SS,VS,SV,VV & SS,SV,ST\end{bmatrix}, \tag{2.6b}$$

where the blocks in the matrix are the same as those in Eq. (2.6a).

Thus, we have six separate problems to solve (SS,SV,ST,VS,VV,TS), but for convenience will group them into three ($SS,SV+VS+VV,ST+TS$) and refer to these loosely as scalar, vector, and tensor problems.

The physical interpretation of the various components is basically determined by the nature of the d-dimensional part of the spatial dependence, since that is where future observers of the perturbations will reside. Thus the TS part, with spatial dependence $q^{(2)i}{}_jQ^{(0)}$ represents gravitational waves in the ordinary space. The ST part, representing gravitational waves in the extra space, will be interpreted as matter perturbations in the ordinary dimensions. So the physical classification will be *TS* for gravitational waves, *VS*,*VV* for vector matter perturbations, and *SS*,*SV*,*ST* for scalar matter perturbations. It will be observed that this classification differs from the way in which the problem naturally splits from the purely mathematical point of view.

We may now parametrize the perturbations to the metric and energy-momentum tensors. The spatial dependence of the perturbations is given by one or other of the above possibilities, while the amplitudes will be time-dependent functions. We consider each of the scalar, vector, and tensor problems in turn.

A. The scalar problem

The background space-time metric is [from Eq. (1.1)]

$$\begin{aligned} g_{00}&=-1\ ,\\ g_{0i}&=g_{0a}=g_{ia}=0\ ,\\ g_{ij}&=r^{2\,d}g_{ij}\ ,\\ g_{ab}&=R^{2\,D}g_{ab}\ . \end{aligned} \tag{2.7a}$$

This gives a background Ricci tensor

$$R^0{}_0=-\left[d\frac{\ddot r}{r}+D\frac{\ddot R}{R}\right],$$

$$R^i{}_j=-\delta^i{}_j\left[\left[\frac{\dot r}{r}\right]^{\cdot}+\frac{\dot r}{r}\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]+(d-1)\frac{K_r}{r^2}\right], \tag{2.7b}$$

$$R^a{}_b=-\delta^a{}_b\left[\left[\frac{\dot R}{R}\right]^{\cdot}+\frac{\dot R}{R}\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]+(D-1)\frac{K_R}{R^2}\right],$$

where an overdot denotes a time derivative. The perturbed metric may be parametrized as

$$\begin{aligned} g_{00}&=-(1+2Aq^{(0)}Q^{(0)})\ ,\\ g_{0i}&=-rb^{(0)}q_i^{(0)}Q^{(0)}\ ,\\ g_{0a}&=-RB^{(0)}q^{(0)}Q_a^{(0)}\ ,\\ g_{ij}&=r^2[(1+2h_Lq^{(0)}Q^{(0)})\,{}^dg_{ij}+2h_T^{(0)}q_{ij}^{(0)}Q^{(0)}]\ ,\\ g_{ab}&=R^2[(1+2H_Lq^{(0)}Q^{(0)})\,{}^Dg_{ab}+2H_T^{(0)}q^{(0)}Q_{ab}^{(0)}]\ ,\\ g_{ia}&=2rRG^{(0)}q_i^{(0)}Q_a^{(0)}\ , \end{aligned} \tag{2.7c}$$

where $A,\dots,G^{(0)}$ are (assumed small for linearized theory to be valid) functions of t. Note that in contrast with Ref. 1 we do not use conformal time.

Now consider the energy-momentum tensor. In the background we take this to be given by

$$T^\mu{}_\nu=p\delta^\mu{}_\nu+(\rho+p)u^\mu u_\nu\ , \tag{2.8a}$$

where u^μ is the fluid velocity, ρ the energy density, and p the pressure. The Einstein equations are then given by

$$\delta G^\mu{}_\nu\equiv\delta R^\mu{}_\nu-\tfrac{1}{2}\delta^\mu{}_\nu\delta R^\lambda{}_\lambda=-8\pi\overline{G}\delta T^\mu{}_\nu\ , \tag{2.8b}$$

where $\overline{G}$ is the $(d+D)$-dimensional Newton's constant and henceforth we set $8\pi\overline{G}=1$. This generalization of the perfect-fluid form is sufficient for illustrative purposes; we do not wish to make detailed assumptions about the matter content of such models. One obvious further generalization would be to use different pressures p and P in the ordinary and extra dimensions. Here we restrict attention to the case of a common *background* pressure in all dimensions, and where necessary assume a background equation of state $p=\rho/n$. We do, however, allow different pressure *perturbations* in the different spaces, as will be seen below. The background equation of motion for the matter is

$$\frac{\dot\rho}{\rho+p}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]=0\ . \tag{2.8c}$$

In Sec. IV we shall see that considerations of the matter content are unimportant in determining the dominant behavior of the perturbations.

The perturbed fluid velocity is given by

$$\begin{aligned} u^0&=1-Aq^{(0)}Q^{(0)}\ ,\\ u^i&=\frac{v^{(0)}}{r}q^{(0)i}Q^{(0)}\ ,\\ u^a&=\frac{V^{(0)}}{R}q^{(0)}Q^{(0)a} \end{aligned} \tag{2.9a}$$

so that

$$\begin{aligned} u_0&=-(1+Aq^{(0)}Q^{(0)})\ ,\\ u_i&=r(v^{(0)}-b^{(0)})q_i^{(0)}Q^{(0)}\ ,\\ u_a&=R(V^{(0)}-B^{(0)})q^{(0)}Q_a^{(0)}\ , \end{aligned} \tag{2.9b}$$

where

$$\begin{aligned} v^{(0)}q^{(0)i}Q^{(0)}&=r\frac{dx^i}{dt}\ ,\\ V^{(0)}q^{(0)}Q^{(0)a}&=R\frac{dX^a}{dt}\ . \end{aligned} \tag{2.9c}$$

So we have

$$\begin{aligned} T^0{}_0&=-\rho(1+\delta\cdot q^{(0)}Q^{(0)})\ ,\\ T^0{}_i&=r(\rho+p)(v^{(0)}-b^{(0)})q_i^{(0)}Q^{(0)}\ ,\\ T^0{}_a&=R(\rho+p)(V^{(0)}-B^{(0)})q^{(0)}Q_a^{(0)}\ ,\\ T^i{}_j&=p(1+\pi_Lq^{(0)}Q^{(0)})\delta^i{}_j+p\pi_T^{(0)}q^{(0)i}{}_jQ^{(0)}\ ,\\ T^a{}_b&=p(1+\Pi_Lq^{(0)}Q^{(0)})\delta^a{}_b+p\Pi_T^{(0)}q^{(0)}Q^{(0)a}{}_b\ ,\\ T^i{}_a&=p\frac{R}{r}\pi_x^{(0)}q^{(0)i}Q_a^{(0)}\ . \end{aligned} \tag{2.10}$$

For a perfect fluid we can expect the various transverse pressure terms $\pi_T^{(0)}$, $\Pi_T^{(0)}$, and $\pi_X^{(0)}$ to vanish, while $\pi_L=\Pi_L=\delta$. The transverse pressure contrasts also vanish in the vector and tensor cases.

B. The vector problem

Since we cannot construct a scalar from solutions to Eq. (2.3), we have the following perturbed metric:

$$
\begin{aligned}
&g_{00}=-1\ ,\\
&g_{0i}=-rb^{(1)}q_i^{(1)}Q^{(0)}\ ,\\
&g_{0a}=-RB^{(1)}q^{(0)}Q_a^{(1)}\ ,\\
&g_{ij}=r^2({}^dg_{ij}+2h_T^{(1)}q_{ij}^{(1)}Q^{(0)})\ ,\\
&g_{ab}=R^2({}^Dg_{ab}+2H_T^{(1)}q^{(0)}Q_{ab}^{(1)})\ ,\\
&g_{ia}=2rR(G^{(01)}q_i^{(0)}Q_a^{(1)}+G^{(10)}q_i^{(1)}Q_a^{(0)}+G^{(11)}q_i^{(1)}Q_a^{(1)})\ .
\end{aligned}
\tag{2.11}
$$

The fluid velocity is

$$
\begin{aligned}
&u^0=1\ ,\\
&u^i=\frac{v^{(1)}}{r}q^{(1)i}Q^{(0)}\ ,\\
&u^a=\frac{V^{(1)}}{R}q^{(0)}Q^{(1)a}\ ,
\end{aligned}
\tag{2.12a}
$$

thus giving the energy-momentum tensor

$$
\begin{aligned}
&T^0{}_0=-\rho\ ,\\
&T^0{}_i=r(\rho+p)(v^{(1)}-b^{(1)})q_i^{(1)}Q^{(0)}\ ,\\
&T^0{}_a=R(\rho+p)(V^{(1)}-B^{(1)})q^{(0)}Q_a^{(1)}\ ,\\
&T^i{}_j=p\delta^i{}_j+p\pi_T^{(1)}q^{(1)i}{}_jQ^{(0)}\ ,\\
&T^a{}_b=p\delta^a{}_b+p\Pi_T^{(1)}q^{(0)}Q^{(1)a}{}_b\ ,\\
&T^i{}_a=p\frac{R}{r}(\pi_X^{(01)}q^{(0)i}Q_a^{(1)}+\pi_X^{(10)}q^{(1)i}Q_a^{(0)}+\pi_X^{(11)}q^{(1)i}Q_a^{(1)})\ .
\end{aligned}
\tag{2.12b}
$$

C. The tensor problem

For the tensor problem there are still fewer nontrivial terms in the metric tensor, as we cannot construct scalars or vectors from the solutions of Eq. (2.5). Here we have

$$
\begin{aligned}
&g_{00}=-1\ ,\\
&g_{0i}=g_{0a}=g_{ia}=0\ ,\\
&g_{ij}=r^2({}^dg_{ij}+2h_T^{(2)}q_{ij}^{(2)}Q^{(0)})\ ,\\
&g_{ab}=R^2({}^Dg_{ab}+2H_T^{(2)}q^{(0)}Q_{ab}^{(2)})\ .
\end{aligned}
\tag{2.13}
$$

Also

$$
\begin{aligned}
&u^0=1\ ,\\
&u^i=u^a=0\ ,
\end{aligned}
\tag{2.14a}
$$

and so finally

$$
\begin{aligned}
&T^0{}_0=-\rho\ ,\\
&T^0{}_i=T^0{}_a=T^i{}_a=0\ ,\\
&T^i{}_j=p\delta^i{}_j+p\pi_T^{(2)}q^{(2)i}{}_jQ^{(0)}\ ,\\
&T^a{}_b=p\delta^a{}_b+p\Pi_T^{(2)}q^{(0)}Q^{(2)a}{}_b\ .
\end{aligned}
\tag{2.14b}
$$

We are now in a position to be able to work out the perturbations in the Ricci tensor $\delta R^\mu{}_\nu$. Expressions for the various pieces of this tensor for the different problems may be found in the Appendix. It should be noted again that our conventions regarding choice of time coordinate, overall sign of $R^\mu{}_\nu$, etc., differ from those of Ref. 1; however, it may readily be checked that our expressions reduce to the correct ones on taking $d=3$ and $D=0$.

