Clustering in a quark gas

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In an infinite one-dimensional quark gas it is shown that a static color force, which increases at large distance, leads to a density fluctuation in the ground state. A self-consistent mean field can only be found for an effectively attractive quark-quark interaction that increases less than linearly at large distances. For a fixed coupling constant, the clustering disappears at high quark density.

It is an established fact that an infinite Fermi system cannot have a uniform ground state for an essentially attractive two-body interaction.[1] Clustering solutions have been shown to exist as mean-field solutions[2] which are stable towards particle-hole excitations.[3] In this note we address ourselves to the question of whether or not a quark gas will also exhibit a similar pattern. Two features make this investigation different from previous work:[2] (i) The effective quark-quark interaction increases at large distances and (ii) the color dependence $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ makes a contribution only to the exchange term in a mean-field calculation, if the ground state is assumed to be a color singlet.

A phase transition from nuclear matter to quark matter has been discussed in the literature; there[4] the authors use thermodynamical considerations. We use a microscopic approach when starting from Hartree-Fock equations. Our findings are similar in nature to those of some recent work[5] where the significance of a finite quark amplitude between nuclei is discussed.

We consider a one-dimensional model of a quark gas with the quarks interacting through the static force $V(x) = g\boldsymbol{\lambda}_1\cdot\boldsymbol{\lambda}_2|x|^\nu$, $\nu>0$. This phenomenological potential is motivated by the one-gluon exchange which yields $\nu=1$ from the one-dimensional Fourier transform of the bare gluon propagator q^{-2}. Actually the form of the potential that we should use is $\alpha_s(q)/q^2$ where the running coupling constant vanishes logarithmically (see, for instance, Ref. 6 and papers quoted therein). In our numerical procedure the logarithmic term is insignificant for large values of q since the pattern is essentially governed by the singular behavior at $q=0$. This in turn reflects the confining behavior of $V(x)$ for large $|x|$; in one dimension we find for $|x|^\nu$ the behavior $q^{-1-\nu}$, where a precise mathematical meaning of the singularity at $q=0$ is given in Ref. 7. In three dimensions, $V(x)\sim|x|$ gives rise to a q^{-4} behavior[6] for small values of q; an extension of our calculations to the three-dimensional case is in progress.

We use the relativistic expression for the kinetic energy term, but the static force renders the model nonrelativistic in character. This is similar in spirit to some recent work[8] on hadron structure in quark matter. In our work the emphasis lies on cluster formation by a mean field, while the authors of the quoted paper considered the interaction among model hadrons which are constructed explicitly. The lumps which we obtain in the single-quark distribution function originate by construction from color-singlet conglomerates, i.e., from clusters of three, six, etc., quarks as they may exist in nuclear matter.[9] A residual interaction among these clusters goes beyond the scope of this work. In particular, we believe that its long-range part requires a genuine relativistic treatment to reproduce the expected meson-exchange terms; the short-range repulsive core seems to be explainable by the Pauli principle.[8,10]

We employ the zero-temperature formalism, but investigate the dependence on the coupling constant g which can be related directly to a density dependence by a scaling property of the Hartree-Fock equation. As expected[4,8] the cluster formation is less pronounced the higher the density. The high-density limit corresponds to the noninteracting limit, i.e., a uniform quark distribution.

In the calculation we look for a self-consistent solution under the assumption of a static periodic density distribution. The periodicity assumption is technically advantageous and physically acceptable as it means a homogeneous hadron distribution if there is clustering. The appropriate quasiparticle operators are then given by the canonical transformation[2]

$$\psi_n^\dagger(\bar{k}) = \sum_{r=-\infty}^{\infty} \alpha_r^{(n)}(\bar{k})\, c_{\bar{k}+rQ}^\dagger \ , \tag{1}$$

where the $c_k^\dagger$ are the plane-wave operators. The fixed momentum Q determines the "lattice spacing" $d=2\pi/Q$. The total ground-state energy is minimized for $Q=2k_F$, where the Fermi momentum k_F is determined by the density of the system. This situation corresponds to a filled first Brillouin zone $|\bar{k}|\leq Q/2$ [$n=1$ in Eq. (1)]. The real $\alpha_r^{(n)}(\bar{k})$ are given by the eigenvectors of the coupled system:[2]

