# Quantum-chromodynamic evolution of the baryon system

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We present a general method for solving the QCD evolution equations which govern relativistic multiquark wave functions. In the case of three-quark systems, we generate a light-cone basis of completely antisymmetric wave functions. This provides a general covariant classification of baryonic states. We then calculate the spin-orbit mixing generated by the QCD evolution kernel in the basis of completely antisymmetric representations. We are consistent with previous results, but additionally, we obtain a distinctive classification of nucleon and delta wave functions. The corresponding $Q^2$ dependence of the baryon distribution amplitudes distinguishes the nucleon and delta form factors.

## I. INTRODUCTION

The form of the short-distance behavior of a baryon wave function can be computed systematically in perturbative quantum chromodynamics (QCD).[1] The leading behavior of the baryon three-quark wave function at large momentum transfer or short distances is controlled through an evolution equation with an irreducible hard-scattering kernel which, in lowest order, is identical to the gluon-exchange potential. Since the running coupling constant $\alpha_s(Q^2)=4\pi/[\beta\ln(Q^2/\Lambda^2)]$ ($\beta=11-\frac{2}{3}n_f$, where $n_f$ is the number of flavors) is small for large momentum transfer $Q$, a perturbative calculation of the short-distance part of the wave function can be justified. The anomalous dimensions of the three-quark amplitude can also be predicted by the operator-product expansion and the renormalization group.[2]

A particularly convenient and physical formalism for studying processes with large momentum transfer is light-cone quantization, as discussed in Ref. 1. A systematic analysis of exclusive processes and hadron distribution amplitudes has been given, including solutions of the evolution equation of the three-quark system. Thus far the analysis of evolution equations has not been sufficiently detailed to give a complete classification of the proton, neutron, and delta states. In particular, the distinction between QCD predictions for the baryon wave functions and form factors needs to be clarified.

In this paper, we develop a systematic basis for the baryon system by constructing completely antisymmetric three-quark representations. The calculation of the QCD evolution kernel matrix in the basis of completely antisymmetric representations is given by a straightforward method. The solutions obtained by the present method are consistent with the preceding results,[1,2] but additionally we obtain a distinctive classification of the $N$ and $\Delta$ wave functions and the corresponding $Q^2$ dependence which discriminates the $N$ and $\Delta$ form factors.

The methods used in this paper have general applicability to the problem of analyzing the short-distance dynamics of multiquark systems. The analysis for four-quark systems in $SU(2)_C$ and six-quark systems in $SU(3)_C$ will presented in subsequent publications.[3]

In Sec. II we classify the baryon state constructing completely antisymmetric representations. In Sec. III we describe several properties of the three-quark evolution equation associated with the various quantum numbers and match the antisymmetric representations with the evolution equation.

To construct a basis of completely antisymmetric representations, we define products of spin and orbit representations in analogy to nonrelativistic wave functions. Mixings of spin and orbital representations are described in Sec. IV. The results for the anomalous dimensions and the eigensolutions are presented and discussed in Sec. V. Conclusions follow in the last section.

## II. ANTISYMMETRIC REPRESENTATIONS

Two identical fermions cannot occupy the same physical state; thus one describes interacting fermions in terms of antisymmetric wave functions. The systematic classification of the bound states of a fermion (including the baryons) is generated on the basis of antisymmetrized constituent representations. In order to describe relativistic systems we always refer to the valence Fock component of the bound-state wave function defined at equal light-cone time in the light-cone gauge. Further details may be found in Ref. 1.

A fermionic system in QCD is classified by the assignment of four quantum numbers: color ($C$), isospin ($T$), spin ($S$), and orbital ($O$). Each quantum sector of the wave function can be classified using irreducible representations with permutation symmetry denoted by Young diagrams.[4] The explicit construction of totally antisymmetric representations in terms of an orbital index-power basis will be described in the next subsection.

### A. Color ($C$), isospin ($T$), and spin ($S$) states

We can classify the quantum numbers of $C$, $T$, and $S$ by the group of $G=SU(3)_C\times SU(2)_T\times SU(2)_S$ without loss of generality. Each quantum state assigned by $C$, $T$, and $S$ is the irreducible representation of $G$, and each irreducible representation is denoted by the corresponding

 

Young diagram. Once a Young diagram is given, the explicit representation can be constructed from its permutation symmetry[4] and Schmidt orthogonalization.