III. GAUGE DEPENDENCE AND GAUGE-INVARIANT QUANTITIES

Before approaching the solutions to these problems, the matter of choice of coordinate gauge must be addressed. It is by now well understood that such a choice of gauge can lead to difficulties of interpretation and spurious nonphysical effects. Therefore, the relevant differential equations will be derived without making a particular choice of gauge, as in Ref. 1. We will shortly combine the metric perturbations to define sets of gauge-invariant quantities for the various cases, based again on Ref. 1, but appropriately generalized. These quantities are then related to gauge-invariant combinations of the Ricci curvature tensor components. Complications arise here for the scalar case, as the gauge-invariant quantities which naturally appear in the equations are not simply related to those which seem more fundamental from the point of view of the metric. However, in the next section we will show that in the region of interest, the differential equations simplify sufficiently to allow easy solution. These results may be confirmed by working in a particular gauge and using answers obtained there to determine the time dependence of the gauge-invariant quantities from the expressions given below. The vector and tensor cases do not suffer from the same problems, as the Einstein equations are readily expressed in terms of the relevant gauge-invariant quantities.

Consider first the question of gauge transformations for each of the three cases.

A. The scalar problem. Coordinate transformations here are parametrized in terms of three arbitrary functions of time:

$$
\begin{aligned}
&\tilde t=t+Tq^{(0)}Q^{(0)}\ ,\\
&\tilde x^i=x^i+l^{(0)}q^{(0)i}Q^{(0)}\ ,\\
&\tilde X^a=X^a+L^{(0)}q^{(0)}Q^{(0)a}\ ,
\end{aligned}
\tag{3.1}
$$

where T,l,L are functions of t. Using these we readily find (see Ref. 1 for details)

$$
\begin{aligned}
&\tilde A=A-\dot T\ ,\\
&\tilde b^{(0)}=b^{(0)}+\frac{k_r^{(0)}}{r}T+r\dot l^{(0)}\ ,\\
&\tilde B^{(0)}=B^{(0)}+\frac{k_R^{(0)}}{R}T+R\dot L^{(0)}\ ,\\
&\tilde h_L=h_L-\frac{k_r^{(0)}}{d}l^{(0)}-\frac{\dot r}{r}T\ ,\\
&\tilde H_L=H_L-\frac{k_R^{(0)}}{D}L^{(0)}-\frac{\dot R}{R}T\ ,\\
&\tilde h_T^{(0)}=h_T^{(0)}+k_r^{(0)}l^{(0)}\ ,\\
&\tilde H_T^{(0)}=H_T^{(0)}+k_R^{(0)}L^{(0)}\ ,\\
&\tilde G^{(0)}=G^{(0)}+\frac{k_R^{(0)}}{2}\frac{r}{R}l^{(0)}+\frac{k_r^{(0)}}{2}\frac{R}{r}L^{(0)}\ .
\end{aligned}
\tag{3.2}
$$

Regarding the energy-momentum tensor perturbations, we similarly find that

$$\tilde{v}^{(0)}=v^{(0)}+r\dot{l}^{(0)}\,,$$
$$\tilde{V}^{(0)}=V^{(0)}+R\dot{L}^{(0)}\,,$$
$$\tilde{\delta}=\delta-\frac{\dot{\rho}}{\rho}T\,,\qquad(3.3)$$
$$\tilde{\pi}_L=\pi_L-\frac{\dot{p}}{p}T\,,$$
$$\tilde{\Pi}_L=\Pi_L-\frac{\dot{P}}{P}T\,.$$

The other quantities in $T^{\mu}{}_{\nu}$ are gauge invariant. The last three of the above could be rewritten in terms of r and R using the equations of motion and the equation of state for the matter, if desired.

B. The vector problem. For the vectors we have two possible arbitrary coordinate transformations,

$$\tilde{x}^i=x^i+l^{(1)}q^{(1)i}Q^{(0)}\,,$$
$$\tilde{X}^a=X^a+L^{(1)}q^{(0)}Q^{(1)a}\,,\qquad(3.4)$$

and similarly to the above we use these to find

$$\tilde{b}^{(1)}=b^{(1)}+r\dot{l}^{(1)}\,,$$
$$\tilde{B}^{(1)}=B^{(1)}+R\dot{L}^{(1)}\,,$$
$$\tilde{h}_T^{(1)}=h_T^{(1)}+k_r^{(1)}l^{(1)}\,,$$
$$\tilde{H}_T^{(1)}=H_T^{(1)}+k_R^{(1)}L^{(1)}\,,\qquad(3.5)$$
$$\tilde{G}^{(10)}=G^{(10)}+\frac{k_R^{(0)}}{2}\frac{r}{R}l^{(1)}\,,$$
$$\tilde{G}^{(01)}=G^{(01)}+\frac{k_r^{(0)}}{2}\frac{R}{r}L^{(1)}\,,$$
$$\tilde{G}^{(11)}=G^{(11)}\,.$$

The energy-momentum tensor perturbations are simpler than in the scalar case:

$$\tilde{v}^{(1)}=v^{(1)}+r\dot{l}^{(1)}\,,$$
$$\tilde{V}^{(1)}=V^{(1)}+R\dot{L}^{(1)}\,,\qquad(3.6)$$

with all other quantities gauge invariant.

C. The tensor problem. All associated quantities are gauge invariant for this case, so the problem does not arise.

We now turn to the issue of choosing appropriate gauge-invariant quantities to study for each case. This choice of gauge-invariant quantities (especially for the scalar problem) goes hand in hand with choosing gauge-invariant combinations of the various pieces of the Einstein equations, so we will consider those first.

A. The scalar problem

The following combinations are found to be gauge invariant, using the background Einstein equations [Eq. (2.8b)] where necessary (note that we expect to find five such combinations as we start with eight independent pieces but have three arbitrary coordinate transformations):

$$\delta G^i{}_j-\frac{1}{d}\delta^i{}_j\delta G^k{}_k\,,$$
$$\delta G^a{}_b-\frac{1}{D}\delta^a{}_b\delta G^c{}_c\,,$$
$$\delta G^i{}_a\,,\qquad(3.7)$$
$$\delta G^0{}_0-\frac{d}{k_r^{(0)2}}\frac{\dot{r}}{r}\delta G^{(0)}{}_i{}^{|i}-\frac{D}{k_R^{(0)2}}\frac{\dot{R}}{R}\delta G^0{}_a{}^{|a}\,,$$
$$\delta G^\lambda{}_\lambda\,.$$

The first three of these pick out the various transverse parts of the Ricci tensor, and generalize one of Bardeen's choices.[1] The fourth generalizes the other combination used in Ref. 1, and the fifth is simply a convenient choice.

We now turn to choosing gauge-invariant combinations of the metric perturbations, which we wish to relate to the quantities in Eq. (3.7). Clearly, many such combinations are possible, but the ones defined below again use Ref. 1 as a starting point, and seem to us to be suitable. The first two come directly from Ref. 1:

$$\Phi_h=h_L+\frac{h_T^{(0)}}{d}+\frac{r}{k_r^{(0)}}\frac{\dot{r}}{r}b^{(0)}-\frac{r^2}{k_r^{(0)2}}\frac{\dot{r}}{r}\dot{h}_T^{(0)}\,,$$
$$\Phi_H=H_L+\frac{H_T^{(0)}}{D}+\frac{R}{k_R^{(0)}}\frac{\dot{R}}{R}B^{(0)}-\frac{R^2}{k_R^{(0)2}}\frac{\dot{R}}{R}\dot{H}_T^{(0)}\,.\qquad(3.8)$$

These are taken[1] to physically represent curvature perturbations in the ordinary and extra dimensions, respectively. Note that on reduction to three dimensions, Φ_H vanishes and Φ_h is precisely Bardeen's Φ_H.

