Finite-dimensional Einstein-Maxwell-scalar field system

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The finite-dimensional Einstein-Maxwell-scalar field system, characterized by a spatially homogeneous or spatially self-similar gravitational field, is approached from a group-theoretical point of view. The field equations are expressed as a parametrized finite-dimensional mechanical system whose singularity behavior is readily apparent, using a method which allows one to consider all possible symmetry types simultaneously.

I. INTRODUCTION

"Finite-dimensional" cosmological models have fascinated many relativists at one time or another in their careers. Here finite dimensional means that the configuration space of variables which evolve in time is finite dimensional, with the result that the evolution of those variables can be described by ordinary differential equations. The reduction to finite dimension, an enormous simplification, is a direct consequence of the high symmetry of the class of models. However, the symmetry is not so overwhelming that most interesting features of the field equations are frozen out as in the most symmetric cosmological models of the Friedmann-Robertson-Walker type. In fact the symmetry is just right, very often leading to a close finite-dimensional analog of many features of the infinite-dimensional system lacking any symmetry. Not surprisingly, there are some differences, differences which enrich the structure of the finite-dimensional system as a dynamical system in its own right. In particular this system is remarkably rich in the numerous ways in which Lie groups play an important role, both at the spacetime level and in the dynamics itself.

The finite-dimensional cosmological models are spacetimes which are spatially homogeneous or spatially self-similar, characterized by the existence of a symmetry group G acting transitively on three-dimensional spacelike orbits, the symmetry being isometries of the spacetime metric in the case of spatial homogeneity and homothetic motions in the case of self-similarity. The energy-momentum tensor of the source must share this symmetry, while the source variables themselves are restricted only by the condition that their energy-momentum tensor have the symmetry. The spatially homogeneous symmetry was introduced in 1951 by Taub[1] who constructed cosmological models out of the homogeneous Riemannian three-manifolds studied by the Italian mathematician Bianchi[2] at the turn of the century. Spatial self-similarity was introduced some 20 years later by Eardley,[3] who presented a classification of the possible symmetry types, extending the modern version of the Bianchi classification given by Behr.[4,5] Both the spatially homogeneous "Bianchi models" and their spatially self-similar generalizations generically have a three-dimensional symmetry group acting simply transitively on its orbits. The only models not of this type are the Kantowski-Sachs spacetimes[6] and their spatially self-similar generalizations. These models have a four-dimensional symmetry group but no three-dimensional subgroup acting simply transitively on the orbits of the group. This case will not be treated in the present paper; however, its omission is not serious since it is related by analytic continuation to a particular class of spacetimes with Bianchi type III symmetry.[7]

In order to express the field equations for a specific cosmological model (choice of gravitational theory and sources) as ordinary differential equations, the spacetime frame used to express those equations in component form must be adapted to the symmetry. Since these spacetimes have a natural slicing by the spacelike orbits of the symmetry group, they are ideally suited to the 3-plus-1 approach to the problem, introduced for general relativity by Arnowitt, Deser, and Misner[8,9] (ADM). Refined by many others, this ADM formulation can be described in a geometrical way involving two basic features.[10,11] First one chooses a slicing of the spacetime by a family of spacelike hypersurfaces and then a threading of that slicing by a congruence of curves nowhere tangent to the elements of the slicing, thus establishing a "global reference frame." The slices correspond to hypersurfaces of constant time and the congruence of curves to the world lines of points of "space." The actual choice of the time function t can be specified by a 1-1 parametrization of the elements of the slicing such that the differential of the resulting time function is always nonzero (the value of the time function at a slice equals the parameter of the slice). The parametrization of the slicing picks out a unique vector field on the spacetime whose congruence of integral curves is the congruence used to thread the slicing. This vector field is called an ADM generator for the slicing; it is nowhere tangent to the slicing and one can recover the entire slicing from any slice by Lie dragging along this vector field. Next one must choose a spatial frame on one slice and local coordinates $\{x^i \mid i=1,2,3\}$ on that slice which are Lie dragged along the congruence by the ADM generator (spatial frame means that each frame vector is tangent to the slice). One thus obtains local coordinates (ADM coordinates) adapted to the slicing and a spacetime frame (a comoving ADM frame) adapted to the slicing which consists of the ADM generator (just $\partial/\partial t$ in the

ADM coordinates) and the spatial frame vectors (which in the ADM coordinates have no time component and spatial components which are independent of t). Thus in the case where the spacetime has a natural slicing, the freedom in the choice of time is reduced to the choice of a parametrization of that slicing. The choice of a particular spatial frame at one element of the slicing, as well as the choice of the threading of the slicing, involves the spatial diffeomorphism freedom.

For the spatially homogeneous and spatially self-similar spacetimes, the spatial diffeomorphism freedom compatible with the symmetry is related to the automorphism group of the Lie algebra of generators of the action of the symmetry group on the orbit. These generators are just the Killing or homothetic Killing vector fields of the spacetime associated with the spatial homogeneity or spatial self-similarity. In order that the field equations become ordinary differential equations when expressed in terms of the adapted ADM spacetime frame consisting of the ADM generator and a Lie dragged spatial frame, the spatial frame must be invariant under the symmetry group on each element of the slicing. This restricts the choice of spatial frame on any given slice to a finite-dimensional space and restricts the threading of the slicing to those congruences which keep the spatial frame in this finite-dimensional space on each slice under dragging along. The freedom in the choice of global reference frame is best described in terms of the lapse function and shift vector field which result from the projection of the ADM generator onto the unit normal vector field to the slicing and into the slicing itself, respectively. The lapse can only be a function of the parametrization of the slicing of the spacetime by the orbits of the symmetry group, while the shift vector field must be confined to the Lie algebra generating the automorphism group of the Lie algebra of generators of the symmetry group G in the spatially homogeneous case and a certain Lie subalgebra of generators of the automorphism group in the spatially self-similar case. (In the latter case, not only the Lie brackets of the symmetry group Lie algebra must be preserved but also the values of the linear form on that Lie algebra associated with the homothetic condition.)

Expressing the field equations in such a frame and projecting them into perpendicular and parallel pieces with respect to the slicing[11] then leads to ordinary differential equations which reflect both the gauge freedom associated with the choice of shift through the action of the matrix representation of the above automorphism group or subgroup, as well as the freedom of choice of the initial spatial frame, involving the same matrix group but acting as a symmetry group of the initial data. These ordinary differential equations contain the components of the structure constant tensor of the Lie algebra of generators of the symmetry group (as well as the components of the homothetic linear form) as parameters whose choice corresponds to a gauge condition on the choice of initial spatial frame, preserved by the matrix representation of the automorphism group (respectively, the subgroup also preserving the homothetic linear form). Usually these parameters are set equal to their canonical values given in the Bianchi-Behr or Eardley classification, and one studies the particular symmetry type represented by that choice.

However, by allowing enough freedom in the specification of these constants so that one can pass freely from one symmetry type to another without complicating the simplified form of the ordinary differential equations obtained by choosing the canonical values, one can study the system for all symmetry types simultaneously. In this way apparently isolated exact solutions can be seen as particular members of families of solutions which contain many symmetry types and qualitative methods of studying particular symmetry types can be seen as part of a qualitative treatment of the system as a whole (including all symmetry types). Obviously certain properties will differentiate between the approach one takes in the details of treating each symmetry type, but by avoiding specialization until absolutely necessary one understands what one is doing in a global context rather than as an isolated special case.

This can be achieved by working with the parametrized simply connected three-dimensional Lie group and a related multivalued choice of parametrization of the variables of the problem adapted to the automorphism gauge freedom.[12,13] The parameter space consists of a particular subspace of the space $\mathscr{C}$ of components of possible structure constant tensors for a three-dimensional Lie algebra. This subspace, called the standard diagonal form subspace $\mathscr{C}_D$, is essentially discussed by Ellis and MacCallum[5] (except for a permutation which singles out the first direction rather than the third as is customary in physics) and is particularly suited to the discussion of isometry classes of the spatial three-metrics[14] and therefore to the structure of the differential equations which govern the evolution of three-metric.

This paper will discuss the above ideas in the following way. First the symmetry of the metric and sources will be discussed, together with the spaces of symmetry adapted variables and the explicit construction of the spacetimes themselves. Then the $\mathscr{C}_D$-parametrized diagonal gauge will be discussed in general and finally the Einstein-Maxwell-scalar field system will be discussed within the context of zero cosmological constant general relativity. The discussion of the vector potential and variational principle for Maxwell's equations will be relegated to the Appendix.

II. SYMMETRY TYPE

As shown by Eardley,[3] a spatially self-similar spacetime can be obtained from a spatially homogeneous spacetime by a particular conformal transformation of the spacetime metric. Similarly the sources of a spatially self-similar spacetime can be obtained by scaling the sources for the spatially homogeneous case by the appropriate power of the conformal factor. Each field undergoes the scaling associated with its dimension q assigned so that the spacetime metric has dimension $q=2$ while the covariant energy-momentum tensor of the source and the covariant Ricci and Einstein tensors both have $q=0$ (the gravitational constant does not scale here so it has $q=0$); this scaling operation has been called "stretching" in the case of a perfect fluid.[15] The sources of the spatially homogeneous spacetimes which are not themselves spatially homogeneous can be obtained in all cases (excluding gauge-dependent gauge fields) by a complex phase transformation of the spatially homogeneous fields provided the appropriate variables are chosen; this complex phase transformation has been called "twisting" in the case of a Dirac spinor source.[16] Thus all the variables which enter into the problem can be expressed as at most an inhomogeneous complex scaling of the spatially homogeneous variables. The classification of spatially self-similar spacetimes arises from the study of the modulus of this complex scale factor while the classification of less symmetric sources for the gravitational field deals with its phase. Not surprisingly this has a very simple group theoretical interpretation. The logarithmic differential of the modulus (the homothetic one-form) and the differential of the phase (the twist one-form) must be exact one-forms on the spacetime which are invariant under the symmetry group; in fact with certain identifications they must belong to the adjoint invariant subspace of the dual of the Lie algebra of generators of the action of the symmetry group. The classification both of spatially self-similar spacetimes and of less symmetric sources for spatially homogeneous or spatially self-similar spacetimes then corresponds to the classification of adjoint invariant or equivalently bi-invariant one-forms on the abstract symmetry group G.

A given one-form associated either with the spatially self-similar conformal scaling or the independent phase transformation of a given complex source field can be initially classified as exceptional or nonexceptional depending on whether or not it is linearly independent of the one-form associated with the trace of the adjoint representation of the symmetry group Lie algebra (automatically nonexceptional if the latter form vanishes, as it does in the unimodular case). Let this latter form be called the structure constant trace one-form. A source-filled spacetime is called exceptional if the homothetic one-form or any of the twist one-forms is exceptional or if the set of such one-forms together with the structure constant trace one-form contains more than one linearly independent element. The exceptional case is a set of measure zero in the space of all possible variables and symmetry types, which is fortunate since its corresponding spatial gauge group has at most one dimension rather than at least three as in the nonexceptional case; hence, one is unable to eliminate as many degrees of freedom as is possible in the nonexceptional case, leaving the system in a rather complicated form. Reduction of the number of degrees of freedom is accomplished by exploiting the constraint equations which arise in the 3-plus-1 approach, together with the gauge symmetry of the field equations. The exceptional case can in fact occur only for symmetry groups G of the special Bianchi types I, II, and III for which the adjoint representation is not faithful due to the existence of a nontrivial center.