All physical baryons are color-singlet states. The corresponding Young diagram is given by

$$\begin{array}{|c|}\hline r \\ \hline y \\ \hline b \\ \hline \end{array}$$

in $SU(3)_C$. Thus the explicit color representation of the baryon is fixed:

$$\begin{array}{|c|}\hline r \\ \hline y \\ \hline b \\ \hline \end{array} = \frac{1}{\sqrt{6}}(ryb + ybr + bry - byr - rby - yrb)$$
$$\equiv \frac{1}{\sqrt{6}}\epsilon_{ijk} , \tag{2.1}$$

where the completely antisymmetric Cartesian tensor $\epsilon_{ijk}$ ($i$, $j$, and $k$ correspond to one of $r$, $y$, and $b$) defines the color-singlet representation. The quantum state (color in this case) of the first, second, and third quark is represented by the first, second, and third location of every term in Eq. (2.1). Hereafter, we will use this convention for each quantum number unless we specifically denote the particle number.

The classification of the baryon into $N$ and $\Delta$ is given by the isospin label: i.e., $T=\frac{1}{2}$ and $\frac{3}{2}$, and the corresponding Young diagrams are

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array}$$

for $N$ and $\Delta$, respectively. The mixed symmetry

$$\begin{array}{|c|c|}\hline \ & \ \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1} \end{array}$$

has two orthogonal permutation symmetries, represented by two different Yamanouchi labels

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} .$$

As an example, we present the explicit representation of $T_Z=\frac{1}{2}$ for $T=\frac{1}{2}$ and $\frac{3}{2}$:

$$\begin{array}{|c|c|}\hline u & u \\ \hline \multicolumn{1}{|c|}{d} \\ \cline{1-1} \end{array} = \begin{cases} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{2}}(duu - udu) , \\ \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{6}}(duu + udu - 2uud) , \end{cases}$$
$$(T,T_Z)=(\tfrac{1}{2},\tfrac{1}{2}), \tag{2.2a}$$

$$\begin{array}{|c|c|c|}\hline u & u & d \\ \hline \end{array} = \frac{1}{\sqrt{3}}(uud + udu + duu) ,$$
$$(T,T_Z)=(\tfrac{3}{2},\tfrac{1}{2}) , \tag{2.2b}$$

where $(T,T_Z)=(\frac{1}{2},\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2})$ correspond to $p$ and $\Delta^+$, respectively.

The spin states of the three-quark system are classified by the Young diagrams for $S=\frac{1}{2}$ and $\frac{3}{2}$. The explicit representations are obtained from the isospin representations with the replacement of $u$ and $d$ by $\uparrow$ and $\downarrow$.

### B. Orbital ($O$) states

The orbital states are normally defined by the quantum numbers of angular momentum $L$ and $L_Z$. On the light cone, the quark distribution amplitude $\phi(x_i,Q)$ is defined by

$$\phi(x_i,Q) = \left[\ln\frac{Q^2}{\Lambda^2}\right]^{-3C_F/2\beta} \times \int^Q \left[\prod_{i=1}^{3}\frac{d^2\mathbf{k}_{\perp i}}{16\pi^3}\right] 16\pi^3\delta^2\left[\sum_{i=1}^{3}\mathbf{k}_{\perp i}\right]\psi^{(Q)}(x_i,\mathbf{k}_{\perp i}) , \tag{2.3}$$

where $\psi^{(Q)}(x_i,\mathbf{k}_{\perp i})$ is the wave function of three quarks which have longitudinal-momentum fractions $x_i=k_i^+/(\sum_{i=1}^{3}k_i^+)=(k_i^0+k_i^3)/[\sum_{i=1}^{3}(k_i^0+k_i^3)]$ and transverse momenta $\mathbf{k}_{\perp i}$. In this definition, the $L_Z=0$ projection defines the amplitude for finding the constituents collinear up to the scale $Q$. We will use as a basis for the orbital dependence of $\phi(x_i,Q)$ the index-power space representations $x_1^{n_1}x_2^{n_2}x_3^{n_3}$ with $n=n_1+n_2+n_3$. The total power is analogous, as far as permutation symmetry is concerned, to the angular momentum $L$ for the nonrelativistic system. In the QCD evolution equation the minimal anomalous dimensions $\gamma_n$ which determine hadronic amplitudes at very short distances are associated with small values of $n$; only the smallest powers of $x_i$ are important for probing the short-distance behavior of $\phi(x_i,Q)$. Thus, we consider the "orbital" symmetry on the index-power space ($n=n_1+n_2+n_3$) which determines the power of $x_1$, $x_2$, and $x_3$ such as $x_1^{n_1}x_2^{n_2}x_3^{n_3}$.