The next three choices are not quite so straightforward, and are chosen in order that the first two gauge-invariant combinations of the Ricci tensor pieces of Eq. (3.7) assume particularly simple forms. The first two are explicitly given by

$$\Phi_A^{(r)}=A+\frac{r}{k_r^{(0)}}\dot{b}^{(0)}+\frac{r}{k_r^{(0)}}\left[\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]b^{(0)}-\frac{r^2}{k_r^{(0)2}}\left[\ddot{h}_T^{(0)}+\left[2\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\dot{h}_T^{(0)}\right]+D\left[H_L+\frac{H_T^{(0)}}{D}\right]\,,$$
$$\Phi_A^{(R)}=A+\frac{R}{k_R^{(0)}}\dot{B}^{(0)}+\frac{R}{k_R^{(0)}}\left[d\frac{\dot{r}}{r}+\frac{\dot{R}}{R}\right]B^{(0)}-\frac{R^2}{k_R^{(0)2}}\left[\ddot{H}_T^{(0)}+\left[d\frac{\dot{r}}{r}+2\frac{\dot{R}}{R}\right]\dot{H}_T^{(0)}\right]+d\left[h_L+\frac{h_T^{(0)}}{d}\right]\,.\qquad(3.9)$$

Note that on reduction to $D=0$, $\Phi_A^{(r)}$ becomes Bardeen's

Φ_A. $\Phi_A^{(R)}$ is not gauge invariant if one naively sets $D=0$, but this does not matter since it is always multiplied by a coefficient which vanishes in this limit. The final gauge-invariant combination is based on $G^{(0)}$, which does not appear in the ones defined so far:

$$\Phi_G = G^{(0)} - \frac{1}{2}\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}h_T^{(0)} - \frac{1}{2}\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}H_T^{(0)} \tag{3.10}$$

which vanishes on reduction to $D=0$. In terms of these Φ's, then, we readily find that

$$\begin{aligned} \delta G^i{}_j - \frac{1}{d}\delta^i{}_j\delta G^k{}_k &= \frac{k_r^{(0)2}}{r^2}\left[\Phi_A^{(r)} + (d-2)\Phi_h + 2\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\Phi_G\right] q^{(0)i}{}_j Q^{(0)} , \\ \delta G^a{}_b - \frac{1}{D}\delta^a{}_b\delta G^c{}_c &= \frac{k_R^{(0)2}}{R^2}\left[\Phi_A^{(R)} + (D-2)\Phi_H + 2\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\Phi_G\right] q^{(0)} Q^{(0)a}{}_b , \end{aligned} \tag{3.11}$$

which should be compared with Eq. (4.2) of Ref. 1.

Before turning to the other parts of Eq. (3.7) it is convenient to define an auxiliary gauge-invariant quantity which will be used in these expressions. We set

$$\frac{\dot r}{r}\frac{\dot R}{R}\Phi_6 = \frac{\dot R}{R}\left[h_L + \frac{h_T^{(0)}}{d}\right] - \frac{\dot r}{r}\left[H_L + \frac{H_T^{(0)}}{D}\right] . \tag{3.12a}$$

We may express this in terms of the earlier variables:

$$\begin{aligned} \dot\Phi_6 + \left[d\frac{\dot r}{r} + D\frac{\dot R}{R}\right]\Phi_6 = {} & \left[\frac{r}{\dot r}\right]^2 \left\{\frac{\dot r}{r}\dot\Phi_h - \left[\left[\frac{\dot r}{r}\right]^{\cdot} - D\frac{\dot r}{r}\frac{\dot R}{R}\right]\Phi_h\right\} - \Phi_A^{(r)} \\ & - \left[\frac{R}{\dot R}\right]^2 \left\{\frac{\dot R}{R}\dot\Phi_H - \left[\left[\frac{\dot R}{R}\right]^{\cdot} - d\frac{\dot r}{r}\frac{\dot R}{R}\right]\Phi_H\right\} + \Phi_A^{(R)} . \end{aligned} \tag{3.12b}$$

Consider next the fourth quantity in Eq. (3.7), which is a direct generalization from Ref. 1. Using the expressions given above and in the Appendix, we find that

$$\begin{aligned} \delta G^0{}_0 - \frac{d}{k_r^{(0)2}}\frac{\dot r}{r}\delta G^0{}_i{}^{|i} - \frac{D}{k_R^{(0)2}}\frac{\dot R}{R}\delta G^0{}_a{}^{|a} = {} & \left[(d-1)\frac{k_r^{(0)2}}{r^2}\left[1 - \frac{dK_r}{k_r^{(0)2}}\right]\Phi_h \right. \\ & \left. + (D-1)\frac{k_R^{(0)2}}{R^2}\left[1 - \frac{DK_R}{k_R^{(0)2}}\right]\Phi_H + \Xi_1\right] q^{(0)} Q^{(0)} , \end{aligned} \tag{3.13a}$$

where

$$\begin{aligned} \Xi_1 = {} & \left[d\frac{\dot r}{r}\frac{k_R^{(0)2}}{R^2} + D\frac{\dot R}{R}\frac{k_r^{(0)2}}{r^2}\right]\left[\frac{1}{2}\frac{r}{\dot r}\Phi_h + \frac{1}{2}\frac{R}{\dot R}\Phi_H - \left[\frac{rR}{k_r^{(0)}k_R^{(0)}}\Phi_G\right]\right] \\ & + 2\frac{rR}{k_r^{(0)}k_R^{(0)}}\Phi_G\left[\frac{k_r^{(0)2}k_R^{(0)2}}{r^2R^2} + d\left[\frac{\dot r}{r}\right]^2\frac{k_R^{(0)2}}{R^2} + D\left[\frac{\dot R}{R}\right]^2\frac{k_r^{(0)2}}{r^2}\right] \\ & + \tfrac{1}{2}\Phi_6\left\{d\frac{\dot r}{r}\left[\frac{k_R^{(0)2}}{R^2} + 2D\left[\frac{\dot R}{R}\right]^2\right] - D\frac{\dot R}{R}\left[\frac{k_r^{(0)2}}{r^2} + 2d\left[\frac{\dot r}{r}\right]^2\right]\right\} . \end{aligned} \tag{3.13b}$$

Note that Ξ_1 vanishes on reduction to $D=0$, so that in this limit we reproduce Eq. (4.1) of Ref. 1.

A representation for the third expression in Eq. (3.7) is given by the combination

$$\delta G^i{}_a = \frac{1}{2}\frac{R}{r}\frac{k_r^{(0)}k_R^{(0)}}{rR}[\Phi_A^{(r)} + \Phi_A^{(R)} + (d-2)\Phi_h + (D-2)\Phi_H - 2\Xi_2] q^{(0)i} Q_a^{(0)} , \tag{3.14a}$$

where we have introduced a second auxiliary gauge-invariant quantity:

$$\Xi_2=\frac{rR}{k_r^{(0)}k_R^{(0)}}\left\{\ddot{\Phi}_G+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot{\Phi}_G\right.$$

$$+\left[\left[\frac{\dot r}{r}\right]^{\cdot}-\left[\frac{\dot R}{R}\right]^{\cdot}+(d-1)\left[\frac{\dot r}{r}\right]^2(D-d+2)\frac{\dot r}{r}\frac{\dot R}{R}-(D+1)\left[\frac{\dot R}{R}\right]^2-2(D-1)\frac{K_R}{R^2}\right]\Phi_G\right\}$$

$$-\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]\left[\frac{r}{\dot r}\Phi_h-\frac{R}{\dot R}\Phi_H-\Phi_6\right]. \qquad (3.14b)$$

This, like $\Phi_A^{(R)}$, is not gauge invariant after reduction to $D=0$, but again always appears with vanishing coefficient under these conditions. Note that we may use the background equations of motion to rewrite the quantity in square brackets above in the simpler, manifestly symmetric $r\leftrightarrow R$ form:

$$-\left[(d-1)\frac{K_r}{r^2}+(D-1)\frac{K_R}{R^2}+\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]^2\right]. \qquad (3.14c)$$

We will, in the future, make similar simplifications.

The final equation, coming from the trace, is by far the most complicated in the general situation. We will see in the next section that in the region of primary interest it simplifies greatly. Explicitly in terms of the various gauge-invariant quantities defined above we have

$$\delta G^{\lambda}{}_{\lambda}=(n-1)q^{(0)}Q^{(0)}\left[d\left\{\ddot{\Phi}_h+\left[(d+1)\frac{\dot r}{r}+2D\frac{\dot R}{R}\right]\dot{\Phi}_h-\frac{r}{\dot r}\left[\left[\frac{\dot r}{r}\right]^{\cdot\cdot}+(d+1)\frac{\dot r}{r}\left[\frac{\dot r}{r}\right]^{\cdot}-D(d+1)\left[\frac{\dot r}{r}\right]^2\frac{\dot R}{R}\right.\right.\right.$$

$$\left.-D^2\frac{\dot r}{r}\left[\frac{\dot R}{R}\right]^2-D\frac{\dot r}{r}\left[\frac{\dot R}{R}\right]^{\cdot}+\frac{d-1}{d}\frac{\dot r}{r}\frac{k_r^{(0)2}}{r^2}\right]\Phi_h$$

$$\left.-\frac{\dot r}{r}\dot{\Phi}_A^{(r)}-\left[2\left[\frac{\dot r}{r}\right]^{\cdot}+(d+1)\left[\frac{\dot r}{r}\right]^2+D\frac{\dot r}{r}\frac{\dot R}{R}+\frac{k_r^{(0)2}}{dr^2}\right]\Phi_A^{(r)}\right\}$$

$$\left.+D\{\text{l.c.}\leftrightarrow\text{u.c.}\}+\tfrac{1}{2}\Phi_6\left[d\left[\frac{\ddot r}{r}\right]^{\cdot}-D\left[\frac{\ddot R}{R}\right]^{\cdot}-\rho\frac{n+1}{n}\left[d\frac{\dot r}{r}-D\frac{\dot R}{R}\right]\right]\right], \qquad (3.15)$$

where {l.c.$\leftrightarrow$u.c.} signifies $r\leftrightarrow R, d\leftrightarrow D$, etc., from the preceding curly brackets.

Defining gauge-invariant combinations for the matter degrees of freedom is rather easier. We have for the fluid velocity

$$v_s^{(0)}=v^{(0)}-\frac{r}{k_r^{(0)}}\dot h_T^{(0)},$$
$$V_s^{(0)}=V^{(0)}-\frac{R}{k_R^{(0)}}\dot H_T^{(0)}. \qquad (3.16a)$$

For the energy density there are various choices analogous to those of Ref. 1:

$$\epsilon_m=\delta+\frac{n+1}{n}\left[d\frac{\dot r}{k_r^{(0)}}(v^{(0)}-b^{(0)})+D\frac{\dot R}{k_R^{(0)}}(V^{(0)}-B^{(0)})\right],$$

$$\epsilon_g=\delta-\frac{n+1}{n}\left[d\frac{\dot r}{k_r^{(0)}}\left[b^{(0)}-\frac{r}{k_r^{(0)}}\dot h_T^{(0)}\right]+D\frac{\dot R}{k_R^{(0)}}\left[B^{(0)}-\frac{R}{k_R^{(0)}}\dot H_T^{(0)}\right]\right]. \qquad (3.16b)$$

The physical interpretations of these are that they measure energy-density perturbations relative to spacelike hypersurfaces (respectively) at rest with respect to the matter or whose normal unit vectors have zero shear.