The discussion is simplified if we explicitly construct the spacetime manifold and symmetry action as follows.[3,15] Let G be a simply connected three-dimensional Lie group with Lie algebra $\mathfrak{g}$ of left invariant vector fields and let $\tilde{\mathfrak{g}}$ be the Lie algebra of right invariant vector fields on G. [If $X\in\mathfrak{g}$ and $\tilde X\in\tilde{\mathfrak{g}}$, then $X(e)=\tilde X(e)$, where e is the identity of G.] Let $\{e_a\}$ be a basis of $\mathfrak{g}$ with dual basis $\{\omega^a\}$ identified with a basis of the space $\mathfrak{g}^*$ of left invariant one-forms on the group. [Let $\tilde{\mathfrak{g}}^*$ be the corresponding right invariant space with $\tilde\sigma(e)=\sigma(e)$ for $\sigma\in g,\tilde\sigma\in\tilde g^*$.] The components of the structure constant tensor of the Lie algebra with respect to this basis are defined by[4,5]

$$\begin{aligned}
&C^a{}_{bc}=\omega^a([e_b,e_c])=-\tilde\omega^a([\tilde e_b,\tilde e_c])=\epsilon_{bcd}n^{ad}+a_f\delta^{fa}_{bc}\ ,\\
&a_f=\tfrac{1}{2}C^a{}_{fa},\quad C^{ab}=\tfrac{1}{2}C^a{}_{cd}\epsilon^{bcd}\ ,\\
&C^{(ab)}=n^{ab},\quad C^{[ab]}=\epsilon^{abc}a_c\ ,
\end{aligned}\tag{2.1}$$

while $[e_a,\tilde e_b]=0$. The structure constant trace one-form mentioned above is $C^f{}_{af}\omega^a=2a_a\omega^a$.

Take the spacetime manifold to be the product manifold $R\times G$, where R is the real line with global coordinate t which is identified with the parameter of the natural slicing of $R\times G$. Identifying fields on the factor manifolds with corresponding fields on $R\times G$, one can represent a general spatially homogeneous metric ${}^4g$ and a general spatially self-similar metric ${}^4\underset{\sim}{g}$ in the following form

$$\begin{aligned}
&{}^4g=-N^2dt\otimes dt+g_{ab}(\omega^a+N^adt)\otimes(\omega^b+N^bdt)\ ,\\
&{}^4\underset{\sim}{g}=e^{2\psi}\,{}^4g\ ,
\end{aligned}\tag{2.2}$$

while a spatial or spacetime field $\underset{\sim}{X}$ of dimension q associated with a source which is compatible with the spatial self-similarity of the metric ${}^4\underset{\sim}{g}$ can be represented in terms of a spatially homogeneous such field X by the following complex scaling, provided the appropriate variables are chosen

$$\underset{\sim}{X}=e^{q\psi+i\theta^X}X\ . \tag{2.3}$$

For example, the unit normal vector field to the family of orbits of the symmetry group has dimension $q=-1$ so

$$\underset{\sim}{e}_\perp=e^{-\psi}e_\perp,\quad e_\perp=N^{-1}(\partial/\partial t-\vec N)\ . \tag{2.4}$$

The lapse function N and metric component functions g_{ab} are functions only of t, while the "scale function" ψ and "twist function" θ^X are assumed to be time independent and have spatially homogeneous differentials

$$d\psi=b_a\omega^a\in\mathfrak{g}^*\cap\tilde{\mathfrak{g}}^*,\quad d\theta^X=c^X_a\omega^a\in\mathfrak{g}^*\cap\tilde{\mathfrak{g}}^*\ , \tag{2.5}$$

where b_a and c^X_a are real constants and

$$\begin{aligned}
\mathfrak{g}^*\cap\tilde{\mathfrak{g}}^*&=\{\sigma=\sigma_a\omega^a\in\mathfrak{g}^*\mid(d\sigma)_{bc}=-\sigma_aC^a{}_{bc}=0\}\\
&=\ker\operatorname{ad}(\mathfrak{g})^*\\
(\sigma\in\mathfrak{g}^*\cap\tilde{\mathfrak{g}}^*&\rightarrow\sigma=\sigma_a\omega^a=\sigma_a\tilde\omega^a)\ ,
\end{aligned}\tag{2.6}$$

is the fixed point set of the coadjoint representation of G, equivalent to the space of bi-invariant one-forms on G. The coadjoint representation of $\mathfrak{g}$ is defined by

$$\begin{aligned}
&(\operatorname{ad}(X)^*\sigma)(Y)=\sigma(\operatorname{ad}(X)Y)\ ,\\
&\sigma\in\mathfrak{g}^*,\quad X,Y\in\mathfrak{g}\ ,
\end{aligned}\tag{2.7}$$

where

$$\mathrm{ad}(X)Y=[X,Y] \tag{2.8}$$

defines the adjoint representation of $\mathscr{g}$. The subspace $\mathscr{g}^*\cap\tilde{\mathscr{g}}^*$ is also the annihilator of the commutator subalgebra or "first derived subalgebra" $\mathscr{g}^{(1)}=[\mathscr{g},\mathscr{g}]$ of $\mathscr{g}$, which explains why it is a trivial space for a semisimple Lie algebra where $\mathscr{g}^{(1)}=\mathscr{g}$. The dimension of $\mathscr{g}^*\cap\tilde{\mathscr{g}}^*$ is $3-\mathrm{rank}(C^{ab})$, since the condition (2.6) on σ is equivalent to $\sigma_a C^{ab}=0$; the assignment of roman numerals in the Bianchi classification is in fact based on the rank of the matrix (C^{ab}). If $\sigma=\sigma_a\omega^a\in\mathscr{g}^*\cap\tilde{\mathscr{g}}^*$ is a bi-invariant one-form, it is called exceptional if $\epsilon^{abc}a_c\sigma_b\neq0$, i.e., if the two one-forms σ and $a_a\omega^a=[\mathrm{Tr\,ad}(e_a)]\omega^a$ are linearly independent; otherwise σ is called nonexceptional. The differentials of the scale and twist functions will be called, respectively, the homothetic one-form and the twist one-forms. The trace one-form itself is the differential of the determinant of the adjoint representation of the group.

The shift vector field is restricted first by the condition that the linear transformation of the spatial vectors $\{e_a\}$ induced by the action of its Lie derivative depend only on the time; the same must be true of the Lie derivative of the scale and twist functions:

$$\begin{aligned}
&\text{(i) } \mathscr{L}_{\vec N}\, e_a\equiv\mathrm{ad}_e(\vec N)^b{}_a e_b,\quad {}^3d(\mathrm{ad}_e(\vec N)^a{}_b)=0\ ,\\
&\text{(ii) } \mathscr{L}_{\vec N}\,\psi=d\psi(\vec N)=b_aN^a,\quad {}^3d(b_aN^a)=0\ ,\\
&\text{(iii) } \mathscr{L}_{\vec N}\,\theta^X=d\theta^X(\vec N)=c^X_aN^a,\quad {}^3d(c^X_aN^a)=0\ .
\end{aligned} \tag{2.9}$$

These conditions must be imposed in order that the field equations reduce to ordinary differential equations. Since ψ is time independent, $d\psi={}^3d\psi$ holds and conditions (ii) and (iii) may be rewritten

$$0={}^3d(\mathscr{L}_{\vec N}\,\psi)=\mathscr{L}_{\vec N}\,d\psi,\quad 0=\cdots=\mathscr{L}_{\vec N}\,d\theta^X\ , \tag{2.10}$$

which may be expressed in component form:

$$\begin{aligned}
0&=-(\mathscr{L}_{\vec N}\,d\psi)(e_b)=-b_a(\mathscr{L}_{\vec N}\,\omega^a)(e_b)\\
&=b_a\omega^a(\mathscr{L}_{\vec N}\,e_b)=b_a\mathrm{ad}_e(\vec N)^a{}_b\ ,\\
0&=-(\mathscr{L}_{\vec N}\,d\theta^X)(e_b)=\cdots=c^X_a\mathrm{ad}_e(\vec N)^a{}_b\ .
\end{aligned} \tag{2.11}$$

The operator $\mathscr{L}_{\vec N}$ is already an inner derivation of the infinite-dimensional Lie algebra of vector fields on G. The condition (2.9) (i) requires the shift vector field to act as a derivation of the Lie subalgebra $\mathscr{g}$ into itself and therefore belong to the generating Lie algebra $\mathscr{g}\oplus_s\mathscr{aut}(G)=\tilde{\mathscr{g}}\oplus_s\mathscr{aut}(G)$ of the group $L_G\otimes_s Aut(G)=R_G\otimes_s Aut(G)$ of automorphisms and translations of G into itself. The matrix $[\mathrm{ad}_e(\vec N)^a{}_b]$ belongs to the Lie algebra of the matrix group of automorphisms of the Lie algebra $\mathscr{g}$ with respect to the basis $\{e_a\}$. The conditions (2.10) or (2.11) restrict the shift to the Lie subalgebra which Lie derives the homothetic one-form $d\psi$ and the twist one-forms $d\theta^X$ associated with the source variables, i.e., so that $[\mathrm{ad}_e(\vec N)^a{}_b]$ belongs to the Lie algebra of that subgroup of the Lie-algebra automorphism matrix group which leaves the covectors (b_a) and (c^X_a) invariant. In the nonexceptional case with $a_a\neq0$ (G a class B Lie group), the entire generating Lie algebra satisfies this condition, while if $a_a=0$ (G a class A Lie group), this subalgebra of the generating Lie algebra is the entire such Lie algebra for a certain class B Lie group.

However, unless the shift Lie derives the homothetic and twist functions themselves

$$\mathscr{L}_{\vec N}\,\psi=\mathscr{L}_{\vec N}\,\theta^X=0\ , \tag{2.12}$$

they will depend on the choice of spatial gauge. Rescaled spatially homogeneous fields like ${}^4g$ and X will also be gauge dependent and will therefore undergo additional conformal and phase gauge transformations when transforming between different spatial gauges. These complications can be avoided simply by imposing (2.12) as an additional condition on the shift vector field.

For any homothetic Killing vector field of the spacetime metric ${}^4\underset{\sim}{g}$ of the form $Z=Z^a(t)\tilde e_a$, the following relations hold for the Lie derivatives of the metric and the source field $\underset{\sim}{X}$:

$$\begin{aligned}
&\mathscr{L}_Z\,{}^4\underset{\sim}{g}=2\,d\psi(Z)\,{}^4\underset{\sim}{g}\ ,\\
&\mathscr{L}_Z\underset{\sim}{X}=[q\,d\psi(Z)+i\,d\theta^X(Z)]\underset{\sim}{X}\ .
\end{aligned} \tag{2.13}$$

Such a vector field Z Lie derives all spatially homogeneous fields; in particular its time dependence is determined by the condition

$$0=\mathscr{L}_Z e_\perp\leftrightarrow\dot Z^a=\tilde\omega^a(\mathscr{L}_{\vec N}\,\tilde e_b)Z^b\ . \tag{2.14}$$

Note that such vector fields will in general be time dependent unless $\vec N$ itself is spatially homogeneous (i.e., $\mathscr{L}_{\vec N}\,\tilde e_b=0$). When this is so, the left action of G on $R\times G$ reduces to the natural left action in which each copy of G in $R\times G$ undergoes the same left translation of G into itself. Otherwise the symmetry action of G on the spacetime is by a time-dependent left translation of G into itself. However, the condition (2.14), together with the bi-invariance of the homothetic and twist one-forms, the equality $\omega^a(\mathscr{L}_{\vec N}\,e_b)=\tilde\omega^a(\mathscr{L}_{\vec N}\,\tilde e_b)$ which holds for symmetry compatible shifts, and (2.11), implies the constancy of the coefficients $d\psi(Z)=b_aZ^a$ and $d\theta^X(Z)=c^X_aZ^a$ which appear in (2.13) even if Z is time dependent.