$$\omega^{(n)}(\bar{k})\alpha_t^{(n)}(\bar{k}) = \epsilon_t(\bar{k})\alpha_t^{(n)}(\bar{k}) - \tfrac{16}{3}g \sum_{s=-\infty}^{\infty}\sum_r \int_{-Q/2}^{Q/2} d\bar{q}\; V(\bar{k}-\bar{q}-rQ) \sum_{n=1}^{N_F} [\alpha_{r+t}^{(n)}(\bar{q})\alpha_{r+s}^{(n)}(\bar{q})]\alpha_s^{(n)}(\bar{k}) \ , \tag{2}$$

where the eigenvalue $\omega^{(n)}(\bar{k})$ and $\epsilon_t(\bar{k})$ are the single-particle and relativistic free-particle energies, respectively. Here we consider $N_F=1$ in accordance with $Q=2k_F$.

The Fourier transform $V(q)$ of the confining potential $|x|^\nu$ is defined by[7]

$$V(q) = \lim_{\mu\to 0}\int_{-\infty}^{\infty} dx\, |x|^\nu e^{-\mu|x|} e^{-iqx} \ , \tag{3}$$

where, for numerical convenience, small finite values of μ have been used, from which the actual limit of the solution of Eq. (2) has been extrapolated.

The expectation value of this interaction with respect to the uniform ground state (plane waves) is finite only for $\nu < 1$. Using[7]

$$V(q)=\frac{i}{\nu}\Gamma(\nu+1)\lim_{\mu\to 0}\frac{d}{dq}\left[\frac{1}{(\mu+iq)^\nu}-\frac{1}{(\mu-iq)^\nu}\right] \quad (4)$$

we obtain

$$\int_{-k_F}^{k_F}\int dk\,dk'\,V(k-k')=\frac{4\Gamma(\nu+1)}{\nu(1-\nu)}\sin\left[\frac{\pi\nu}{2}\right](2k_F)^{1-\nu}\ , \quad (5)$$

which becomes infinite for $\nu\to 1$. Note that, while the right-hand side of Eq. (5) is finite for $\nu > 1$, the integral does not exist in the classical sense.[7] This carries over to the nonuniform state, where self-consistent solutions of Eq. (2) are found by iteration for $\nu < 1$, while no solutions seem to exist when ν approaches unity. This is exhibited in the numerical procedure, where the extrapolation of the screening parameter μ to zero value is straightforward for $\nu < 1$, while for $\nu = 1$ no limit can be attained.

The parameters used in the calculation were $k_F = 2\ \text{fm}^{-1}$, $m = 0.1\ \text{fm}^{-1}$ and $10^{-2}\ \text{fm}^{-(\nu+1)} \leq g \leq 10\ \text{fm}^{-(\nu+1)} (\hbar = 1)$. We record the main results obtained.

(i) Self-consistent solutions are only obtainable for $g < 0$, i.e., for interactions that appear effectively attractive in the exchange term.

(ii) As discussed, the numerical solution of Eq. (2) becomes increasingly more difficult as $\nu\to 1$. For $\nu = 0$, $V(q)$ is a δ function[7] and we obtain the free solutions ($g = 0$) with shifted single-particle energies.

(iii) The single-particle energy spectrum $\omega^{(n)}(\bar{k})$ shows the same structure as the one obtained in the band model in solid-state physics, with k being a wave number in a Brillouin zone. Note, however, that this pattern is brought about by a self-consistent density distribution.

(iv) In contrast with the situation with finite-range potentials,[2] the critical interaction strength is $g_c = 0$, i.e., the total energy for the nonuniform ground state is immediately lower than that of the plane-wave ground state when the interaction is switched on. This is illustrated in Fig. 1. Furthermore, this feature does not depend on the density of the system as can be seen from the scaling law

$$\frac{E}{N}(\alpha g, Q)=\alpha^{1/(\nu+1)}\frac{E}{N}(g,\alpha^{-1/(\nu+1)}Q)\ , \quad (6)$$

which follows from Eq. (2) provided mass scaling is ignored. An immediate consequence is that higher densities correspond to weaker interaction, i.e., the clusters dissolve at high density.