So we may now write down the Einstein equations for the scalar case, in the order given in Eq. (3.7), but omitting the spatial dependence which simply gives an overall factor

$$\frac{k_r^{(0)2}}{r^2}\left[\Phi_A^{(r)}+(d-2)\Phi_h+2\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\Phi_G\right]=p\pi_T^{(0)}\,,$$

$$\frac{k_R^{(0)2}}{R^2}\left[\Phi_A^{(R)}+(D-2)\Phi_H+2\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\Phi_G\right]=p\Pi_T^{(0)}\,,$$

$$\frac{1}{2}\frac{k_r^{(0)}k_R^{(0)}}{rR}[\Phi_A^{(r)}+\Phi_A^{(R)}+(d-2)\Phi_h+(D-2)\Phi_H-2\Xi_2]=p\pi_X^{(0)}\,, \quad (3.17a)$$

$$(d-1)\frac{k_r^{(0)2}}{r^2}\left[1-d\frac{K_r}{k_r^{(0)2}}\right]\Phi_h+(D-1)\frac{k_R^{(0)2}}{R^2}\left[1-D\frac{K_R}{k_R^{(0)2}}\right]\Phi_H+\Xi_1=-\rho\epsilon_m\,,$$

$$\delta G^\lambda{}_\lambda=(\rho\delta-dp\pi_L-Dp\Pi_L)=\rho\left[\delta-\frac{d}{n}\pi_L-\frac{D}{n}\Pi_L\right].$$

Compare the first of the above to Eq. (4.4) and the fourth to Eq. (4.3) of Ref. 1. In a perfect fluid, we can apply the remarks that followed Eq. (2.10) to find that

$$\Phi_A^{(r)}+(d-2)\Phi_h=-2\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\Phi_G\,,$$

$$\Phi_A^{(R)}+(D-2)\Phi_H=-2\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\Phi_G\,,$$

$$\Xi_2=-\left[\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}+\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\right]\Phi_G\,,$$

$$\delta G^\lambda{}_\lambda=0\,. \quad (3.17b)$$

B. The vector problem

Here all of the pieces of the Ricci tensor are separately gauge invariant so some of the problems encountered in the scalar case do not arise. As explained in Sec. II, the spatial dependence of the various perturbation amplitudes allows us to separate the vector problem into the three pieces *VS*, *SV*, and *VV*, which we shall consider in turn.

The nonzero Ricci tensor pieces contributing to the *VS* case are (see the Appendix) $\delta R^0{}_i$, $\delta R^i{}_j$, and part of $\delta R^i{}_a$ which we shall denote by $\delta R^i{}_a{}^{(VS)}$. We define the following gauge-invariant combinations of the metric perturbation amplitudes:

$$\Psi_r=b^{(1)}-\frac{r}{k_r^{(1)}}\dot h_T^{(1)}\,,$$

$$J_r=G^{(10)}-\frac{1}{2}\frac{k_R^{(0)}}{k_r^{(1)}}\frac{r}{R}h_T^{(1)}\,. \quad (3.18a)$$

The first of these is precisely Bardeen's Ψ; the second vanishes on reduction to $D=0$.

Gauge-invariant versions of the matter velocity perturbation are easily derived. There are two basic choices: namely,

$$v_s^{(1)}=v^{(1)}-\frac{r}{k_r^{(1)}}\dot h_T^{(1)}\,,$$

$$v_c=v^{(1)}-b^{(1)}=v_s^{(1)}-\Psi_r\,. \quad (3.18b)$$

All other quantities in the energy-momentum tensor are gauge invariant. So, in terms of these quantities the Einstein equations $\delta G^\mu{}_\nu=\delta T^\mu{}_\nu$ may be written as

$$\delta G^0{}_i=-rq_i{}^{(1)}Q^{(0)}\left\{\frac{1}{2}\left[\frac{k_r^{(1)2}}{r^2}\left[1-(d-1)\frac{K_r}{k_r^{(1)2}}\right]+\frac{k_R^{(0)2}}{R^2}\right]\Psi_r-\frac{k_R^{(0)}}{R}\left[\dot J_r-\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]J_r\right]\right\}$$

$$=r(\rho+p)v_cq_i^{(1)}Q^{(0)}\,,$$

$$\delta G^i{}_j=q^{(1)i}{}_jQ^{(0)}\frac{k_r^{(0)}}{r}\left[\dot\Psi_r+\left[(d-1)\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\Psi_r+2\frac{k_R^{(0)}}{R}J_r\right]$$

$$=p\pi_T^{(1)}q^{(1)i}{}_jQ^{(0)}\,, \quad (3.19)$$

$$\delta G^i{}_a{}^{(VS)}=\frac{R}{r}q^{(1)i}Q_a^{(0)}\frac{k_R^{(0)}}{R}\left[\tfrac{1}{2}\dot\Psi_r+\frac{1}{2}\left[(d+1)\frac{\dot r}{r}+(D-2)\frac{\dot R}{R}\right]\Psi_r-\frac{R}{k_R^{(0)}}\left\{\ddot J_r+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot J_r+\left[\frac{k_r^{(1)2}}{r^2}-(D-1)\frac{K_R}{R^2}-\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]^2\right]J_r\right\}\right]$$

$$=\frac{R}{r}p\pi_X^{(10)}q^{(1)i}Q_a^{(0)}\,.$$

Note that the first of these corresponds to Eq. (4.12) of Ref. 1.

We may repeat a similar process for the SV case. Here we have

$$\begin{aligned}&\Psi_R=B^{(1)}-\frac{r}{k_R^{(1)}}\dot{H}_T^{(1)},\quad J_R=G^{(01)}-\frac{1}{2}\frac{k_r^{(0)}}{k_R^{(1)}}\frac{R}{r}H_T^{(1)}\,,\\&V_s^{(1)}=V^{(1)}-\frac{R}{k_R^{(1)}}\dot{H}_T^{(1)},\quad V_c=V^{(1)}-B^{(1)}=V_s^{(1)}-\Psi_R\,,\end{aligned} \tag{3.20}$$

which lead to the Einstein equations

$$\begin{aligned}\delta G^0{}_a&=-Rq^{(0)}Q_a^{(1)}\left\{\frac{1}{2}\left[\frac{k_r^{(0)2}}{r^2}+\frac{k_R^{(1)2}}{R^2}\left[1-(D-1)\frac{K_R}{k_R^{(1)2}}\right]\right]\Psi_R-\frac{k_r^{(0)}}{r}\left[\dot{J}_R+\left[\frac{\dot{r}}{r}-\frac{\dot{R}}{R}\right]J_R\right]\right\}\\&=R(\rho+p)V_cq^{(0)}Q_a^{(1)}\,,\\\delta G^a{}_b&=q^{(0)}Q^{(1)a}{}_b\frac{k_R^{(1)}}{R}\left[\dot{\Psi}_R+\left[d\frac{\dot{r}}{r}+(D-1)\frac{\dot{R}}{R}\right]\Psi_R+2\frac{k_r^{(0)}}{r}J_R\right]\\&=p\Pi_T^{(1)}q^{(0)}Q^{(1)a}{}_b\,,\\\delta G^i{}_a{}^{(SV)}&=\frac{R}{r}q^{(0)i}Q_a^{(1)}\frac{k_r^{(0)}}{r}\left[\tfrac{1}{2}\dot{\Psi}_R+\frac{1}{2}\left[(d-2)\frac{\dot{r}}{r}+(D+1)\frac{\dot{R}}{R}\right]\Psi_R\right.\\&\qquad\left.-\frac{r}{k_r^{(0)}}\left\{\ddot{J}_R+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\dot{J}_R+\left[\frac{k_R^{(1)2}}{R^2}-(d-1)\frac{K_r}{r^2}-\left[\frac{\dot{r}}{r}-\frac{\dot{R}}{R}\right]^2\right]J_R\right\}\right]\\&=\frac{R}{r}p\pi_X^{(01)}q^{(0)i}Q_a^{(1)}\,.\end{aligned} \tag{3.21}$$

Regarding the VV case all of the relevant amplitudes are already gauge invariant, so we simply have the one equation

$$\begin{aligned}\delta G^i{}_a{}^{(VV)}&=-\frac{R}{r}q^{(1)i}Q_a^{(1)}\left\{\ddot{G}^{(11)}+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\dot{G}^{(11)}+\left[\frac{k_r^{(1)2}}{r^2}+\frac{k_R^{(1)2}}{R^2}-\left[\frac{\dot{r}}{r}-\frac{\dot{R}}{R}\right]^2\right]G^{(11)}\right\}\\&=\frac{R}{r}p\pi_X^{(11)}q^{(1)i}Q_a^{(1)}\,.\end{aligned} \tag{3.22}$$

C. The tensor problem

As before, problems of gauge invariance do not arise, so we have the two following differential equations to consider:

$$\begin{aligned}\delta G^i{}_j&=-q^{(2)i}{}_jQ^{(0)}\left[\ddot{h}_T^{(2)}+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\dot{h}_T^{(2)}+\left[\frac{k_r^{(2)2}}{r^2}+2\frac{K_r}{r^2}+\frac{k_R^{(0)2}}{R^2}\right]h_T^{(2)}\right]\\&=p\pi_T^{(2)}q^{(2)i}{}_jQ^{(0)}\,,\\\delta G^a{}_b&=-q^{(0)}Q^{(2)a}{}_b\left[\ddot{H}_T^{(2)}+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\dot{H}_T^{(2)}+\left[\frac{k_r^{(0)2}}{r^2}+\frac{k_R^{(2)2}}{R^2}+2\frac{K_R}{R^2}\right]H_T^{(2)}\right]\\&=p\Pi_T^{(2)}q^{(0)}Q^{(2)a}{}_b\,.\end{aligned} \tag{3.23}$$

The first of these corresponds to Eq. (4.14) of Ref. 1.