The symmetry types of the vacuum and nonvacuum spacetimes considered here are classified by considering the action of the general linear group on the parameter spaces which characterize the symmetry. Introduce first the space of possible structure constant tensors, parametrized by the Behr decomposition[4] [letting $S_2(R^3)$ be the space of symmetric 3×3 matrices]

$$\mathscr{C}=\{(n^{ab},a_b)\in S_2(R^3)\times R^3\mid a_an^{ab}=0\} \tag{2.15}$$

and its obvious analog for the spatial self-similarity actions

$$\mathscr{C}^{sss}=\{(n^{ab},a_a,b_a)\in\mathscr{C}\times R^3\mid b_aC^{ab}=0\} \tag{2.16}$$

and its further generalization to allow one symmetry compatible source

$$\mathscr{C}^{sss,X}=\{(n^{ab},a_a,b_a,c^X_a)\in\mathscr{C}^{sss}\times R^3\mid c^X_aC^{ab}=0\} \tag{2.17}$$

with obvious generalizations to the case of multiple sources. The space $\mathscr{C}$ is divided into two categories called class A ($a_a=0$) and class B ($a_a\neq 0$) which are included as subspaces of $\mathscr{C}^{sss}$ ($b_a\neq 0$) and of $\mathscr{C}^{sss,X}$ ($b_a=0=c_a^X$). The space $\mathscr{C}^{sss}$ contains in addition two nontrivial categories ($b_a\neq 0$) called class C ($a_a=0$) and class D ($a_a\neq 0$) and a subcategory class D_0 for which $b_a=a_a\neq 0$.

$GL(3,R)$ acts on these spaces of components in a way which reflects the transformation properties of the corresponding tensor densities and covectors over the Lie algebra $\mathscr{g}$ of G under a change of basis

$$\begin{aligned} &e_a\mapsto A^{-1b}{}_a e_b,\ \ \omega^a\mapsto A^a{}_b\omega^b,\ \ \mathbf{A}=(A^a{}_b)\in GL(3,R)\ ,\\ &(n^{ab},a_a)\mapsto((\det\mathbf{A}^{-1})A^a{}_cA^b{}_dn^{cd},a_bA^{-1b}{}_a)\ , \end{aligned} \tag{2.18}$$

with b_a and c_a^X transforming like a_a. A symmetry classification amounts to a description of the orbit spaces $\mathscr{C}/GL(3,R)$, $\mathscr{C}^{sss}/GL(3,R)$, and $\mathscr{C}^{sss,X}/GL(3,R)$. For the space $\mathscr{C}$, the isotropy group

$$GL(3,R)_{(\mathbf{n},a)}=\{\mathbf{A}\in GL(3,R)\mid A^a{}_dC^d{}_{fg}A^{-1f}{}_bA^{-1g}{}_c=C^a{}_{bc}\}\equiv Aut_e(\mathscr{g}) \tag{2.19}$$

at a point $(n^{ab},a_a)\in\mathscr{C}$ is the matrix representation of the automorphism group of a Lie algebra $\mathscr{g}$ with respect to a basis $\{e_a\}$ in which the structure constant tensor components are exactly this point. The isotropy groups for $\mathscr{C}^{sss}$ are subgroups of those at corresponding points of $\mathscr{C}$, with a similar statement holding for $\mathscr{C}^{sss,X}$. For example, on $\mathscr{C}^{sss}$ one has

$$GL(3,R)_{(\mathbf{n},a)}=\{\mathbf{A}\in Aut_e(\mathscr{g})\mid b_bA^{-1b}{}_a=b_a\}\ .$$

It is worth noting that the isotropy group on $\mathscr{C}^{sss}$ at a nonexceptional point (n^{ab},a_a,b_a) of $\mathscr{C}^{sss}$ is the same as that at (n^{ab},a_a) of $\mathscr{C}$ or if $a_a=0$ then it is the same as at the point (n^{ab},b_b) of $\mathscr{C}$, etc.

Invariants of this action of GL(3,R) are helpful in the classification.[3] When $a_a\neq 0$, one has an invariant h defined by

$$a_aa_b=\tfrac{1}{2}h\epsilon_{acd}\epsilon_{bfg}n^{cf}n^{dg}\ . \tag{2.20}$$

When $a_a=0$, one has invariants f and f^X defined by

$$\begin{aligned} b_ab_b&=\tfrac{1}{2}f\epsilon_{acd}\epsilon_{bfg}n^{cf}n^{dg}\ ,\\ c_a^Xc_b^X&=\tfrac{1}{2}f^X\epsilon_{acd}\epsilon_{bfg}n^{cf}n^{dg}\ , \end{aligned} \tag{2.21}$$

and when $a_a\neq 0$ and b_a and c_a^X are nonexceptional

$$b_a=fa_a,\ \ c_a^X=f^Xa_a\ . \tag{2.22}$$

For the exceptional one-forms no such invariants exist. The easiest way of displaying the symmetry classification is to pick a canonical representative of each orbit by reducing the symmetric matrix (n^{ab}) to its standard Sylvester form and aligning a_a with the third direction when nonzero, etc., leading to a table[18] of canonical representatives for the space $\mathscr{C}^{sss}$. [Other tables in the literature[3,19] only list the signs of the nonvanishing eigenvalues of (n^{ab}); converting these signs to ± 1 gives a table of canonical representatives.]

How do the various familiar sources fit into this scheme? Let us consider the Maxwell ($s=1$), Dirac ($s=\frac{1}{2}$), and scalar ($s=0$) fields and a perfect fluid, following the conventions of Misner, Thorne, and Wheeler.[9] Let $F_{\alpha\beta}$ be the components of a spatially homogeneous electromagnetic two-form with respect to the spacetime frame $\{e_\alpha\mid\alpha=0,1,2,3\}=\{\partial/\partial t,e_a\}$ with dual frame $\{\omega^\alpha\}=\{dt,\omega^a\}$ and introduce the electric and magnetic field densities $\mathscr{E}^a=g^{1/2}E^a=g^{1/2}F^{\perp a}$ and $\mathscr{B}^a=g^{1/2}B^a=1/2\epsilon^{abc}F_{bc}$, all of which have dimension $q=1$, and the complex fields $Z_{\alpha\beta}=F_{\alpha\beta}+i\,{}^*F_{\alpha\beta}$ and $\mathscr{Z}^a=g^{1/2}(F^{\perp a}+i\,{}^*F^{\perp a})=\mathscr{E}^a-i\mathscr{B}^a$, where ${}^*F_{\alpha\beta}=1/2\eta_{\alpha\beta\gamma\delta}F^{\gamma\delta}$ are the components of the dual of F and $\eta_{0123}=Ng^{1/2}$. The energy-momentum tensor is $T_{\alpha\beta}=(8\pi)^{-1}\mathrm{Re}(Z^*_{\alpha\gamma}Z^\gamma{}_\beta)$, where $Z^*_{\alpha\beta}$ is the complex conjugate of $Z_{\alpha\beta}$. Let Ψ be a spatially homogeneous Dirac spinor field with dimension $q=-\frac{1}{2}$ and the energy-momentum tensor $T_{\alpha\beta}=-\frac{1}{2}i\bar{\Psi}\gamma_{\alpha}\overleftrightarrow{\nabla}_{\beta}\Psi$ and let ϕ be a real or complex scalar field (dimension $q=0$) with energy-momentum tensor $T_{\alpha\beta}=(4\pi)^{-1}(\partial_{(\alpha}\phi^*\partial_{\beta)}\phi-\frac{1}{2}g_{\alpha\beta}\partial_\gamma\phi^*\partial^\gamma\phi)$. Since the energy-momentum tensor is real, it is invariant under constant phase transformations of the complex fields which occur in it. In the electromagnetic case there are just duality transformations

$$(Z,\mathscr{Z})\mapsto(e^{i\varphi}Z,e^{i\varphi}\mathscr{Z})\leftrightarrow F\mapsto\cos\varphi F-\sin\varphi\,{}^*F\ . \tag{2.23}$$

Thus one can introduce the phase factors θ^{em}, θ^D, and θ^S and the parameters c_a^{em}, c_a^D, and c_a^S for electromagnetic, Dirac, and complex scalar field sources. The electromagnetic case has been discussed extensively by McIntosh.[20] A perfect fluid, on the other hand, is completely determined by its energy-momentum tensor and so must have the symmetry of the metric. The fluid variables (ρ,p,u_a,n,μ,l,v_a) have respective dimensions $(-2,-2,1,-2\gamma^{-1},2(\gamma^{-1}-1),3-2\gamma^{-1},2\gamma^{-1}-1)$ for a perfect fluid with equation of state $p=(\gamma-1)\rho$ with $\gamma\in[1,2]$ a constant.[21]

III. DIAGONAL GAUGE

It is most convenient to work with a basis $\{e_a\}$ of $\mathscr{g}$ in which the components of the structure constant tensor and of the one-forms associated with the symmetry action are assumed to be in a certain simple form called standard diagonal form in which the symmetric matrix (n^{ab}) is diagonal and the covector (a_a) is aligned with the third direction. Introduce the standard diagonal form subspace $\mathscr{C}_D\subset\mathscr{C}$ of points which satisfy these conditions

$$\mathscr{C}_D=\{(n^{ab},a_a)\in\mathscr{C}\mid(n^{ab})=\mathrm{diag}(n^{(1)},n^{(2)},n^{(3)}),a_b=a\delta^3{}_b,a\geq 0,an^{(3)}=0\}\ . \tag{3.1}$$

Correspondingly introduce the space

$$\mathscr{C}_D^{sss}=\{(n^{ab},a_a,b_b)\in\mathscr{C}^{sss}\mid(n^{ab},a_a)\in\mathscr{C}_D,b_a=b^{\mathrm{exc}}\delta_{\mathrm{III}}E_a+b\delta^3{}_a,bn^{(3)}=0\}\ , \tag{3.2}$$

where

$$E_a = |n^{(2)}|^{1/2}\delta^1{}_a + \mathrm{sgn}\, n^{(1)}\, |n^{(1)}|^{1/2}\delta^2{}_a \tag{3.3}$$

is an R^3-valued function on $\mathscr{C}_D$ which satisfies $E_a C^{ab}=0$ on the Bianchi type III orbit (characterized by $h=-1$) and δ_{III} is the characteristic function of the Bianchi type III orbit having the value 1 on the orbit and zero elsewhere. For a single source introduce the space

$$\mathscr{C}_D^{\mathrm{sss},X} = \{(n^{ab},a_a,b_a,c_a^X) \mid (n^{ab},a_a,b_a)\in \mathscr{C}_D^{\mathrm{sss}},\ c_a^X = c_1^X\delta_{\mathrm{I}}\delta^1{}_a + c_2^X(\delta_{\mathrm{I}}+\delta_{\mathrm{II}})\delta^2{}_a + c^{X\mathrm{exc}}\delta_{\mathrm{III}}E_a + c^X\delta^3{}_a, c^X n^{(3)} = c_2^X n^{(2)} = 0\}\ , \tag{3.4}$$

while for multiple sources one simply includes each c_a^X in the definition.

The explanation of these definitions is as follows. The semisimple Bianchi types VIII and IX [for which $\det(n^{ab})\neq 0$] do not admit bi-invariant forms and require all of the covectors to vanish. For the nonsemisimple Bianchi types, assuming $(n^{ab},a_a)\in\mathscr{C}_D$ and $n^{(3)}=0$, the one-form $\omega^3=\tilde{\omega}^3=dx^3$ is always bi-invariant, but the space of bi-invariant one-forms is larger than span $\{\omega^3\}$ for the "degenerate" Bianchi types I, II, and III whose adjoint groups have dimension less than the generic dimension 3. For these three types the space $\mathscr{g}^*\cap\tilde{\mathscr{g}}^*$ of bi-invariant one-forms has dimension 3, 2, and 2, respectively, which leads to the terms involving the Bianchi-type characteristic functions. The most general type II twist one-form is obtained only for the case $n^{(1)}\neq 0$. For $\mathscr{C}_D^{\mathrm{sss}}$ one can use the matrix automorphism group to align the homothetic one-form with the third direction in the Bianchi type I case; similarly for one of the source spaces one can assume $c_1^X=0$. Of the three Bianchi types I, II, and III, only the case B type III admits exceptional one-forms, but all these types allow the possibility of exceptional nonvacuum spacetimes.