(v) Typical self-consistent density distributions

$$\langle\psi^\dagger(x)\psi(x)\rangle=\sum_{n=-\infty}^{\infty}a_n(Q)\cos(nQx)\ , \quad (7)$$

where

$$a_n(Q)=\sum_{k'}\langle c_{k'}^\dagger c_{k'-nQ}\rangle,\quad n=0,\pm 1,\pm 2,\ldots, \quad (8)$$

are plotted in Fig. 2 for $Q = 2k_F$ and two different coupling strengths. For sufficiently large coupling strengths or low densities, the probability of finding a quark between clusters is virtually zero.

(vi) There is a singularity of the solution at $g = 0$ in that no perturbative expansion (Taylor series) exists around $g = 0$. This becomes obvious from Fig. 3, where the free solution is compared with a solution for $g < 0$. The single-

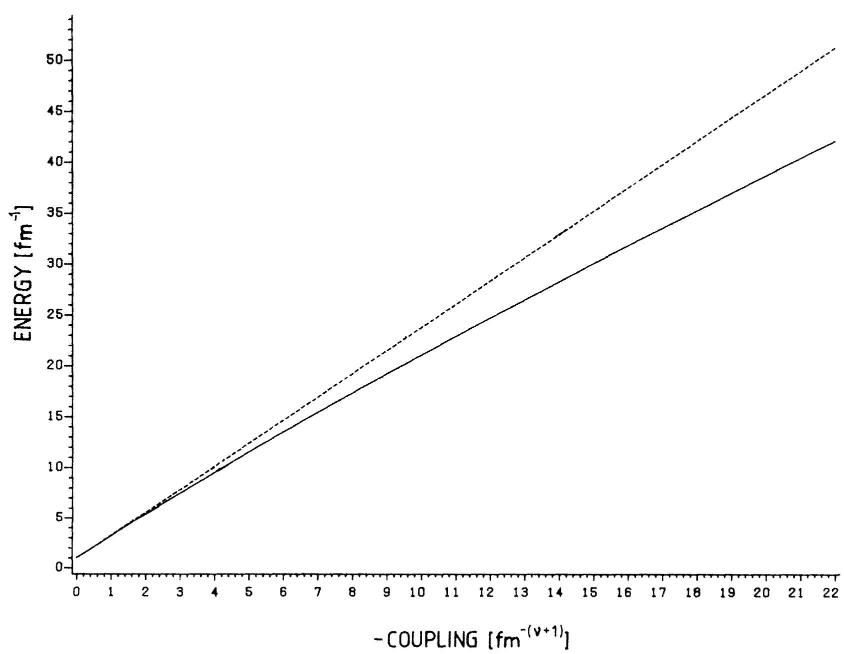

FIG. 1. The total energy per particle for the nonuniform solution vs negative coupling strength for $\nu = 0.2$ (solid curve). The dashed line indicates the corresponding plane-wave solution. The clustering solution is preferred immediately when the interaction strength is switched on. Qualitatively, this picture does not change when ν is varied between 0 and 1, nor if the density of the system changes.