IV. KALUZA-KLEIN INFLATION AND THE GROWTH OF PERTURBATIONS

In this section we find solutions to the above equations assuming that the behavior of r and R is as described in earlier work.[2,3] The main features of this scenario are the following.

(1) After an initial big bang with $r=R=0$, we have $r\sim R\sim t^{2/(n+1)}$, where $n=d+D$. For illustrative purposes[2,3] we will from time to time in this section choose $d=3, D=40$.

(2) Both r and R then remain roughly constant for some time—this will be referred to as the "resting phase." During this epoch the horizon length grows to be larger than r, so that causal processes can homogenize matter in our dimensions.

(3) R then collapses to its final value $R_{\rm KK}$ and is presumed constant thereafter. This drives an extremely rapid inflation of r. We presume that the value of $R_{\rm KK}$ is given by the same unknown processes (quantum gravity?) which halt the collapse, and regard it simply as an input parameter to the classical model described here. For simplicity we will set $R_{\rm KK}=1$ (all lengths and time intervals are expressed in Planck units), but this is not essential. During this epoch we have

$$R\sim B\tau^{\gamma},\quad \gamma=\frac{1+\left[\dfrac{d(n-1)}{D}\right]^{1/2}}{n}\,,$$
$$r\sim b\tau^{\eta},\quad \eta=\frac{1-\left[\dfrac{D(n-1)}{d}\right]^{1/2}}{n}\,, \tag{4.1}$$

where b and B are constants, $\tau=t_0-t$ and t_0 is the time at which R would reach zero were the collapse not to stop. For $d=3$ and $D=40$, $\gamma\approx0.0645$ and $\eta\approx-0.5271$. There are serious conceptual problems with assuming that a classical approximation remains valid in this period, and we shall return to discuss this later.

(4) Rapid inflation of r continues even after $R=R_{\rm KK}$ during an interval in which the temperature drops sufficiently for the radiation to "freeze-out" of the extra dimensions. Once this has happened, normal Robertson-Walker behavior with $r\sim(t-\bar{t})^{1/2}$ ensures ($\bar{t}$ is an arbitrary constant of integration) and asymptotically the only relics of the earlier multi-dimensional era are the consequences of inflation.

Particularly relevant to solving the perturbation equations are the following considerations. First, the fact that r and R vary as powers of τ in period (3), where $\tau\to0$ as $R\to0$, reverses the conventional ideas concerning the growth of perturbations. The usual result[1,6] is that there are two modes (considered as powers of t)—a "decaying" mode typically going as t^{-1}, and a "growing" mode which is constant or grows as a positive power of t. Clearly, in the "time-reversed" scenario here a mode which behaves as, say, τ^{-1} will dominate over a mode which is constant or goes as a positive power of τ, thus leading to the reversal of roles. It will be necessary to find those solutions of the equations which are most singular as $\tau\to0$. With $R_{\rm KK}=1$, $r\approx50$ characterizes the middle of the resting phase,[2,3] and the collapse halts around $\tau\approx10^{-31}$.

Also important is the fact that the inflation is driven by the terms in Einstein's equations which represent the geometric coupling between the two spaces.[2] Thus, the general form of the inflation is already contained in Einstein's equations once we have included our basic assumptions concerning the geometry of space, in particular, that M_D has positive curvature. The explicit factorization between M_d and M_D ensures that the evolution of the two spaces is spatially uniform in this approximation. All details of the microphysics which may be responsible for the dynamics of the compactification[4] of the D extra dimensions is hidden in the effective curvature K_R. This situation is to be contrasted with the more standard scenarios[7] of inflation where the responsible microphysics is more explicit. In the present approximation it can be shown[2,3] that during inflation the matter or curvature terms in Eqs. (2.7b) and (2.8b) are less singular than the derivative terms. Thus, to leading order it is sufficient to solve the simpler equations $\delta G^{\mu}{}_{\nu}=0$ and completely ignore the matter terms. This is, of course, only possible because of the first point, that most of the growth of the perturbations happens in epoch (3). Computer calculations are currently in progress confirming the belief that this epoch is the one in which most of the growth occurs. During the resting phase, the matter terms presumably play an important physical role. We can, in fact, use the equations of the previous section to relate the perturbations of interest to the statistical fluctuations presumed to be present during the resting phase in, for example, the "matter" energy density of Eq. (3.16). This connection can then yield an estimate of the initial conditions for the perturbations at the onset of the inflationary phase. The details of this connection are not relevant for the present discussion. We will need only that the statistical fluctuations and the perturbations are of about the same order (to within a factor of 10^3). We treat the "matter" as a relativistic gas of temperature[3] 10^{-2} (in Planck units). Hence, if we consider a volume containing enough energy to subsequently condense into a galaxy, $\sim10^{48}$ Planck masses, and thus $\sim10^{50}$ "particles," we might expect energy density contrasts of order $1/\sqrt{N}\sim10^{-25}$. This is clearly only a crude estimate but, as will be seen shortly, our general conclusions will be unchanged whether this number is 10^{-20} or 10^{-40}.

A. The scalar problem

We wish to solve for the time dependence of Φ_h, etc., as $\tau\to0$. To do this, we take the expressions given in the previous section and use the following approximations which are valid in this limit:

$$\frac{\dot r}{r}=-\frac{\eta}{\tau},\quad \frac{\dot R}{R}=-\frac{\gamma}{\tau}\,,$$
$$d\eta+D\gamma=d\eta^2+D\gamma^2=1\,, \tag{4.2}$$
$$\frac{1}{r^2},\frac{1}{R^2}\ll\frac{1}{\tau}\,.$$

Note that with the last approximation, a leading-order result will not give the spectrum of k_r and k_R. Thus, in this approximation, the dominant contribution to the growth of the perturbations is independent of the wavelength characterizing the perturbation. The inflation process will preserve the initial spectrum in k_r. Then the five parts of Eq. (3.17) give the following equations in this region (note that derivatives are still with respect to t, not τ):

$$
\begin{aligned}
&\Phi_A^{(r)}+(d-2)\Phi_h+2\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\Phi_G=0\ ,\\
&\Phi_A^{(R)}+(D-2)\Phi_H+2\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\Phi_G=0\ ,\\
&\frac{k_r^{(0)}k_R^{(0)}}{rR}\Xi_2+\left[\frac{k_R^{(0)2}}{R^2}+\frac{k_r^{(0)2}}{r^2}\right]\Phi_G=0\ ,\\
&(d-1)\frac{k_r^{(0)2}}{r^2}\left[1-\frac{dK_r}{k_r^{(0)2}}\right]\Phi_h+(D-1)\frac{k_R^{(0)2}}{R^2}\left[1-\frac{DK_R}{k_R^{(0)2}}\right]\Phi_H+\Xi_1=0\ ,\\
&d\left[\ddot{\Phi}_h-\frac{2+\eta-d\eta}{\tau}\dot{\Phi}_h-\frac{(2-d\eta)(1-\eta)}{\tau^2}\Phi_h+\frac{\eta}{\tau}\dot{\Phi}_A^{(r)}+\frac{\eta(1-\eta)}{\tau^2}\Phi_A^{(r)}\right]\\
&\qquad+D\left[\ddot{\Phi}_H-\frac{2+\gamma-D\gamma}{\tau}\dot{\Phi}_H-\frac{(2-D\gamma)(1-\gamma)}{\tau^2}\Phi_H+\frac{\gamma}{\tau}\dot{\Phi}_A^{(R)}+\frac{\gamma(1-\gamma)}{\tau^2}\Phi_A^{(R)}\right]\\
&\qquad+\Phi_6\left[\frac{d\eta(1-\eta)}{\tau^3}-\frac{D\gamma(1-\gamma)}{\tau^3}\right]=0\ .
\end{aligned}
\tag{4.3}
$$

It is straightforward to show that the *most* singular solutions of the above equations are (recall that $d/dt=-d/d\tau$)

$$
\begin{aligned}
&\Phi_A^{(r)}=-(d-2)\Phi_h\sim\tau^{-2+2\eta}\ ,\\
&\Phi_A^{(R)}=-(D-2)\Phi_H\sim\tau^{-2+2\gamma}\ ,\\
&\Phi_G\sim\tau^{\eta-\gamma}\ .
\end{aligned}
\tag{4.4}
$$

In a similar manner one can determine the asymptotic τ dependence of the auxiliary quantities Φ_6, Ξ_1, and Ξ_2 by using the expressions given above and in the previous section.

All of the above results, including those concerning the auxiliary quantities, may be confirmed by working within a specific gauge ($b^{(0)}=B^{(0)}=G^{(0)}=0$ is particularly convenient). One may then determine the behavior of the various Φ's from that of the original metric perturbations A, etc., and using the expressions given in Sec. III.

It remains to consider the consequences of this growth rate. We have

$$
\begin{aligned}
\frac{\Phi_{h(\text{final})}}{\Phi_{h(\text{initial})}}&=\left[\frac{\tau_{(\text{final})}}{\tau_{(\text{initial})}}\right]^{-2+2\eta}\\
&\approx\left[\frac{10^{-31}}{50}\right]^{-3.054}\approx10^{99}\ .
\end{aligned}
\tag{4.5}
$$

Thus, as noted in the Introduction and above, the growth in the perturbation amplitudes for any reasonable initial conditions (such as those discussed above) is such that the linearized perturbation theory developed here will likely have become invalid. The growth of the other amplitudes is not so extreme, e.g.,

$$
\begin{aligned}
&\left[\frac{\tau_{(\text{final})}}{\tau_{(\text{initial})}}\right]^{-2+2\gamma}\sim10^{61}\ ,\\
&\left[\frac{\tau_{(\text{final})}}{\tau_{(\text{initial})}}\right]^{\eta-\gamma}\sim2\times10^{19}\ .
\end{aligned}
\tag{4.6}
$$

The above analysis ignores the fact that the inflation of r continues after the collapse has stopped and while the extra dimensions are decoupling from the radiation. In this epoch[3] the scale factor behaves like $r\sim(t-\hat{t})^{2n/d(n+1)}$ and it would be possible to perform similar calculations to the above with this behavior. In view of the results obtained, however, we do not consider this worthwhile.