For each point of $\mathscr{C}_D$ let G be the simply connected Lie group with the basis $\{e_a\}$ of its Lie algebra $\mathscr{g}$ explicitly parametrized by its expression in canonical coordinates of the second kind $\{x^a\}$ which are global coordinates on $G\sim R^3$ for all but the type IX points of $\mathscr{C}_D$ where $G\sim S^3$ and their range must be restricted.[14] In this way we obtain from (2.2) a $\mathscr{C}_D^{\mathrm{sss}}$-parametrized spacetime $(R\times G, {}^4\underset{\sim}{g})$ which includes all spatially homogeneous and spatially self-similar spacetimes except the Kantowski-Sachs spacetimes and their spatially self-similar generalizations. For a given point of $\mathscr{C}_D^{\mathrm{sss}}$ or $\mathscr{C}_D^{\mathrm{sss},X}$, the explicit basis $\{e_a\}$ of $\mathscr{g}$ can be changed by any element of the invariance group of that point under the action of GL(3,R), a subgroup of the matrix automorphism group $Aut_e(\mathscr{g})$ with respect to that basis, without changing the parameters which characterize the symmetry type of the given class of vacuum or nonvacuum spacetimes. This symmetry may be used to simplify initial data.

For vanishing shift vector field, the Einstein equations for a spatially self-similar spacetime can be written in Hamiltonian form with a nonpotential force arising from the spatial curvature and lapse derivative terms in the evolution equations exactly as in the spatially homogeneous case.[15] The nonzero shift case may also be treated exactly as in the spatially homogeneous case.[22] One finds an additional contribution $2^3 d\psi(\vec{N})\pi^{ab}dg_{ab}$ to the momentum-dependent force which arises in the nonzero shift case, provided one adds the term

$$P_{\vec{N}} = e^{-2\psi}(\mathscr{L}_{\vec{N}}\, {}^3\underset{\sim}{g})_{ab}\pi^{ab} = P(\mathrm{ad}_e(\vec{N})) \tag{3.5}$$

to the zero shift Hamiltonian, using the notation of Ref. 22. [Note the misplaced factor of 2 and missing $g^{1/2}$ in Eq. (3.7) of this reference.] The additional term involving the homothetic one-form generates the conformal scaling associated with the gauge dependence of the spatially homogeneous rescaled fields due to the gauge dependence of ψ. By imposing the additional gauge condition $\mathscr{L}_{\vec{N}}\psi=0$ on the shift vector field, one makes ψ gauge invariant and avoids this complication. This will be assumed here.

Two different choices of spatial gauge lead to symmetry adapted spatial frames which are related to each other by a time-dependent matrix belonging to the automorphism group $Aut_e(\mathscr{g})$ (Ref. 21). The usual and most transparent gauge is the zero shift gauge or "orthogonal gauge," but a much more convenient choice of shift maintains the diagonality of the spatial metric. Such a diagonal gauge, intimately connected to the standard diagonal form of the structure constant tensor, offers the advantage of simplifying all expressions involving the spatial metric. However, rather than working directly in diagonal gauge one can obtain the diagonal gauge field equations by a certain gauge decomposition of the orthogonal gauge variables, an approach which has the advantage of incorporating the shift variables into the Hamiltonian system on the same footing as the other gravitational variables.

First one introduces the transformation matrix from the orthogonal gauge spatial frame $\{e_a\}$ to the diagonal gauge spatial frame $\{e'_a\}$. This matrix assumes values in a particular 3-dimensional subgroup of the special automorphism matrix group of the Lie algebra $\mathscr{g}$ and may be parametrized by canonical coordinates of the second kind with respect to the basis $\{\boldsymbol{\kappa}_a\}$:

$$e'_a = S^{-1b}{}_a e_b,\quad \omega'^a = S^a{}_b\omega^b\ ,$$

$$\mathbf{S} = e^{\theta^1\boldsymbol{\kappa}_1}e^{\theta^2\boldsymbol{\kappa}_2}e^{\theta^3\boldsymbol{\kappa}_3}\in SAut_e(\mathscr{g})\ ;$$

$$\left.\begin{aligned} \boldsymbol{\kappa}_a &= e^{-\alpha^a}(n^{(b)}\mathbf{e}^c{}_b + n^{(c)}\mathbf{e}^b{}_c) \\ e^{\alpha^a} &= 2^{-1/2}(n^{(b)2}+n^{(c)2})^{1/2} \\ \hat{n}^{(a)} &= n^{(a)}e^{\alpha^a-\alpha^b-\alpha^c} \end{aligned}\right\}\ (a,b,c)=\sigma^+(1,2,3)\ ; \tag{3.6}$$

$$[\boldsymbol{\kappa}_a,\boldsymbol{\kappa}_b] = \hat{C}^c{}_{ab}\boldsymbol{\kappa}_c\ ,$$

$$\hat{C}_{ab} = \epsilon_{abd}\hat{n}^{ad},\quad (\hat{n}^{ab}) = \mathrm{diag}(\hat{n}^{(1)},\hat{n}^{(2)},\hat{n}^{(3)})\ .$$

Here $\mathbf{e}^a{}_b$ is the 3×3 matrix whose only nonzero component is a one in the ath column and bth row, while the notation $\sigma^+(1,2,3)$ means an even permutation of the triplet (1,2,3). The matrices $\boldsymbol{\kappa}_a$ are well-defined functions everywhere on $\mathscr{C}_D$ except where rank $(\mathbf{n})<2$ and the scale matrix $e^{\boldsymbol{\alpha}}=\text{diag}(e^{\alpha^1},e^{\alpha^2},e^{\alpha^3})$ is singular; at these points they have direction-dependent limits from points of higher rank. The set of such limiting values are interpreted as possible values of $\boldsymbol{\kappa}_a$, which are everywhere nonvanishing and linearly independent multivalued functions on $\mathscr{C}_D$.

The spatial metric matrix and the source variables are then decomposed into diagonal gauge variables denoted by a prime superscript and the transformation between the two sets of variables

$$\mathbf{g}=\mathbf{S}^T\mathbf{g}'\mathbf{S},\quad X=\sigma(\mathbf{S}^{-1})X' . \tag{3.7}$$

Here σ is the tensor density representation under which the source variable X transforms, while the diagonality of $\mathbf{g}'$ characterizes diagonal gauge

$$\begin{aligned}&\mathbf{g}'=e^{2\boldsymbol{\beta}} ,\\ &\boldsymbol{\beta}=\text{diag}(\beta^1,\beta^2,\beta^3)=\beta^0\mathbf{1}+\beta^+\text{diag}(1,1,-2)\\ &\qquad\qquad+\beta^-\text{diag}(\sqrt{3},-\sqrt{3},0) .\end{aligned} \tag{3.8}$$

Diagonal gauge is not unique when G is not semisimple, but the choice of transformation (3.6) picks out a particularly useful gauge in this class. The interpretation of (3.7) in terms of the change of spatial gauge is valid only when the additional gauge condition $\mathscr{L}_{\vec{N}}\psi=0$ holds since otherwise an additional conformal scaling would be involved. The variables $\{\beta^0=-\Omega,\beta^+,\beta^-\}\equiv\{\beta^A\}_{A=0,+,-}$ were introduced by Misner.[9] $\beta^{\pm}$ parametrize the space of trace-free diagonal matrices, while β^0 parametrizes the pure trace matrices and hence the determinant g' of $\mathbf{g}'$. Since $\mathbf{S}$ is unimodular this agrees with the determinant g of $\mathbf{g}$:

$$g^{1/2}=g'^{1/2}=e^{3\beta^0} . \tag{3.9}$$

The geometrical configuration space variables $\{\beta^A,\theta^a\}$ have associated velocities $\{\dot\beta^A,\dot\theta^a\}$ and conjugate momenta $\{p_A,p_a\}$. The automorphism variables $\{\theta^a\}$ are local coordinates on the group manifold $\hat{G}$, where it is more convenient to use noncanonical velocities and conjugate momenta which correspond to expressing components with respect to a right invariant frame on that group manifold[21]:

$$\dot{\mathbf{S}}\mathbf{S}^{-1}=\boldsymbol{\kappa}_a\dot{\tilde{W}}{}^a,\quad \dot{\tilde{W}}{}^a=\tilde{W}{}^a{}_b(\theta)\dot\theta^b,\quad \tilde{P}_a=\tilde{E}{}^b{}_a p_b . \tag{3.10}$$

The matrices $(\tilde{W}{}^a{}_b)$ and $(\tilde{E}{}^b{}_a)$ are inverse matrices and represent the coordinate components of the right invariant one-forms $\tilde{W}{}^a$ and vector fields $\tilde{E}_a$ associated with the basis $\{\boldsymbol{\kappa}_a\}$ of the matrix Lie algebra of $\hat{G}$; one can easily derive the Poisson brackets relations[21]:

$$\{\tilde{P}_a,\tilde{P}_b\}=\hat{C}{}^c{}_{ab}\tilde{P}_c,\quad \{\mathbf{S},\tilde{P}_a\}=\boldsymbol{\kappa}_a\mathbf{S} . \tag{3.11}$$

Any member of the equivalence class of shift vector fields which satisfy $[\text{ad}_e(\vec{N})^a{}_b]=\boldsymbol{\kappa}_a\dot{\tilde{W}}{}^a$ will induce the change of spatial frame (3.6) to diagonal gauge. The conditions (2.11) are identically satisfied for arbitrary values of $\dot{\tilde{W}}{}^a$ except for Bianchi types I, II, and III:

$$b_c\kappa_a{}^c{}_b=0=c_c^X\kappa_a{}^c{}_b\leftrightarrow b'_a=b_a,\quad c_a^{X'}=c_a^X . \tag{3.12}$$

For types I and II the condition involving b_a selects those values of $\boldsymbol{\kappa}_a$ whose one-parameter groups leave b_a invariant. This condition is automatically true for the type III nonexceptional case, but in the exceptional case b_a is invariant under only a one-parameter subgroup of $\hat{G}$ so b'_a necessarily depends on two independent linear combinations of the variables $\{\theta^a\}$, leading to many complications. In particular diagonal gauge is incompatible with the assumption leading to (2.11). The same is true for any exceptional case source-filled spacetime: at least one of the set of homothetic and twist one-forms is time dependent in the diagonal gauge, leading to explicit dependence of its primed components on the transformation matrix $\mathbf{S}$.