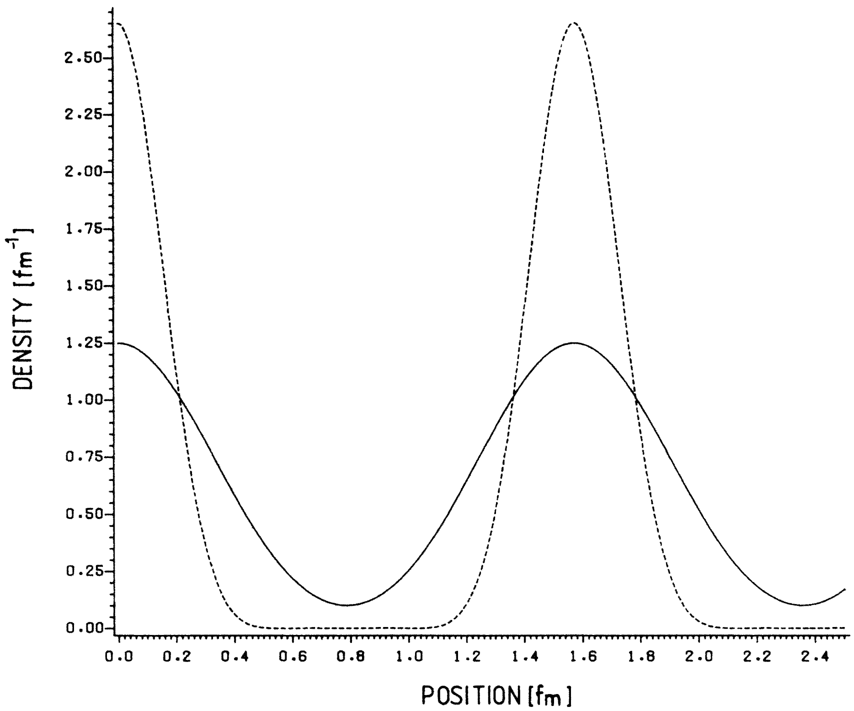

FIG. 2. Periodic density fluctuation for $\nu=0.2$ about the average $\rho_0=2/\pi$ fm^{-1}, i.e., for $Q=4$ fm^{-1}, for two coupling strengths $g=-2\pi$ fm$^{-(\nu+1)}$ (solid line) and -10π fm$^{-(\nu+1)}$ (dashed line).

particle density distribution is given by $n(k)=\alpha^2(k)$. While we do not know the precise form of $\alpha(k)$ in analytic terms, the curves are reminiscent of Fermi distributions at $T=0$ and $T>0$, respectively, displaying the well-known essential singularity in the variable T at zero. Note that the singularity at $g=0$ is in line with $g_c=0$ as discussed under (v).

In concluding, we note that despite great simplifications, the model does provide a mechanism for quark clustering. While at this level a more quantitative analysis does not seem appropriate—the most important results are listed under items (iv), (v), and (vi). We recall that these results are brought about by a confining force in combination with the $\boldsymbol{\lambda}\cdot\boldsymbol{\lambda}$ term.

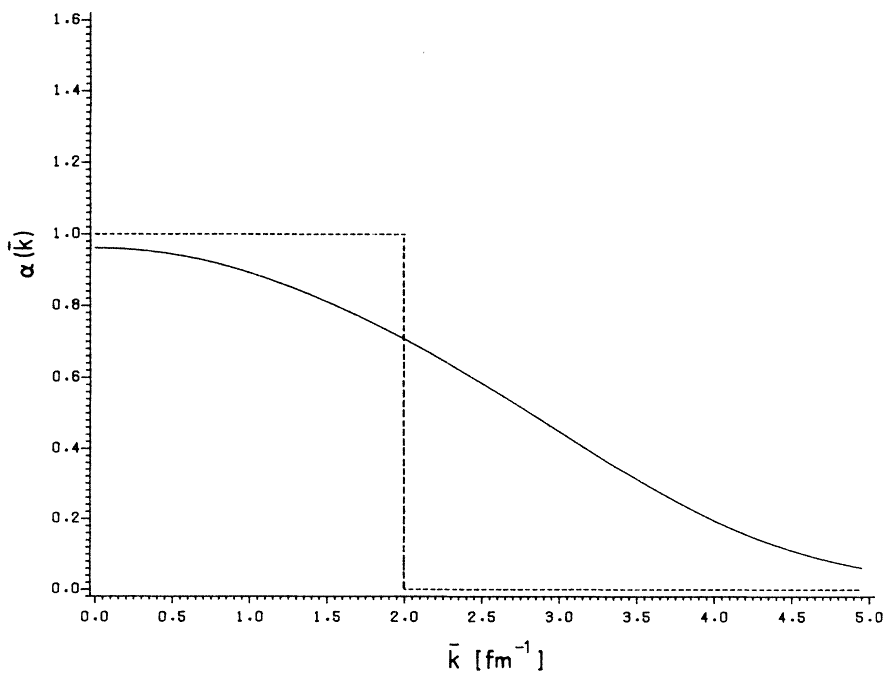

FIG. 3. $\alpha(k)$ for zero coupling (dashed curve) and $g=-2\pi$ fm$^{-(\nu+1)}$ (nonuniform ground state).

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