B. The vector problem

Consider first the *VS* part. Near the collapse time, Eq. (3.19) becomes [$\dot{J}_r=(d/dt)J_r$, etc.]

$$\dot{J}_r + \frac{\eta-\gamma}{\tau} J_r - \frac{R}{2k_R^{(0)}} \left[\frac{k_r^{(1)2}}{r^2} \left[1-(d-1)\frac{K_r}{k_r^{(1)2}} \right] + \frac{k_R^{(0)2}}{R^2} \right] \Psi_r = 0 ,$$

$$J_r + \frac{R}{2k_R^{(0)}} \left[\dot{\Psi}_r - \frac{(d-1)\eta + D\gamma}{\tau} \Psi_r \right] = 0 , \quad (4.7a)$$

$$\ddot{J}_r - \frac{1}{\tau}\dot{J}_r - \frac{(\eta-\gamma)^2}{\tau^2} J_r - \frac{k_R^{(0)}}{2R} \left[\dot{\Psi}_r - \frac{1+\eta-2\gamma}{\tau} \Psi_r \right] = 0 .$$

The third of these may, in fact, be derived from the others, so we need only consider the first two. These may be manipulated in such a way as to get separate equations for J_r and Ψ_r,

$$\ddot{J}_r - \frac{1-2(\eta-\gamma)}{\tau}\dot{J}_r + \frac{(\eta-\gamma)^2}{\tau^2} J_r = 0 ,$$
$$\ddot{\Psi}_r - \frac{1-2(\eta-\gamma)}{\tau}\dot{\Psi}_r - \frac{(1-\eta)(1+\eta-2\gamma)}{\tau^2}\Psi_r = 0 , \quad (4.7b)$$

which have most singular solutions

$$J_r \sim \tau^{\eta-\gamma}, \quad \Psi_r \sim \tau^{\eta-1} . \quad (4.8)$$

So for the *VS* case we have

$$\frac{J_{r(\text{final})}}{J_{r(\text{initial})}} \approx 2\times 10^{19} ,$$
$$\frac{\Psi_{r(\text{final})}}{\Psi_{r(\text{initial})}} \approx 9\times 10^{49} . \quad (4.9)$$

Once again it is likely that at least one of the amplitudes will have grown to the extent that linearized theory will no longer be valid. Note that the overall tendency is for the vector amplitudes to be less singular than the scalar ones; this trend will continue so that the tensor amplitudes are least singular.

Next consider the *SV* part of the vector problem. In a similar way to the above we may manipulate Eq. (3.21) to give the following differential equations in the limit $\tau \to 0$:

$$\ddot{J}_R - \frac{1}{\tau}\dot{J}_R - \frac{(\eta-\gamma)^2}{\tau^2} J_R = 0 ,$$
$$\ddot{\Psi}_R - \frac{1+2(\eta-\gamma)}{\tau}\dot{\Psi}_R - \frac{(1-\gamma)(1+\gamma-2\eta)}{\tau^2}\Psi_R = 0 , \quad (4.10a)$$

which have most singular solutions

$$J_R \sim \tau^{\eta-\gamma}, \quad \Psi_R \sim \tau^{\gamma-1} . \quad (4.10b)$$

This gives an overall growth of

$$\frac{J_{R(\text{final})}}{J_{R(\text{initial})}} \approx 2\times 10^{19} ,$$
$$\frac{\Psi_{R(\text{final})}}{\Psi_{R(\text{initial})}} \approx 4\times 10^{30} . \quad (4.10c)$$

The *VV* part has only the one equation for $G^{(11)}$ and so is simpler than the above cases. Near collapse we have

$$\ddot{G}^{(11)} - \frac{1}{\tau}\dot{G}^{(11)} - \frac{(\eta-\gamma)^2}{\tau^2} G^{(11)} = 0 \quad (4.11a)$$

with a most singular solution

$$G^{(11)} \sim \tau^{\eta-\gamma} . \quad (4.11b)$$

Thus we find

$$\frac{G^{(11)}_{(\text{final})}}{G^{(11)}_{(\text{initial})}} \sim 2\times 10^{19} . \quad (4.11c)$$

Note that, of all the vector amplitudes, only Ψ_r and Ψ_R are likely to cause problems for initial conditions such as those discussed above, J_r, J_R, and $G^{(11)}$ all stay sufficiently small not to cause difficulties. However, the different amplitudes are only decoupled by the spatial dependence to leading order. Hence, if one of the amplitudes cannot be treated by linearized theory, useful conclusions regarding the other amplitudes cannot be drawn either.

C. The tensor problem

As for the *VV* part of the vector problem, finding the solution here is straightforward. Equation (3.23) gives us, near collapse,

$$\ddot{h}_T^{(2)} - \frac{1}{\tau}\dot{h}_T^{(2)} = 0 ,$$
$$\ddot{H}_T^{(2)} - \frac{1}{\tau}\dot{H}_T^{(2)} = 0 , \quad (4.12a)$$

with most singular solutions

$$h_T^2, H_T^{(2)} \sim \ln\tau . \quad (4.12b)$$

Thus, there is negligible growth during the inflationary period. One may similarly show that during the decoupling epoch (and indeed during the subsequent Robertson-Walker phase) these amplitudes remain constant. Hence the tensor amplitudes never create difficulties regarding nonlinearities; on the other hand, they never grow large enough to play a useful role in galaxy formation.

V. DISCUSSION

The most outstanding feature of these calculations is the continued (indeed, phenomenal) growth of perturbations outside the particle horizon. This is in direct contradiction with conventional inflationary models,[7] where physical effects are forbidden to be coherent over scales larger than the horizon. This contrast exemplifies the fundamentally different nature of our model, or at least our approximation, and requires some explanation.

As we noted earlier the Kaluza-Klein—type inflation is quite different from the more usual picture where the inflation is typically "driven" by a scalar field. The explicit dependence on the microphysics of this field, especially its propagation, guarantees that no process can be coherent on scales larger than the horizon. Hence, physical perturbations cannot grow while outside of the horizon. In the present Kaluza-Klein model the inflation is "driven" instead by the geometric coupling between the compactifying D-dimensional space and the usual d-dimensional flat space. By *assumption* the collapse of the D dimensions and the inflation of the d dimensions occurs uniformly in all space *at the same time*. Questions of coherence of the underlying microphysics which might possibly produce the compactification simply cannot arise. This assumed spatial uniformity of the inflation carries over directly to the uniform inflation of the perturbation amplitudes. Moreover, the most singular part of this inflation is explicitly independent of the wave numbers k_r and k_R and nothing prevents the perturbations from continuing to grow outside of the horizon. A more realistic description of this process employing an explicit dynamical compactification mechanism would presumably display features more similar to the usual inflation scenario. Without such an explicit model we can say little more about this issue but turn instead to another question.

We have earlier alluded to the fact that the classical framework described here cannot reasonably be expected to remain valid throughout the collapse. This is for the following reason:[2] the Ricci curvature tensor includes derivative terms like $\dot{r}/r$ which as $\tau \rightarrow 0$ are of order $1/\tau$. By the time the collapse stops, $\tau \sim 10^{-31}$ and so we are exciting frequencies of order 10^{31}. Recall that throughout we work in units of the Planck length, so one might expect quantum effects to become important when the curvature components become of order 1 and certainly before it is of order 10^{31}. Thus, the classical results can at best be suggestive. Had these results been promising as regards final size of the perturbation amplitudes, then one would still have to verify that quantum effects did not disrupt the general picture.[8] However, the classical results are disappointing in the sense that the scheme as described appears to be ruled out.

We can, however, ask the following question. How sensitive are the results obtained to different assumptions about the behavior of the scale factors r and R? If the results from using different growth rates are invariably too large, then Kaluza-Klein inflation must be ruled out as a viable scheme. If, on the other hand, the perturbation amplitudes need not grow so large (with different behavior but the same total amount of inflation of r) then the scheme retains viability. It then, however, becomes a rather more delicate problem to handle correctly. As an illustrative example consider

$$r \sim b e^{\beta_r t}, \quad R \sim R_{\mathrm{KK}} + B e^{-\beta_R t} , \tag{5.1}$$

where b, B, β_r, β_R are constants and the dimensionful scale is presumably supplied by the Planck length. The motivation for this comes from the considerations mentioned last paragraph. There the problem was that the $\dot{r}/r$ terms in the curvature tensor became too large; it is not unreasonable to suppose that quantum gravity effects limit their size, and this is the effect of Eq. (5.1). We may now substitute these into the various equations for the perturbation amplitudes. We again assume (this time with rather less justification) that we may neglect the matter terms and simply solve $\delta G^{\mu}{}_{\nu} = 0$. We also work in the long-wavelength limit in which we may neglect terms like ${k_R}^2/R^2$ or ${k_r}^2/r^2$ in comparison with constants. Note, in particular, that, whereas the original singular power behavior ($\dot{r}/r$ and $\dot{R}/R \sim 1/\tau$ as $\tau \rightarrow 0$) led to differential equations with singular coefficients ($1/\tau$ or $1/\tau^2$) and hence singular power solutions, this "moderated" scenario ($\dot{r}/r, \dot{R}/R \sim$const) leads to constant coefficients and exponential behavior (growing, damped, or oscillating).

A. The scalar problem. Once again, we have the five parts of Eq. (3.17a) to consider, this time in a different regime. We find that all of the fundamental gauge-invariant quantities Φ_h, Φ_H, $\Phi_A^{(r)}$, $\Phi_A^{(R)}$, and Φ_G have solutions which either oscillate or decay with time. Thus the solution here is entirely different from that found earlier, where the scalar amplitudes were extremely singular. The initial statistical amplitudes assumed in this paper will not grow to the size currently required for galaxy formation.