Evaluating the ADM gravitational Lagrangian density and the Legendre transformation and then the associated gravitational Hamiltonian H^G leads to the results

$$\begin{aligned}
&\mathscr{L}^G=N(\mathscr{T}-U^G),\quad \mathscr{T}=N^{-2}e^{3\beta^0}(6\eta_{AB}\dot\beta^A\dot\beta^B+\overline{\mathscr{G}}_{ab}\dot{\tilde{W}}{}^a\dot{\tilde{W}}{}^b),\quad p_A=N^{-1}(12e^{3\beta^0}\eta_{AB}\dot\beta^B,\quad \tilde{P}_a=N^{-1}(12e^{3\beta^0})\tfrac{1}{6}\overline{\mathscr{G}}_{ab}\dot{\tilde{W}}{}^b ,\\
&\dot\beta^A=N(12e^{3\beta^0})^{-1}\eta^{AB}p_b,\quad \dot{\tilde{W}}{}^a=N(12e^{3\beta^0})^{-1}6\overline{\mathscr{G}}{}^{-1ab}\tilde{P}_b,\quad H^G=N\mathscr{H}^G=N(\mathscr{T}+U^G) ,\\
&N\mathscr{T}=N(12e^{3\beta^0})^{-1}(\tfrac{1}{2}\eta^{AB}p_Ap_B+3\overline{\mathscr{G}}{}^{-1ab}\tilde{P}_a\tilde{P}_b),\quad U^G=e^{\beta^0}V^*+2e^{3\beta^0}(e^{-2\beta})^{ab}(3a_aa_b-4a_ab'_b+b'_ab'_b) ,\\
&V^*=\tfrac{1}{2}[(n^{(1)})^2e^{-8\beta_1^+}+(n^{(2)})^2e^{-8\beta_2^+}+(n^{(3)})^2e^{-8\beta_3^+}]-(n^{(2)}n^{(3)}e^{4\beta_1^+}+n^{(3)}n^{(1)}e^{4\beta_2^+}+n^{(1)}n^{(2)}e^{4\beta_3^+}) ,\\
&\left.\begin{aligned}&\overline{\mathscr{G}}_{aa}=(\overline{\mathscr{G}}{}^{-1aa})^{-1}=\tfrac{1}{2}e^{-2\alpha^a}(n^{(b)}e^{2\sqrt{3}\beta_a^-}-n^{(c)}e^{-2\sqrt{3}\beta_a^-})^2\\ &(\eta_{AB})=(\eta^{AB})=\text{diag}(-1,1,1),\quad 6\beta_a^+\equiv-2\beta^a+\beta^b+\beta^c,\quad 2\sqrt{3}\beta_a^-\equiv\beta^b-\beta^c\end{aligned}\right\}\ (a,b,c)=\sigma(1,2,3) .
\end{aligned} \tag{3.13}$$

When $b_a=b\delta^3{}_a$, the gravitational potential and the nonpotential force are explicitly[13,15]

$$\begin{aligned}&U^G=e^{\beta^0}V^*+2e^{\beta^0+4\beta^+}(a-b)(3a-b) ,\\ &Q=4(a-b)e^{\beta^0+4\beta^+}[2(3a-2b)d\beta^++4b\,d\beta^0+e^{\alpha^3}\overline{\mathscr{G}}_{33}\tilde{W}{}^3]\equiv Q_+d\beta^++Q_0d\beta^0+Q_3\tilde{W}{}^3 ,\end{aligned} \tag{3.14}$$

while the gravitational supermomentum components are

$$\mathscr{H}^G_{a'}=-\tilde{P}_b\rho^b{}_a-(a-2b/3)\delta^3{}_ap_++b/3\delta^3{}_ap_0 , \tag{3.15}$$

where the constant matrix ρ and its determinant are

$$\rho=e^{\alpha}+2^{-1/2}(3a-2b)(-\text{sgn}n^{(1)}\mathbf{e}^2{}_1+\text{sgn}n^{(2)}\mathbf{e}^1{}_2),\quad \det\rho=\tfrac{1}{2}\text{sgn}n^{(1)}n^{(2)}[n^{(1)}n^{(2)}+(3a-2b)^2]e^{\alpha^3}\ . \tag{3.16}$$

Forming the total super-Hamiltonian and supermomentum

$$H=N\mathscr{H}-N(\mathscr{H}^G+\mathscr{H}^{\text{em}}+\mathscr{H}^S)\ ,$$
$$\mathscr{H}_{a'}=(\mathscr{H}^G_{a'}+\mathscr{H}^{\text{em}}_{a'}+\mathscr{H}^S_{a'})\ , \tag{3.17}$$

one obtains the gravitational equations of motion

$$\begin{bmatrix}\beta^A\\ p_A\end{bmatrix}^{\cdot}=\left\{\begin{bmatrix}\beta^A\\ p_A\end{bmatrix},H\right\}+\begin{bmatrix}0\\ NQ_A\end{bmatrix}\ ,$$
$$\begin{bmatrix}\theta^a\\ \widetilde{P}_a\end{bmatrix}^{\cdot}=\left\{\begin{bmatrix}\theta^a\\ \widetilde{P}_a\end{bmatrix},H\right\}+\begin{bmatrix}0\\ \delta^3{}_a NQ_3\end{bmatrix}\ , \tag{3.18}$$

and the gravitational constraints

$$\mathscr{H}=0=\mathscr{H}_{a'}\ . \tag{3.19}$$

In the most general case, the matrix ρ is invertible and one may solve the supermomentum constraints (3.15) for the momenta $\widetilde{P}_a$ in terms of the source variables. One is left with a reduced system involving the diagonal gauge gravitational variables $\{\beta^0,\beta^+,\beta^-\}$ and their momenta which is driven by the spatial curvature and the source energy-momentum as well as the kinetic energy of the variables θ^a which remains as an effective potential. The reduced Hamiltonian equations including the nonpotential force contribution are equivalent to part of the Einstein equations expressed in diagonal gauge. The source equations themselves can also be expressed in diagonal gauge; the velocities $\dot{\widetilde{W}}{}^a$ expressed in terms of the momenta $\widetilde{P}_a$ and in turn expressed in terms of the source variables parametrize the shift vector field associated with diagonal gauge. Using the primed source variables, the variables θ^a do not appear explicitly in the nonexceptional case and so have been eliminated from the system. In the exceptional case this is not true, and some or all of the variables θ^a remain explicit, since $\widehat{G}$ is no longer a gauge group. The case in which ρ is degenerate requires special handling exactly as in the perfect-fluid case.[13,21]

The final remaining constraint on the reduced system, namely, the super-Hamiltonian constraint, may be removed by choosing β^0 or Ω as the new time variable.[13–15] The only component of the nonpotential force which directly affects the unconstrained gravitational variables $\beta^+\beta^-$ in the nonexceptional case is Q_+ which may be eliminated by redefining the gravitational potential:

$$U^G=e^{\beta^0}V^*+U^{G\text{bad}},\quad U^{G,\text{eff}}\equiv e^{\beta^0}V^*+U^{G\text{bad,eff}}\ ,$$
$$U^{G\text{bad}}\equiv 2e^{\beta^0+\beta^+}(a-b)(3a-b)\ , \tag{3.20}$$
$$U^{G\text{bad,eff}}\equiv 2e^{\beta^0+\beta^+}b(a-b)\ ,$$
$$-\partial_+U^{G\text{bad}}+Q^+=-\partial_+U^{G\text{bad,eff}}\ .$$

The potential $U^{G\text{bad,eff}}$ generates the correct contribution to the $\beta^+\beta^-$ equations of motion without a nonpotential force correction. Thus for the purposes of the fully reduced system of gravitational variables, it is $U^{G,\text{eff}}$ which generates the correct Hamiltonian equations. In the spatially homogeneous case $b=0$ or the class D_0 case $a-b=0$, it is in fact V^* alone which drives the $\beta^+\beta^-$ variables.

IV. SOURCE VARIABLES AND EQUATIONS

A spatially homogeneous electromagnetic field acting as the source of a gravitational field is best described in the Hamiltonian approach using the orthogonal gauge MTW (Ref. 9) field variables $\mathscr{E}^a$ and $\mathscr{B}^a$ already introduced above, or the complex combination $\mathscr{Z}^a=\mathscr{E}^a-i\mathscr{B}^a$. The stretched and twisted electromagnetic field is obtained by the transformation (2.3) with $q=1$:

$$\underset{\sim}{\mathscr{Z}}=e^{\psi+i\theta^{\text{em}}}\mathscr{Z},\quad d\theta^{\text{em}}=c_a^{\text{em}}\omega^a\ , \tag{4.1}$$

with $\mathscr{E}$ and $\mathscr{B}$ given as the real and imaginary parts of the complex conjugate field $\mathscr{Z}^*$. The Lagrangian density, super-Hamiltonian, and supermomentum of the spatially homogeneous electromagnetic field in orthogonal gauge and diagonal gauge are

$$\mathscr{L}^{\text{em}}=\frac{1}{8\pi}(2\kappa)Ng^{-1/2}g_{ab}(\mathscr{E}^a\mathscr{E}^b-\mathscr{B}^a\mathscr{B}^b)$$
$$=\frac{1}{8\pi}(2\kappa)Ng'^{-1/2}g'_{ab}(\mathscr{E}'^a\mathscr{E}'^b-\mathscr{B}'^a\mathscr{B}'^b)\ ,$$
$$\mathscr{H}^{\text{em}}=\frac{1}{8\pi}(2\kappa)g^{-1/2}g_{ab}(\mathscr{E}^a\mathscr{E}^b+\mathscr{B}^a\mathscr{B}^b)$$
$$=\frac{1}{8\pi}(2\kappa)g'^{-1/2}g'_{ab}(\mathscr{E}'^a\mathscr{E}'^b+\mathscr{B}'^a\mathscr{B}'^b)\ , \tag{4.2}$$
$$\mathscr{H}^{\text{em}}_a=-\frac{1}{4\pi}(2\kappa)\epsilon_{abc}\mathscr{E}'^b\mathscr{B}'^c$$
$$=\left[-\frac{1}{4\pi}(2\kappa)\epsilon_{dbc}\mathscr{E}'^b\mathscr{B}'^c\right]S^d{}_a\ ,$$

using the MTW convention of multiplication of the "true" expressions by a factor of 2κ, where κ is the constant appearing in the Einstein equations $G_{\alpha\beta}=\kappa T_{\alpha\beta}$. These expressions are also equal to the rescaled expressions for the stretched and twisted fields since no derivatives occur.

Maxwell's equations for the stretched and twisted field take the form

$$\dot{Z}^a=-iNg^{-1/2}\mathscr{Z}_b[C^{ba}-\epsilon^{bac}(b_c+ic_c)]\ ,$$
$$0=(-2a_a+b_a+ic_a^{\text{em}})\mathscr{Z}^a\ . \tag{4.3}$$

In the nonexceptional case the divergence constraint reduces to the form $0=(-2a+b+ic^{\text{em}})\mathscr{Z}^3$ which forces $\mathscr{Z}^3=0$ unless $b-2a=c^{\text{em}}=0$. To transform these equations to the diagonal gauge frame one must introduce the time derivative of the transformation matrix

$$\dot{\mathbf{S}}\mathbf{S}^{-1}=\kappa_A \dot{\tilde{W}}^a, \quad \mathscr{Z}'^a=S^a{}_b\mathscr{Z}^b ,$$
$$(\mathscr{Z}'^a)^{\cdot}=-iNg^{-1/2}\mathscr{Z}'_b[C^{ba}+\epsilon^{bac}(b'_c+ic_c^{\mathrm{em}'})] + \dot{\tilde{W}}^c\kappa_c{}^a{}_b\mathscr{Z}'^b , \tag{4.4}$$
$$0=(-2a_a+b'_a+ic_a^{\mathrm{em}'})\mathscr{Z}'^a .$$

In the nonexceptional case these equations admit a constant of the motion ζ which involves the Killing form of the Lie algebra $\mathscr{g}$

$$\zeta=\tilde{\gamma}_{ab}(\mathscr{E}^a\mathscr{E}^b+\mathscr{B}^a\mathscr{B}^b)=\tilde{\gamma}_{ab}(\mathscr{E}'^a\mathscr{E}'^b+\mathscr{B}'^a\mathscr{B}'^b) = \sum_{a=1}^{3}\tilde{\gamma}_{aa}\,|\,\mathscr{Z}'^a\,|^2 ,$$
$$\tilde{\gamma}_{ab}\equiv-\tfrac{1}{2}\gamma_{ab}\equiv-\tfrac{1}{2}C^d{}_{ac}C^c{}_{bd}=(1-h)\Delta(\mathbf{n})_{ab} ,$$
$$\Delta(\mathbf{n})_{ab}\equiv\tfrac{1}{2}\epsilon_{acd}\epsilon_{bfg}n^{cf}n^{dg} , \tag{4.5}$$
$$\Delta(\mathbf{n})_{ab}=\mathrm{diag}(n^{(2)}n^{(3)},n^{(3)}n^{(1)},n^{(1)}n^{(2)}) ,$$
$$\tilde{\gamma}_{ab}n^{bc}=(1-h)(\det\mathbf{n})\delta^c{}_a=\tilde{\gamma}_{ab}C^{bc} ,$$
$$\tilde{\gamma}_{ab}\epsilon^{bcd}a_d=0 .$$

In the semisimple case this is related to the electromagnetic parameter e of the Brill "electromagnetized" Taub solution by $\zeta=8\pi\kappa^{-1}e^2$ (Refs. 15 and 23). However, the divergence constraint in the nonexceptional case requires ζ to vanish unless $b-2a=c^{\mathrm{em}}=0$. At points of $\mathscr{C}_D$ where ζ is identically zero independent of the values of $|\,\mathscr{Z}'^a\,|$, one may often obtain a nonzero constant of the motion by suitably rescaling ζ and taking the limit as one approaches these points from more general points of $\mathscr{C}_D$. This has already been described in detail for the corresponding perfect-fluid constant of the motion.[24] For example, scaling γ_{ab} to $(\mathrm{Tr}\,\mathbf{n}^2)^{-1}\gamma_{ab}$ in ζ is relevant to the semisimple values of κ_a for the Bianchi type I case.