In this power space, the orbital states are determined by filling up the possible Young diagrams with the powers of $x_i$. For example, if we consider $n=n_1+n_2+n_3=0$ case, then the only possible Young diagram

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \end{array}$$

gives the representation

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline \end{array} = 1 . \tag{2.4}$$

For $n=1$ case, the possible diagrams and representations are

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline \end{array} = \frac{1}{\sqrt{3}}(x_1+x_2+x_3) , \tag{2.5}$$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline \multicolumn{1}{|c|}{1} \\ \cline{1-1} \end{array} = \begin{cases} \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \multicolumn{1}{|c|}{2} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{2}}(x_1-x_2), & \text{(2.6a)} \\ \\ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \multicolumn{1}{|c|}{3} \\ \cline{1-1} \end{array} = \dfrac{1}{\sqrt{6}}(x_1+x_2-2x_3) . & \text{(2.6b)} \end{cases}$$

However, the representations given by Eq. (2.5) are not independent of the representation given by Eq. (2.4) because of the conservation of momentum $\sum_{i=1}^{3} x_i = 1$. Generally, the orbital representations can overlap each other between the same diagrams. Thus we use the Gram-Schmidt orthogonalization procedure and normalize the states by the following rule between the orbital representations $\phi_n(x_i,Q_0)$ and $\phi_m(x_i,Q_0)$ with the same Young diagram:

$$\begin{aligned}\langle \phi_m(x_i,Q_0) \mid \phi_n(x_i,Q_0)\rangle &= \int [dx]\omega(x_i)\phi_m^*(x_i,Q_0)\phi_n(x_i,Q_0) \\ &= \delta_{mn}\ , \end{aligned} \tag{2.7}$$

where

$$[dx] = dx_1 dx_2 dx_3 \delta\left[1 - \left[\sum_{i=1}^{3} x_i\right]\right]$$

and $\omega(x_i) = x_1 x_2 x_3$.

After orthonormalization, we obtain the basis set of orbital states. The explicit representations and Young diagrams up to $n=2$ are presented in Table I. We note that the orbital representations in power space are independent of any dynamics, and any model-dependent representation can be projected onto our representation. A state which has arbitrary angular momentum $L$ can be projected on the corresponding index-power space.

### C. Antisymmetrization

In Secs. II A and II B, we showed that the quantum states for each $C$, $T$, $S$, and $O$ quantum number are explicitly represented by the permutation symmetry of the Young diagrams. In particular, the completely antisymmetric representation of a quark system is obtained by the inner product of $C$, $T$, $S$, and $O$ quantum states represented by the corresponding Young diagrams. As an example, let us construct the antisymmetric representation of the excited state of the proton with $(S,S_Z) = (\frac{3}{2},\frac{1}{2})$. For this state, $C$ and $T$ representations are given by Eqs. (2.1) and (2.2a), respectively, and the $S$ representation is given by Eq. (2.2b) with the replacement of $u$ and $d$ by $\uparrow$ and $\downarrow$. To construct the completely antisymmetric representations, we combine the possible orbital symmetries as given by the Clebsh-Gordan series of the permutation group $S_3$. In this case, the only possible orbital Young diagram is

$$\begin{array}{|c|c|}\hline \phantom{0} & \phantom{0} \\ \hline \phantom{0} \\ \cline{1-1}\end{array}\ .$$

The lowest state is $0^2 1(n=1)$ and the representation is given in Table I. If we consider the Clebsch-Gordan coefficients of the permutation group[4]

TABLE I. The orbital representations in the index-power space for the baryon system; the normalization constant is multiplied by the representation for the correct normalization.

| Index–Power | Young Diagram and Representation | Normalization |
|---|---|---|
| $n=0$ | $\begin{array}{\|c\|c\|c\|}\hline 0&0&0\\ \hline\end{array} = 1$ | $\sqrt{5!}$ |
| $n=1$ | $\begin{array}{\|c\|c\|}\hline 0&0\\ \hline 1\\ \cline{1-1}\end{array} = \left\{\begin{array}{l} \begin{array}{\|c\|c\|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array} = \frac{1}{\sqrt{2}}(x_1 - x_2) \\ \begin{array}{\|c\|c\|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} = \frac{1}{\sqrt{6}}(x_1 + x_2 - 2x_3) \end{array}\right.$ | $\sqrt{21} \times \sqrt{5!}$ |
| | $\begin{array}{\|c\|c\|c\|}\hline 0&1&1\\ \hline\end{array} = \frac{1}{\sqrt{3}}\left(x_2 x_3 + x_3 x_1 + x_1 x_2 - \frac{2}{7}\right)$ | $42\sqrt{5!}$ |
| $n=2$ | | |
| | $\begin{array}{\|c\|c\|}\hline 0&1\\ \hline 1\\ \cline{1-1}\end{array} = \left\{\begin{array}{l} \begin{array}{\|c\|c\|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array} = \frac{1}{\sqrt{2}}\left\{(x_2 x_3 - x_1 x_3) + \frac{1}{4}(x_1 - x_2)\right\} \\ \begin{array}{\|c\|c\|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array} = \frac{1}{\sqrt{6}}\left\{(x_2 x_3 + x_1 x_3 - 2x_1 x_2) + \frac{1}{4}(x_1 + x_2 - 2x_3)\right\} \end{array}\right.$ | $6\sqrt{28} \times \sqrt{5!}$ |