B. The vector problem. For this case we need to consider Eqs. (4.7a), (4.10a), and (3.22) in the appropriate limit. It is straightforward to find that the most singular solution grows as $e^{\beta_r t}$ [ignoring $(k/R)^2$ terms], i.e., no faster than the scale r itself. Thus, for example,

$$\frac{J_{r(\text{final})}}{J_{r(\text{initial})}} = \exp[\beta_r (t_{(\text{final})} - t_{(\text{initial})})] = \frac{r_{(\text{final})}}{r_{(\text{initial})}} . \tag{5.2}$$

Now, within the framework described in Refs. 2 and 3, $r_{(\text{initial})} \approx 100$ and $r_{(\text{final})} \sim 10^{31}$. This latter figure is the order of magnitude by the time the radiation has decoupled from the extra dimensions and ordinary evolution has commenced. So we have

$$\frac{J_{r(\text{final})}}{J_{r(\text{initial})}} \sim 10^{29} . \tag{5.3}$$

Thus, although the amplitude grows by a large factor, it may remain sufficiently small that linearized theory is still reliable.

C. The tensor problem. Here consideration of Eq. (3.23) shows that both $h_T^{(2)}$ and $H_T^{(2)}$ have one decaying and one constant solution.

So we see that results using Eq. (5.1) as possible behavior of r and R are rather more encouraging than the earlier ones with power-law behavior given by Eq. (4.1). The perturbation amplitudes may never become so large as to render linearized theory invalid and for initial perturbations of order 10^{-33} the growth rate of at least one of them gives final amplitudes of a magnitude compatible with current requirements. On the other hand, we do not wish to overemphasize these results. We regard them only as an illustration that the various amplitudes need not grow as much as the earlier ones indicated, and that in particular their final magnitudes can be of a useful size. Evidently the results are sensitive to the detailed behavior of r and R during inflation, and not just to the overall change of scale.

VI. CONCLUSIONS

In this paper we have described a general framework in which to analyze cosmological perturbations in Kaluza-Klein models. The scheme does not assume any particular behavior of the background geometry, other than that it may be described classically by the two scale factors r and R. The method uses and generalizes Bardeen's gauge-invariant formalism[1] which was originally developed for an ordinary Friedmann-Robertson-Walker (FRW) geometry. Within linearized perturbation theory one may split a general perturbation into several distinct pieces according to their spatial dependence and then solve for each piece separately. Just as in Ref. 1, various differential equations may be derived for gauge-invariant combinations of the perturbation amplitudes—here, however, both the equation and the expressions for the gauge-invariant quantities in terms of the metric perturbations are more complicated than in the original work.

Having derived the equations, we have substituted in the behavior of r and R found in earlier work.[2,3] During an epoch in which R collapses to its final value R_{KK} and r inflates very rapidly, the perturbation amplitudes also grow extremely rapidly. We find that some of the amplitudes grow sufficiently that for any reasonable initial size of perturbation the linearized theory developed here will have broken down. Not all of the amplitudes are so singular, and indeed most will remain very much less than of order 1 throughout. The decoupling of modes is only valid to leading order, however, and so the entire analysis must be called into question. It is extremely doubtful that nonlinear effects could keep *all* of the amplitudes as small as the size currently favored for galaxy formation ($\sim 10^{-4}$) and so the scheme as described must be ruled out.

However, there are additional poorly understood effects in operation in this epoch, and we have argued that quantum gravity cannot be ignored. In particular, it is the regime when $\dot r/r, |\dot R/R| \gg 1$ which accounts for the large perturbations and which is most suspect in the classical analysis. As a preliminary investigation into how sensitive the results for the perturbations are to different behavior of the background space, we have considered what would happen if the inflation were of exponential ($\dot r/r, \dot R/R \sim$const) rather than singular power form. We find that the overall growth of the perturbations is much less, and in fact can give answers which are not unreasonably far from the desired result. We therefore conclude that results are very sensitive to the detailed behavior of the scale factors. This is fortunate in that the basic idea of Kaluza-Klein inflation need not be discarded, but it makes it very difficult to draw broad conclusions. One interesting fact to emerge is that initial statistical fluctuations can grow in these models to become important; in the standard FRW model this is not the case. Clearly, the problem demands a careful analysis of behavior during the collapse and decoupling periods.

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APPENDIX

Here we give the expressions for the perturbed Ricci tensor pieces for each of the scalar, vector, and tensor problems. The background expressions are given in Eq. (2.7b).

A. The scalar problem

In this case we have the following components, expressed in a general gauge:

$$\delta R^0{}_0 = -q^{(0)}Q^{(0)}\left\{\left[\frac{k_r^{(0)2}}{r^2}+\frac{k_R^{(0)2}}{R^2}-2\left[d\frac{\ddot r}{r}+D\frac{\ddot R}{R}\right]\right]A-\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot A+\frac{k_r^{(0)}}{r}\left[\dot b^{(0)}+\frac{\dot r}{r}b^{(0)}\right]\right.$$
$$\left.+\frac{k_R^{(0)}}{R}\left[\dot B^{(0)}+\frac{\dot R}{R}B^{(0)}\right]+d\left[\ddot h_L+2\frac{\dot r}{r}\dot h_L\right]+D\left[\ddot H_L+2\frac{\dot R}{R}\dot H_L\right]\right\},$$

$$\delta R^0{}_i = -k_r^{(0)}q_i^{(0)}Q^{(0)}\left[\left[(d-1)\frac{\dot r}{r}+D\frac{\dot R}{R}\right]A+\frac{r}{k_r^{(0)}}\left[\frac{k_R^{(0)2}}{2R^2}-\frac{(d-1)K_r}{r^2}\right]b^{(0)}-\frac{k_R^{(0)}}{2R}B^{(0)}-(d-1)\dot h_L\right.$$
$$\left.-\frac{d-1}{d}\left[1-\frac{dK_r}{k_r^{(0)2}}\right]\dot h_T^{(0)}+D\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]H_L-D\dot H_L+\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]G^{(0)}-\frac{k_R^{(0)}}{k_r^{(0)}}\frac{r}{R}\dot G^{(0)}\right],$$

$$\delta R^0{}_a=-k_R^{(0)}q^{(0)}Q_a^{(0)}\left[\left[d\frac{\dot r}{r}+(D-1)\frac{\dot R}{R}\right]A-\frac{k_r^{(0)}}{2r}b^{(0)}+\frac{R}{k_R^{(0)}}\left[\frac{k_r^{(0)2}}{2r^2}-\frac{(D-1)K_R}{R^2}\right]B^{(0)}-d\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]h_L\right.$$

$$\left.-d\dot h_L-(D-1)\dot H_L-\frac{D-1}{D}\left[1-\frac{DK_R}{k_R^{(0)2}}\right]\dot H_T^{(0)}-\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]G^{(0)}-\frac{k_r^{(0)}}{k_R^{(0)}}\frac{R}{r}\dot G^{(0)}\right],$$

$$\delta R^i{}_j=-q^{(0)}Q^{(0)}\delta^i_j\left[\left\{\frac{k_r^{(0)2}}{dr^2}-2\left[\left[\frac{\dot r}{r}\right]^{\cdot}+\frac{\dot r}{r}\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\right]\right\}A-\frac{\dot r}{r}\dot A+\frac{k_r^{(0)}}{dr}\left[(2d-1)\frac{\dot r}{r}+D\frac{\dot R}{R}\right]b^{(0)}\right.$$

$$+\frac{k_r^{(0)}}{dr}\dot b^{(0)}+\frac{k_R^{(0)}}{R}\frac{\dot r}{r}B^{(0)}+\ddot h_L+\left[2d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot h_L+\left[\frac{k_R^{(0)2}}{R^2}+\frac{2(d-1)k_r^{(0)2}}{dr^2}\left[1-\frac{dK_r}{k_r^{(0)2}}\right]\right]h_L$$

$$\left.+\frac{2(d-1)k_r^{(0)2}}{d^2r^2}\left[1-\frac{dK_r}{k_r^{(0)2}}\right]h_T^{(0)}+D\frac{\dot r}{r}\dot H_L+\frac{Dk_r^{(0)2}}{dr^2}H_L+\frac{2k_r^{(0)}k_R^{(0)}}{drR}G^{(0)}\right]$$

$$-q^{(0)i}{}_jQ^{(0)}\left[-\frac{k_r^{(0)2}}{r^2}A-\frac{k_r^{(0)}}{r}\left[(d-1)\frac{\dot r}{r}+D\frac{\dot R}{R}\right]b^{(0)}-\frac{k_r^{(0)}}{r}\dot b^{(0)}-(d-2)\frac{k_r^{(0)2}}{r^2}h_L-D\frac{k_r^{(0)2}}{r^2}H_L\right.$$

$$\left.+\ddot h_T^{(0)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot h_T^{(0)}+\left[\frac{k_R^{(0)2}}{R^2}-\frac{(d-2)k_r^{(0)2}}{dr^2}\right]h_T^{(0)}-\frac{2k_r^{(0)}k_R^{(0)}}{rR}G^{(0)}\right],$$

$$\delta R^a{}_b=-q^{(0)}Q^{(0)}\delta^a_b\left[\left\{\frac{k_R^{(0)2}}{DR^2}-2\left[\left[\frac{\dot R}{R}\right]^{\cdot}+\frac{\dot R}{R}\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\right]\right\}A-\frac{\dot R}{R}\dot A+\frac{k_r^{(0)}}{r}\frac{\dot R}{R}b^{(0)}\right.$$

$$+\frac{k_R^{(0)}}{DR}\left[d\frac{\dot r}{r}+(2D-1)\frac{\dot R}{R}\right]B^{(0)}+\frac{k_R^{(0)}}{DR}\dot B^{(0)}+d\frac{\dot R}{R}\dot h_L+\frac{dk_R^{(0)2}}{DR^2}h_L+\ddot H_L+\left[d\frac{\dot r}{r}+2D\frac{\dot R}{R}\right]\dot H_L$$