Next consider a massless spatially homogeneous complex scalar field. Since it has dimension $q=0$ one need only twist this field:

$$\underset{\sim}{\phi}=e^{i\theta^S}\phi, \quad d\theta^S=c_a^S\omega^a . \tag{4.6}$$

The rescaled spatially homogeneous Lagrangian density, super-Hamiltonian, and supermomentum of this twisted field in orthogonal and diagonal gauge, respectively, are

$$\mathscr{L}^S=\tfrac{1}{2}(2\kappa/4\pi)Ng^{1/2}(N^{-2}\dot{\phi}^*\dot{\phi}-g^{ab}c_a^Sc_b^S\phi^*\phi)$$
$$=\tfrac{1}{2}(2\kappa/4\pi)Ng'^{1/2}(N^{-2}\dot{\phi}^*\dot{\phi}-g'^{ab}c_a^{S'}c_b^{S'}\phi^*\phi) ,$$
$$\mathscr{H}^S=\tfrac{1}{2}(2\kappa/4\pi)[(2\kappa/4\pi)^{-2}g^{-1/2}\pi_\phi^*\pi_\phi+g^{1/2}g^{ab}c_a^Sc_b^S\phi^*\phi]$$
$$=\tfrac{1}{2}(2\kappa/4\pi)[(2\kappa/4\pi)^{-2}g'^{-1/2}\pi_\phi^*\pi_\phi + g'^{1/2}g'^{ab}c_a^{S'}c_b^{S'}\phi^*\phi] , \tag{4.7}$$
$$\mathscr{H}_a^S=i(\pi_\phi\phi-\pi_\phi^*\phi^*)c_a^S=i(\pi_\phi\phi-\pi_\phi^*\phi^*)c_b^{S'}S^b{}_a ,$$

where $\pi_\phi=(2\kappa/4\pi)N^{-1}g^{1/2}\dot{\phi}^*$ is the conjugate momentum. The equations of motion for the twisted field are just the result of expressing the wave equation in first-order form:

$$\dot{\phi}=(2\kappa/4\pi)^{-1}g^{-1/2}N\pi_\phi^*(2\kappa/4\pi)^{-1}g'^{-1/2}N\pi_\phi^* ,$$
$$\dot{\pi}_\phi=(2\kappa/4\pi)Ng^{1/2}[2i(b_a-a_a)-c_a^S]c^{Sa}\phi^* \tag{4.8}$$
$$=(2\kappa/4\pi)Ng'^{1/2}[2i(b'_a-a_a)-c_a^{S'}]c^{S'a}\phi^* .$$

These may be interpreted as the Hamiltonian equations of motion obtained from $N\mathscr{H}^S$ if ϕ and ϕ^* are treated as independent variables with the usual Poisson brackets $\{\phi,\pi_\phi^*\}=2=\{\phi^*,\pi_\phi\}$ and a nonpotential force is introduced

$$\dot{\phi}=\{\phi,N\mathscr{H}^S\}, \quad \dot{\pi}_\phi=\{\pi_\phi,N\mathscr{H}^S\}+NQ^S , \tag{4.9}$$
$$Q^S=(2\kappa/4\pi)2ig^{1/2}c^{S'a}(b_a-a_a) .$$

The basic Poisson brackets follow from the usual Poisson brackets for the real and imaginary parts of $\phi=\phi_1+i\phi_2$ and their conjugate momenta, in terms of which $\pi_\phi=\pi_{\phi_1}-i\pi_{\phi_2}$. Reality of the scalar field corresponds to the special solution $\phi_2=0=\pi_{\phi_2}$ of the equations of motion when $c_a^S=0$. For an untwisted scalar field, c_a^S must vanish, making π_ϕ a constant and contributing a term proportional to $g^{-1/2}$ to the super-Hamiltonian.

For the gravitational equations of motion (3.17) to be well defined, one must specify the Poisson brackets between the gravitational and source variables. The scalar field variables ϕ and π_ϕ and the contravariant densities $\mathscr{E}^a$ and $\mathscr{B}^a$ commute with the canonical gravitational variables, but from (3.11) one has

$$\{\mathscr{Z}'^a,\tilde{P}_b\}=\kappa_b{}^a{}_c\mathscr{Z}'^c . \tag{4.10}$$

V. QUALITATIVE COSMOLOGY FOR ELECTROMAGNETIC-SCALAR FIELD SPACETIMES

Of the many questions one can ask about the solutions of the equations of motion for the finite-dimensional electromagnetic-scalar field system, one of the most natural to investigate is the question of their asymptotic behavior. If one considers the big bang solutions which start at an initial singularity (which may or may not be physical) and then expand, one may study the limiting behavior approaching the initial singularity, and in those models which continue expanding indefinitely, the asymptotic behavior at large times after the initial singularity. The first case has intrigued many people over the past few decades.

There are various approaches one may take in formulating the problem of describing the behavior of these cosmological solutions near the initial singularity. Spatially homogeneous vacuum models and later perfect fluid models were first studied by Lifshitz, Khalatnikov, and Belinsky (BKL), whose work is summarized in several long articles,[25] and which involves direct analytical approximation, and by Misner[26,27] and Ryan[28] who employed the ADM Hamiltonian formulation to reduce the problem to that of the motion of a particle under the influence of various potentials. Bogoyavlensky and Novikov later applied the qualitative theory of differential equations to the system in its Hamiltonian form for orthogonal fluid flow in a series of papers summarized in a long review article.[29] Their work has recently been extended to the general perfect-fluid case by Rosquist.[30]

Real scalar field sources were considered by Nariai[31] using the Misner-Ryan formalism and by Belinsky and Khalatnikov[32] who included electromagnetic fields as well using a five-dimensional Kaluza-Klein formulation. Electromagnetic fields alone were studied by Bogoyavlensky[33] using the techniques developed by himself and Novikov for the perfect-fluid case and by Spokoiny[34] from the BKL point of view. More recently Waller[35] has discussed electromagnetic fields in the Misner-Ryan approach.

The "qualitative cosmology" approach developed by Misner and Ryan and discussed in detail by Jantzen[13] for the general perfect-fluid spatially homogeneous case is quite helpful in understanding in a pictorial way the asymptotic dynamics approaching the classical initial singularity in these finite-dimensional cosmological models. Since the natural gravitational variables which are essential to the dynamics are logarithmic, the potentials which drive the system involve terms which are exponential in these variables. Because of this exponential dependence, a regime is easily reached when approaching the initial singularity in which at most one or two of these terms tend to be important at any given time and one can idealize their effects on the system in terms of an equivalent collision of the system point with a representative potential contour called a wall.

The analogy with the classical mechanics of a particle moving under the influence of time-dependent potentials is drawn in Misner's supertime time gauge[27] specified by the lapse choice $N=N^{\text{super}}\equiv 12g^{1/2}=12e^{3\beta^0}$. This makes the β contribution to the kinetic energy $N\mathscr{T}$ in the Lagrangian or Hamiltonian of the system equal to that of a point particle in a flat three-dimensional Lorentz spacetime on which the variables $\{\beta^0,\beta^+,\beta^-\}$ are orthonormal coordinates. β^0 is the time coordinate while β^+ and β^- are the two spatial coordinates. The "time dependence" of the potentials which contribute to the total potential-energy function occurs explicitly through factors of e^{β^0} and implicitly through the source variables.

Any potential term whose explicit dependence on β^A involves a single exponential can be assigned a constant velocity in $\beta^+\beta^-$ space associated with that explicit dependence. Since one is interested in running time backward toward the initial singularity, the variable $\Omega=-\beta^0$ is convenient. A contour line of a potential involving such an exponential factor

$$\exp(w_A\beta^A)=\exp(-w_0\Omega+w_+\beta^++w_-\beta^-)$$

will move in $\beta^+\beta^-$ space with an "Ω speed" $(w_+{}^2+w_-{}^2)^{-1/2}|w_0|$ in the direction $(w_+{}^2+w_-{}^2)^{-1/2}(w_+,w_-)$ perpendicular to the straight contour lines if the implicit time dependence is ignored. The latter of course contributes an additional time-dependent component to the velocity of the contour lines of the potential since the contour line associated with a particular value of the potential must change if the value of the potential changes due to the dependence on the variables other than β^+ and β^-. Even if the explicit velocity is zero, which occurs when the potential does not explicitly depend on β^0, the implicit contribution will still have the direction given above. The quantity $(w_+{}^2+w_-{}^2)^{-1/2}$ is the characteristic constant for the above exponential. Superimposing two different exponential potentials whose contours intersect at an angle will lead to noticeable distortion of the true contour from the joined straight line contours at their vertex only on distance scales compared to the larger characteristic constant (associated with the slower rise). The distance is measured in the Euclidean metric of the $\beta^+\beta^-$ plane.

The variables (β_a^+,β_a^-) defined by (3.12) for $a=1,2$ are orthonormal coordinate systems rotated clockwise by $2\pi/3$ and $4\pi/3$ radians, respectively, from the $\beta^+\beta^-$ system which coincides with the case $a=3$. Because the potentials arise from various contractions of the matrix of primed metric components with one or two index objects, their $\beta^+\beta^-$ dependence occurs only through exponential factors involving these three sets of variables. The contour lines associated with potential terms involving a single exponential as described above have an angle of inclination to the β^+ axis which may take one of six independent values equally spaced in the interval from 0 to π (see Fig. 2 of Ref. 13).