$$\left.+\left[\frac{k_r^{(0)2}}{r^2}+\frac{2(D-1)k_R^{(0)2}}{DR^2}\left[1-\frac{DK_R}{k_R^{(0)2}}\right]\right]H_L+\frac{2(D-1)k_R^{(0)2}}{D^2R^2}\left[1-\frac{DK_R}{k_R^{(0)2}}\right]H_T^{(0)}+\frac{2k_r^{(0)}k_R^{(0)}}{DrR}G^{(0)}\right]$$

$$-q^{(0)}Q^{(0)a}{}_b\left[-\frac{k_R^{(0)2}}{R^2}A-\frac{k_R^{(0)}}{R}\left[d\frac{\dot r}{r}+(D-1)\frac{\dot R}{R}\right]B^{(0)}-\frac{k_R^{(0)}}{R}\dot B^{(0)}-d\frac{k_R^{(0)2}}{R^2}h_L-(D-2)\frac{k_R^{(0)2}}{R^2}H_L\right.$$

$$\left.+\ddot H_T^{(0)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot H_T^{(0)}+\left[\frac{k_r^{(0)2}}{r^2}-\frac{(D-2)k_R^{(0)2}}{DR^2}\right]H_T^{(0)}-\frac{2k_r^{(0)}k_R^{(0)}}{rR}G^{(0)}\right],\tag{A1}$$

$$\delta R^i{}_a=-\frac{R}{r}q^{(0)i}Q_a^{(0)}\frac{k_r^{(0)}k_R^{(0)}}{rR}\left\{-A-\frac{r}{2k_r^{(0)}}\left[(d+1)\frac{\dot r}{r}+(D-2)\frac{\dot R}{R}\right]b^{(0)}-\frac{r}{2k_r^{(0)}}\dot b^{(0)}\right.$$

$$-\frac{R}{2k_R^{(0)}}\left[(d-2)\frac{\dot r}{r}+(D+1)\frac{\dot R}{R}\right]B^{(0)}-\frac{R}{2k_R^{(0)}}\dot B^{(0)}-(d-1)h_L$$

$$-\frac{d-1}{d}\left[1-\frac{dK_r}{k_r^{(0)2}}\right]h_T^{(0)}-(D-1)H_L-\frac{D-1}{D}\left[1-\frac{DK_R}{k_R^{(0)2}}\right]H_T^{(0)}+\ddot G^{(0)}$$

$$+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot G^{(0)}+\left[\left[\frac{\dot r}{r}\right]^{\cdot}-\left[\frac{\dot R}{R}\right]^{\cdot}+(d-1)\left[\frac{\dot r}{r}\right]^2+(D-d+2)\frac{\dot r}{r}\frac{\dot R}{R}\right.$$

$$\left.-(D+1)\left[\frac{\dot R}{R}\right]^2-\frac{2(D-1)K_R}{R^2}\right]G^{(0)}\Bigg\}.$$

B. The vector problem

Here the problem naturally splits into the three parts *VS*, *SV*, and *VV*. The only Ricci tensor component which contains a mixture of all three parts is $\delta R^i{}_a$, which we write as the three pieces separately. It will be noticed that the expressions are simpler than in the scalar case:

$$\delta R^0{}_0=0\,,$$

$$\delta R^0{}_i=-rq_i^{(1)}Q^{(0)}\left\{\frac{1}{2}\left[\frac{k_r^{(1)2}}{r^2}\left[1-\frac{(d-1)K_r}{k_r^{(1)2}}\right]+\frac{k_R^{(1)2}}{R^2}\right]b^{(1)}-\frac{k_r^{(1)}}{2r}\left[1-\frac{(d-1)K_r}{k_r^{(1)2}}\right]\dot h_T^{(1)}\right.$$

$$\left.-\frac{k_R^{(0)}}{R}\left[\dot G^{(10)}-\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]G^{(10)}\right]\right\},$$

$$\delta R^0{}_a=-Rq^{(0)}Q_a^{(1)}\left\{\frac{1}{2}\left[\frac{k_r^{(0)2}}{r^2}+\frac{k_R^{(1)2}}{R^2}\left[1-\frac{(D-1)K_R}{k_R^{(1)2}}\right]\right]B^{(1)}-\frac{k_R^{(1)}}{2R}\left[1-\frac{(D-1)K_R}{k_R^{(1)2}}\right]\dot H_T^{(1)}\right.$$

$$\left.-\frac{k_r^{(0)}}{r}\left[\dot G^{(01)}+\left[\frac{\dot r}{r}-\frac{\dot R}{R}\right]G^{(01)}\right]\right\},$$

$$\delta R^i{}_j=-q^{(1)i}{}_jQ^{(0)}\left[-\frac{k_r^{(1)}}{r}\left[(d-1)\frac{\dot r}{r}+D\frac{\dot R}{R}\right]b^{(1)}-\frac{k_r^{(1)}}{r}\dot b^{(1)}+\ddot h_T^{(1)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot h_T^{(1)}\right.$$

$$\left.+\frac{k_R^{(0)2}}{R^2}h_T^{(1)}-\frac{2k_r^{(1)}k_R^{(0)}}{rR}G^{(10)}\right],$$

$$\delta R^a{}_b=-q^{(0)}Q^{(1)a}{}_b\left[-\frac{k_R^{(1)}}{R}\left[d\frac{\dot r}{r}+(D-1)\frac{\dot R}{R}\right]B^{(1)}-\frac{k_R^{(1)}}{R}\dot B^{(1)}+\ddot H_T^{(1)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot H_T^{(1)}\right.$$

$$\left.+\frac{k_r^{(0)2}}{r^2}H_T^{(1)}-\frac{2k_r^{(0)}k_R^{(1)}}{rR}G^{(01)}\right],$$

$$\delta R^i{}_a{}^{(VS)}=-\frac{R}{r}q^{(1)i}Q_a^{(0)}\left\{\ddot G^{(10)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot G^{(10)}\right.$$

$$+\left[\left[\frac{\dot r}{r}\right]^{\cdot}-\left[\frac{\dot R}{R}\right]^{\cdot}+(d-1)\left[\frac{\dot r}{r}\right]^2+(D-d+2)\frac{\dot r}{r}\frac{\dot R}{R}-(D+1)\left[\frac{\dot R}{R}\right]^2\right.$$

$$\left.+\frac{k_r^{(1)2}}{r^2}+\frac{(d-1)K_r}{r^2}-\frac{2(D-1)K_R}{R^2}\right]G^{(10)}$$

$$\left.-\frac{k_R^{(0)}}{2R}\left[\left[(d+1)\frac{\dot r}{r}+(D-2)\frac{\dot R}{R}\right]b^{(1)}+\dot b^{(1)}+\frac{k_r^{(1)}}{r}\left[1-\frac{(d-1)K_r}{k_r^{(1)2}}\right]h_T^{(1)}\right]\right\},\qquad\text{(A2)}$$

$$\delta R^{i}{}_{a}{}^{(SV)}=-\frac{R}{r}q^{(0)i}Q_a^{(1)}\left\{\ddot{G}^{(01)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot G^{(01)}\right.$$
$$+\left[\left[\frac{\dot r}{r}\right]^{\cdot}-\left[\frac{\dot R}{R}\right]^{\cdot}+(d-1)\left[\frac{\dot r}{r}\right]^2+(D-d+2)\frac{\dot r}{r}\frac{\dot R}{R}-(D+1)\left[\frac{\dot R}{R}\right]^2\right.$$
$$\left.+\frac{k_R^{(1)2}}{R^2}-\frac{(D-1)K_R}{R^2}\right]G^{(01)}$$
$$\left.-\frac{k_r^{(0)}}{2r}\left[\left[(d-2)\frac{\dot r}{r}+(D+1)\frac{\dot R}{R}\right]B^{(1)}+\dot B^{(1)}+\frac{k_R^{(1)}}{R}\left[1-\frac{(D-1)K_R}{k_R^{(1)2}}\right]H_T^{(1)}\right]\right\},$$

$$\delta R^{i}{}_{a}{}^{(VV)}=-\frac{R}{r}q^{(1)i}Q_a^{(1)}\left\{\ddot{G}^{(11)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot G^{(11)}\right.$$
$$+\left[\left[\frac{\dot r}{r}\right]^{\cdot}-\left[\frac{\dot R}{R}\right]^{\cdot}+(d-1)\left[\frac{\dot r}{r}\right]^2+(D-d+2)\frac{\dot r}{r}\frac{\dot R}{R}-(D+1)\left[\frac{\dot R}{R}\right]^2\right.$$
$$\left.\left.+\frac{(d-1)K_r}{r^2}-\frac{(D-1)K_R}{R^2}+\frac{k_r^{(1)2}}{r^2}+\frac{k_R^{(1)2}}{R^2}\right]G^{(11)}\right\}.$$

C. The tensor problem

As usual, this is the easiest of the problems, with most components of the curvature tensor vanishing:

$$\delta R^0{}_0=\delta R^0{}_i=\delta R^0{}_a=\delta R^i{}_a=0\;,$$

$$\delta R^i{}_j=-q^{(2)i}{}_jQ^{(0)}\left[\ddot h_T^{(2)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot h_T^{(2)}+\left[\frac{k_r^{(2)2}}{r^2}+\frac{2K_r}{r^2}+\frac{k_R^{(0)2}}{R^2}\right]h_T^{(2)}\right],\qquad\text{(A3)}$$

$$\delta R^a{}_b=-q^{(0)}Q^{(2)a}{}_b\left[\ddot H_T^{(2)}+\left[d\frac{\dot r}{r}+D\frac{\dot R}{R}\right]\dot H_T^{(2)}+\left[\frac{k_r^{(0)2}}{r^2}+\frac{k_R^{(2)2}}{R^2}+\frac{2K_R}{R^2}\right]H_T^{(2)}\right].$$

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