Consider the effective gravitational potential for the nonexceptional case in the supertime time gauge given explicitly in (3.13), (3.14), and (3.20):

$$N^{\text{super}}U^{G,\text{eff}}=e^{4\beta^0}V^*+2b(a-b)e^{4(\beta^0+\beta^+)}\ . \tag{5.1}$$

Each of the six individual terms in $e^{4\beta^0}V^*$ as well as the final term involve a single exponential of a linear combination of the variables β^A. According to the preceding discussion, the first three terms have unit Ω speed $\frac{1}{2}$ in the negative β_a^+ direction, while the second three terms have unit Ω speed in the positive β_a^+ direction, respectively, while the final term has unit Ω speed in the positive β^+ direction. In each case the direction of motion for increasing Ω is the direction of increasing values of the potential. No implicit contributions to the velocity exist since the potentials depend only on β^A. Superimposing all of these individual potentials yields the total effective gravitational potential whose contour lines for large values of Ω when all terms are nonzero essentially form an equilateral triangle associated with the three exponential potentials $U_g^{(a)}=\frac{1}{2}n^{(a)2}e^{4(\beta^0-2\beta_a^+)}$ with Ω speed $\frac{1}{2}$, the vertices of which deviate from the triangle due to their superposition and the four unit Ω speed potentials. Taking the limit as a function on $\mathscr{C}_D$ where some of the terms go to zero pushes some of the sides of the triangle out to infinity.[13]

The effective potential remaining from the θ^a variables consists of three terms $U_c^{(a)}$ called centrifugal potentials, each of which depends only on β^- and $\tilde{P}_a$, respectively:

$$\sum_{a=1}^{3}U_c^{(a)}\equiv\sum_{a=1}^{3}6\tilde{P}_a{}^2(e^{-\alpha^a}n^{(b)}e^{2\sqrt{3}\beta_a^-}-e^{-\alpha^a}n^{(c)}e^{-2\sqrt{3}\beta_a^-})^{-2}\ ,$$
$$(a,b,c)=\sigma^+(1,2,3)\ . \tag{5.2}$$

When $e^{-2\alpha^a}n^{(b)}n^{(c)}\neq 0$, however, the potential $U_c^{(a)}$ does not involve a single exponential term, but the contour lines are still straight lines of constant β_a^-. In either case the explicit velocity of a given contour line is zero since the potential is independent of β^0, but the variable $\tilde{P}_a$ when not constant contributes an "implicit velocity" term of either sign in the β_a^+ direction. These potentials have already been discussed in detail.[13]

The scalar field and electromagnetic super-Hamiltonians contribute source potentials to the total potential driving the $\beta^\pm$ variables. The $\beta^\pm$ dependence occurs only through the factors $g'g'^{aa}=e^{-4(\beta^0+\beta_a^+)}$ and $g'_{aa}=e^{2\beta^0-4\beta_a^+}$, respectively:

$$N^{\rm super}\mathscr{H}^S=6(2\kappa/4\pi)[(2\kappa/4\pi)^{-2}\pi_\phi^*\pi_\phi+g'g'^{ab}c_a^{S'}c_b^{S'}\phi^*\phi]\ ,$$
$$N^{\rm super}\mathscr{H}^{\rm em}=\sum_{a=1}^{3}6(2\kappa/4\pi)g'_{aa}\,|\,\mathscr{E}'^a\,|^2\equiv\sum_{a=1}^{3}U_{\rm em}^{(a)}\ . \tag{5.3}$$

A nonexceptional twisted scalar field, i.e., one for which $c_a^{S'}=c^S\delta^3{}_a$, contributes a single exponential potential with explicit unit Ω speed in the positive β^+ direction. An untwisted scalar field does not directly effect the $\beta^\pm$ equations of motion. A general electromagnetic field contributes three purely exponential potentials $U_{\rm em}^{(a)}$, each of which has an explicit Ω speed of $\frac{1}{2}$ in the negative β_a^+ direction, respectively, the direction of increasing potential values, exactly like the first three terms in the gravitational potential. However, the contour lines of the superposition of all three electromagnetic potentials when all three are present are closed, resembling closed equilateral triangles when the vertex effects of the superposition are small.

The source potential contour lines all have implicit velocity contributions. For example, decreasing $|\,\mathscr{E}'^a\,|^2$ causes the contour line associated with a fixed value of the potential $U_{\rm em}^a$ to "move outward," that is, move in the direction of increasing values of the potential, while decreasing $|\,\mathscr{E}'^a\,|^2$ causes the contour line to "move inward." This leads to a variable velocity depending on the rate of change of $|\,\mathscr{E}'^a\,|^2$ which adds to the constant velocity arising from the explicit β^A dependence of the potential. If $|\,\mathscr{E}'^a\,|^2$ momentarily passes through zero, the contour lines move out to infinity. A similar behavior is exhibited by the three centrifugal potentials $U_c^{(a)}$ where the role of $|\,\mathscr{E}'^a\,|^2$ is played by $\tilde{P}_a{}^2$. In the purely exponential case $e^{-2\alpha^a}n^{(b)}n^{(c)}=0$ the behavior is identical, but squared reciprocals of hyperbolic sines and cosines appear when this parameter combination is nonzero and the contour lines "moving outward" toward increasing values converge from opposite directions on a fixed contour line of infinite or maximum value, respectively, while those "moving inward" diverge from this special contour line.

The gravitational equations of motion are exactly soluble when all but one term in the total potential are neglected, in almost all cases corresponding to specialized initial data and particular values of the symmetry parameters. These exact solutions may be used to relate the incoming and outgoing asymptotic-free states before and after the interaction with a given potential in the limit in which only that potential is effective at a given time. Such an interaction has been called a "bounce."[26–28] The free system with no potentials is just the diagonal Bianchi type I system. The solution is geodesic straight line motion in β space; the super-Hamiltonian constraint requires the geodesic to be null, i.e., have unit Ω speed. Except for (1) a small neighborhood of isotropy in the type IX case where the unit Ω speed terms in the spatial curvature make the total gravitational potential negative and (2) the parameter values $b\in(a,3a)$ in the nonexceptional spatially self-similar case, all the contributions to the super-Hamiltonian are non-negative, making the three-velocity of the system point timelike, corresponding to ever-increasing β^0 from the initial singularity $\beta^0\to-\infty$.

An untwisted scalar field has constant canonical momentum π_ϕ and contributes a constant term to the supertime time gauge Hamiltonian. This makes the geodesic motion timelike rather than null when the remaining potentials are negligible, leading to less than unit Ω speed in the free phase, potentially changing the character of the evolution.[31,32] For a twisted scalar field this term in the super-Hamiltonian becomes implicitly time dependent but has similar consequences.

Misner[26,27] introduced the idea of associating a moving "wall" with each potential by selecting a particular contour line which marks the point at which the potential has a large enough value to significantly affect the motion of the system point, independently of the other potentials. The equation

$$0=-\tfrac{1}{2}p_0{}^2+U(\beta^{\rm wall}) \tag{5.4}$$

locates the contour or "wall" at which a turning point of the motion would occur if U were time independent, p_0 were constant, all other potentials were zero, and the motion were orthogonal to the contour. Because of the exponential cutoffs, the system point will not be affected by this potential until its distance from the wall approaches the characteristic constant associated with the potential. One may then ignore this potential until an actual "collision" or bounce against the wall occurs, which may then be approximately described by the analytic solution associated with that particular potential alone. If the potential has "implicit time dependence" due to dependence on the other variables, such an approximation is valid when the change due to those variables is insignificant during the scattering of the system point. The effect of the potential on the system point may then be reduced to a "bounce law" connecting the free motion before and after the bounce; essentially the influence of the potential is reduced to an equivalent bounce of the free system point against an infinite step function potential located at the wall contour, the result of the collision being described by the bounce law.

A bounce against one of the three Ω speed $\frac{1}{2}$ gravitational potentials is described by the exact diagonal Bianchi type II spatially homogeneous solution (with or without a constant term arising from a scalar field). A rescaling of this solution by a factor of 2 describes a bounce against one of the Ω speed $\frac{1}{2}$ electromagnetic potentials $U_{\rm em}^{(a)}$ with constant $|\,\mathscr{E}'^a\,|^2$, which is the diagonal electromagnetic spatially homogeneous Bianchi type I solution with only one nonzero component of $\mathscr{E}^a$. The bounce law in each case is the result of simply Lorentz transforming reflection from the moving potential in its rest frame.[13,27] A bounce against a unit Ω speed potential is described by the exact Bianchi type VI_0 spatially homogeneous Taub-type solution, a solution which is obtained by introducing null coordinates in β space which comove with the potential.[13,27]

A so-called "mixing bounce" occurs when the system point is affected by two (or more) potentials whose individual contours intersect at a nonzero angle. This occurs when the system point penetrates the vertex at a scale compared to the characteristic constants of the two potentials. This has been studied from several points of view for mixing bounces that occur for the gravitational potential.[25–27,36] For example, the gravitational potential due to V^* alone with $n^{(3)}=0$ and $n^{(1)}n^{(2)}\neq 0$ has a vertex where the contours of $U_g^{(1)}$ and $U_g^{(2)}$ intersect, the character of which is determined by the third nonzero term in V^*. The contour of the total gravitational potential at the vertex[13] is open or closed if $n^{(1)}n^{(2)}$ is positive or negative, respectively; in the open case a narrow channel extends outward to infinity. The vertex between two electromagnetic potentials is always closed from the point of view of the total electromagnetic potential since no analogous third term exists.

The system point is confirmed by the super-Hamiltonian constraint to a region of $\beta^+\beta^-$ space "inside" the potential walls, a region which may be "open" or "closed" depending on the number of potentials which are present (disregarding vertex details). The key feature of the system which makes the above bounce description feasible is that as one approaches the initial singularity $\Omega\to\infty$, the walls all recede from each other due to their explicit β dependence, allowing the system point to move over larger and larger distances in $\beta^+\beta^-$ space before colliding with a potential. Each collision then occurs at distance scales determined by the characteristic constants which become more and more insignificant compared to the distance scale of the free motion phase; similarly vertex effects become increasingly less important for the same reason. Furthermore, the limit $\Omega\to\infty$ tends to freeze out the implicit effects on the system due to the source variables which could conceivably oppose the motion of the walls due to the explicit β dependence. This can be seen at a superficial level by examining the source equations of motion. Rigorous verification of these features requires and deserves a more sophisticated study of the system.

The existence of the so-called "oscillatory approach to the singularity" for a particular symmetry type and source depends first on having enough potentials (with explicit Ω speed $\frac{1}{2}$ or less) present to close up the region of $\beta^+\beta^-$ space to which the system point is confined by the super-Hamiltonian constraint, neglecting vertex effects which are important only for special initial data. The system point then rattles around inside such an expanding closed "trapping region" overtaking and bouncing off the walls since its free phase Ω speed in the direction of the wall it is approaching is always greater than that of the wall itself. A second condition for the existence of this oscillatory approach is the absence of the scalar field, owing to its effect on the Ω speed of the system point. This latter quantity is determined by the super-Hamiltonian constraint

$$0=N^{\text{super}}\mathscr{H}=\tfrac{1}{2}(-p_0{}^2+p_+{}^2+p_-{}^2)+U^{\text{total}}\,,$$

$$\begin{aligned} d\beta/d\Omega &\equiv[(d\beta^+/d\Omega)^2+(d\beta^-/d\Omega)^2]^{1/2}\\ &=[(p_+{}^2+p_-{}^2)/p_0{}^2]^{1/2}\\ &=(1-U^{\text{total}}/p_0{}^2)^{1/2}\,. \end{aligned} \tag{5.5}$$

The bounce laws show that $|p_0|$ decreases through collisions with the potential walls which move with Ω speed $\frac{1}{2}$. Indeed its equation of motion shows that $d|p_0|/d\Omega<0$ necessarily holds except possibly in a neighborhood of isotropy in the type IX case and for the interval of values $b\in(a,3a)$ in the spatially self-similar case. If U^{total} has a constant term then as $|p_0|$ decreases, the free phase Ω speed of the system point will decrease as well until it decreases below $\frac{1}{2}$. At this point it will no longer be able to catch the potentials with Ω speed $\frac{1}{2}$ and should remain in the free phase, thus ending a possible oscillatory phase which might otherwise have continued indefinitely in the absence of the scalar field.

The gravitational potential alone is enough to form closed trapping regions in the semisimple case of Bianchi types VIII and IX. Addition of a general perfect-fluid source in the spatially homogeneous case leads to closed trapping regions for all of the remaining Bianchi types except I and V, where isotropic spatial curvature robs the system of a nontrivial less than unit Ω speed gravitational wall.[13] The same statement holds for the nonexceptional spatially self-similar case, with the same qualification when the symmetry group is of Bianchi type I or V. The trapping regions when they exist are formed by two centrifugal walls and an opposing gravitational wall of Ω speed $\frac{1}{2}$. Without the tilted fluid, the supermomentum constraints eliminate at least two of the three centrifugal walls except for those allowed by the degeneracy of the supermomentum constraints (Bianchi types I, II, and $VI_{-1/9}$ in the homogeneous case).

A general electromagnetic source provides a closed trapping region except when the supermomentum or electromagnetic divergence constraints remove one or more of the three electromagnetic potentials. As long as $-2a_a+b_a+ic_a^{\text{em}}=0$, the divergence constraint [see (4.3) and (4.4)] is automatically satisfied and does not remove any of the electromagnetic potentials. However, when the gravitational supermomenta are not independent, another algebraic constraint restricts the electromagnetic field due to the algebraic constraint on its own supermomentum arising from the supermomentum constraint. For example, consider the spatially homogeneous Bianchi type I case with a spatially homogeneous electromagnetic field, obtained by setting all the symmetry parameters (and the scalar field) equal to zero. Then $\mathscr{E}^a$ are constants and the electromagnetic supermomentum must vanish since the gravitational supermomentum vanishes identically. This in turn requires that the electric and magnetic field densities be aligned along the same direction. This allows two centrifugal walls to form a closed trapping region with the electromagnetic wall or walls which are present. For this it is sufficient that the electric and magnetic fields not be eigenvectors of the extrinsic curvature, a possibility overlooked by Spokoiny.[34] A similar situation exists for the spatially homogeneous Bianchi type II case. When the supermomentum constraints are not degenerate, three centrifugal potentials will in general be present, and closed trapping regions are formed by two centrifugal walls and an electromagnetic wall as in the tilted perfect-fluid case. The discussion of all of these effects for the nonexceptional spatially self-similar case is similar to the class A spatially homogeneous case.

The walls associated with the nonexceptional twisted scalar field have unit explicit Ω speed and so cannot lead to closed trapping regions which might temporarily trap the system point before its "free phase" Ω speed decays due to the effect of the β independent term in the supertime time gauge Hamiltonian. However, the addition of the scalar field supermomentum to the supermomentum constraint may allow the presence of more electromagnetic potentials than otherwise would have been allowed and therefore lead to a temporary oscillatory phase.

VI. CONCLUSIONS

The present paper essentially extends Bogoyavlensky's formulation[33] of Bianchi type IX electromagnetic spacetimes to all of the spatially homogeneous and spatially self-similar symmetry types, allowing a less symmetric electromagnetic field and including a neutral complex scalar field as an additional source. The discussion of the system extends Waller's description of qualitative cosmology for the Bianchi type IX electromagnetic spacetimes[35] to the general finite-dimensional case. By extending Bogoyavlensky's application of the qualitative theory of differential equations to the present case, taking Rosquist's work into consideration,[30] a more rigorous understanding of the qualitative behavior of the system could be obtained. A step in this direction was done by Spokoiny,[34] who also included an untilted perfect-fluid source.

Additional symmetry,[21] either continuous or discrete, can be used to classify the specializations possible for each symmetry type. Some special solutions of the electromagnetic case exist in the literature, sometimes including an untilted perfect fluid.[37–40] These may be organized in a fashion similar to the summary of exact solutions given in Ref. 15; the $\mathscr{C}_D$ parametrization of the system enables one to identify families of solutions containing more than one symmetry type. The Brill solution, for example, is a member of a family of locally rotationally symmetric electromagnetic spacetimes containing Bianchi types I, II, VIII, and IX which allows the inclusion of a scalar field and an untilted stiff perfect fluid,[41] as may be understood from an analysis of the structure of the gravitational constraints.[15]

One need only take seriously the ideas developed over the past two decades toward understanding the structure of gravitational theories to arrive at the present formulation of the finite-dimensional case. These methods can be applied to any gravitational theory or choice of sources. They provide a framework with which one may put into perspective the hodgepodge of particular and seemingly unrelated results which are scattered throughout the literature. All too often particularization of a problem in mathematical cosmology forfeits the possibility of interpreting the results in a wider setting. The subject could benefit from some more understanding rather than the simple presentation of more solutions of particular differential equations. While concern for length has limited the present discussion, the foundation has been laid for a more global perspective of the topic.

APPENDIX: ELECTROMAGNETIC VECTOR POTENTIAL

By introducing a vector potential for an electromagnetic field without symmetry, half the Maxwell equations may be satisfied identically while the remaining equations take the form of a constrained Hamiltonian system.[8,9] Imposing symmetry on this system leads to complications. For the sake of brevity only untwisted electromagnetic fields will be considered, i.e., those for which $c_a^{\rm em}=0$; these share the symmetry of the metric. Suppose one tries to represent the electromagnetic two-form as the differential of a one-form of the same symmetry type in orthogonal gauge

$$\begin{aligned}
&\underset{\sim}{A}=e^{\psi}A=e^{\psi}(A_0dt+A_a\omega^a)\,,\\
&\underset{\sim}{F}=e^{\psi}F=d\underset{\sim}{A}\,,\\
&\mathscr{E}_a=N^{-1}g^{1/2}(-\dot{A}_a+A_0b_a),\quad \mathscr{B}^a=-A_b\tilde{C}^{ba}\,,\\
&\tilde{C}^{ab}\equiv C^{ab}-\epsilon^{abc}b_c=n^{ab}+\epsilon^{abc}(a_c-b_c)\,.
\end{aligned}\tag{A1}$$

In the spatially homogeneous case, A_0 does not contribute to the electromagnetic field and might as well be set equal to zero. There is no reason not to impose this same gauge con ition in the spatially self-similar case as well, so it will be assumed that $A_0=0$.

Two problems may arise. When the matrix $(\tilde{C}^{ab})$ is not symmetric one obtains the incorrect Hamiltonian equation for the momentum π_A^a conjugate to A_a, and when this matrix is degenerate, it does not necessarily produce the most general divergence-free magnetic field of the given symmetry type. (An asymmetric vector potential is needed.) The first problem can be solved by the introduction of a nonpotential force into the electromagnetic Hamiltonian equations. The second requires the addition of certain constant magnetic field density components of the expression derived from the vector potential.

From the electromagnetic Lagrangian in orthogonal gauge, the momentum π_A^a and electromagnetic super-Hamiltonian are evaluated to be

$$\begin{aligned}
&\pi_A^a=(2\kappa/4\pi)N^{-1}g^{1/2}g^{ab}\dot{A}_b=-(2\kappa/4\pi)\mathscr{E}^a\,,\\
&\mathscr{H}^{\rm em}=(2\kappa/4\pi)[\tfrac{1}{2}(2\kappa/4\pi)^{-2}g^{-1/2}g_{ab}\pi_A^a\pi_A^b\\
&\qquad\qquad+\tfrac{1}{2}g^{1/2}g_{cd}\tilde{C}^{ca}\tilde{C}^{db}A_aA_b]\,,
\end{aligned}\tag{A2}$$

while the correct Hamiltonian equations [the source-free Maxwell equations (4.3) for the evolution of the electric field density] require a nonpotential force

$$\begin{aligned}
&\dot{A}_a=\{A_a,N\mathscr{H}^{\rm em}\},\quad \dot{\pi}_A^a=\{\pi_A^a,N\mathscr{H}^{\rm em}\}+NQ^{{\rm em}a}\,,\\
&Q^{{\rm em}a}=(2\kappa/4\pi)g^{1/2}A_d\tilde{C}^d{}_b\tilde{C}^{[ba]}\,,
\end{aligned}\tag{A3}$$

whose sole effect is to reverse the order of the indices on $\tilde{C}^{ab}$ from the incorrect order which appears in the Poisson-brackets term. The usual Poisson-brackets relations $\{A_a, \pi^b_A\} = \delta^b{}_a$ hold.

When $(\tilde{C}^{ab})$ is degenerate, the vector potential can be decomposed into a piece lying in the kernel of the matrix and the remainder

$$A_a = A_a^{\text{cyclic}} + A_a^{\text{not}}, \quad A_b^{\text{cyclic}} \tilde{C}^{ba} = 0 \;, \tag{A4}$$

while the magnetic field density can be decomposed into a piece lying in the range of the matrix $(\tilde{C}^{ab})$ and a remainder which is subject to the divergence constraint

$$\begin{aligned} \mathscr{B}^a &= -A_b^{\text{not}} \tilde{C}^{ba} + \mathscr{B}^a_{\text{not}} \;, \\ (-2a_a + b_a)\mathscr{B}^a &= (-2a_a + b_a)\mathscr{B}^a_{\text{not}} \;. \end{aligned} \tag{A5}$$

Since $\mathscr{B}^a_{\text{not}}$ is not in the range of the matrix $(\tilde{C}^{ab})$, while Maxwell's equations place $\mathscr{B}^a$ in that range, it follows that $\mathscr{B}^a_{\text{not}}$ is constant. On the other hand, since A_a^{cyclic} does not appear in the super-Hamiltonian, its conjugate momentum $\pi^a_{A\,\text{cyclic}}$ is a constant of the motion, assuming no complications arise with the nonpotential force, i.e., assuming the component of that force along A_a^{cyclic} is zero.

In the nonsemisimple case of spatially homogeneous Bianchi type VIII and IX metrics, $\tilde{C}^{ab} = C^{ab} = n^{ab}$ is symmetric and nondegenerate and no problems arise ($A_a^{\text{cyclic}} = 0 = \mathscr{B}^a_{\text{not}}$). For the other extreme case of spatially homogeneous Bianchi type I metrics, $\tilde{C}^{ab} = C^{ab} = 0$ and $\mathscr{B}^a = \mathscr{B}^a_{\text{not}}$, while $A_a^{\text{not}} = 0$ and $\mathscr{E}^a$ is constant. For the remaining symmetry types an intermediate situation exists. The details will not be enumerated here. It is worth noting that for a spatially homogeneous electromagnetic field of class A ($a_a = b_a = c_a^{\text{em}} = 0$), the divergence constraint is identically satisfied by the electric and magnetic field densities and each is allowed three independent components. The finite-dimensional Maxwell equations in this case therefore do not model the infinite-dimensional equations (for fields without symmetry) which lead to only two independent degrees of freedom for the source-free electromagnetic field. This feature as well as the necessity of a nonpotential force in the Hamiltonian equations breaks the naive correspondence between the finite- and infinite-dimensional systems. This is of course due to the nonignorable nonvanishing divergences which occur in the variational formalism[42] and in decompositions associated with the constraints.[18]

Using the canonical variables $\{g_{ab}, \pi^{ab}; A_a, \pi^a_A; \phi, \pi_\phi\}$ one has a totally Hamiltonian system with Hamiltonian $N\mathscr{H}$, driven by the nonpotential forces when they are nonzero, and subject to the constraint equations. The point transformation (3.6)—(3.8) of the canonical gravitational variables must be extended to a noncanonical transformation of the full set of variables in order to solve the supermomentum constraints and simplify the system. One needs to introduce the primed components

$$A'_a = A_b S^{-1b}{}_a, \quad \pi'^a_A = S^a{}_b \pi^b_A \tag{A6}$$

in order to do this. A'_a and π'^a are canonically conjugate but both have nonzero Poisson brackets with the momenta $\tilde{P}_a$ as in (4.10). The electromagnetic constraint which requires that the electric field density be divergence-free

$$(-2a_a + b'_a)\pi'^a_A = 0 \tag{A7}$$

is easily imposed. The supermomentum constraints may then be solved as discussed at the end of Sec. III. Finally the super-Hamiltonian constraint may be eliminated by choosing β^0 as the new time variable. One is left with a reduced system in the remaining unconstrained variables which have canonical brackets among themselves.

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