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array} = \frac{1}{\sqrt{2}} \begin{array}{|c|c|}\hline 1&3\\ \hline 2 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1&3\\ \hline 2 \\ \cline{1-1}\end{array} + \frac{1}{\sqrt{2}} \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array} , \tag{2.8}$$

then we can obtain the completely antisymmetric representation

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}_{\,CTSO} = \begin{array}{|c|}\hline r \\ \hline y \\ \hline b \\ \hline\end{array}_{\,C} \times \begin{array}{|c|c|}\hline u&u\\ \hline d \\ \cline{1-1}\end{array}_{\,T} \times \begin{array}{|c|c|c|}\hline \uparrow&\uparrow&\downarrow \\ \hline\end{array}_{\,S} \times \begin{array}{|c|c|}\hline 0&0\\ \hline 1 \\ \cline{1-1}\end{array}_{\,O}$$

$$= \frac{\sqrt{N}}{18} \epsilon_{ijk}(\uparrow\uparrow\downarrow + \uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow)$$
$$\times [duu(2x_1 - x_2 - x_3) + udu(-x_1 + 2x_2 - x_3) + uud(-x_1 - x_2 + 2x_3)] , \tag{2.9}$$

where $N = 21 \times 5!$ and $\epsilon_{ijk}$ is defined by Eq. (2.1). In a similar way, we can classify all possible three-quark states and obtain the explicit antisymmetric representations. In Table II we present the classification and the representations of the baryon system up to the power $n = 2$.

## III. THE BARYON EVOLUTION EQUATION

The three-quark evolution equation for the three-quark distribution amplitude $\phi(x,Q)$ with $L_Z = 0$ is given by[1]

$$x_1 x_2 x_3 \left[ \frac{\partial}{\partial \xi} + \frac{3C_F}{2\beta} \right] \widetilde{\phi}(x,Q) = \frac{C_B}{\beta} \int_0^1 [dy] V(x,y) \widetilde{\phi}(y,Q) , \tag{3.1}$$

where the reduced amplitude $\widetilde{\phi}(x,Q)$ and the variable $\xi$ are defined by

$$\phi(x,Q) = x_1 x_2 x_3 \widetilde{\phi}(x,Q) ,$$

and

$$\xi(Q^2) = \frac{\beta}{4\pi} \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \alpha_s(k^2) \sim \left[ \frac{\ln(Q^2/\Lambda^2)}{\ln({Q_0}^2/\Lambda^2)} \right] .$$

The color factor $C_B = (n_c + 1)/2n_c = \frac{2}{3}$ is fixed. The evolution kernel $V(x,y)$ is the sum over interactions between quark pairs $i,j$ due to exchange of a single gluon:

$$V(x,y) = 2x_1 x_2 x_3 \sum_{i \neq j} \theta(y_i - x_i) \delta(x_k - y_k) \times \frac{y_j}{x_j} \left[ \frac{\delta_{h_i \bar{h}_j}}{x_i + x_j} + \frac{\Delta}{y_i - x_i} \right] = V(y,x) , \tag{3.2}$$

where $\delta_{h_i \bar{h}_j} = 1(0)$ when the helicities of quark pairs $i,j$ are antiparallel (parallel). The infrared singularity at $x_i = y_i$ is canceled by $\Delta \widetilde{\phi}(y,Q) \equiv \widetilde{\phi}(y,Q) - \widetilde{\phi}(x,Q)$ reflecting the fact that the baryon is a color singlet.

We use the following general properties of the kernel.

(a) The color states are evolved by the operator $\sum_{a=1}^{8} (\lambda_a/2)(\lambda_a/2)$ [the $\lambda_a$ are the Gell-Mann matrices of $SU(3)_C$ group]. Under the action of this operator, the antisymmetric color-singlet representation does not change. The color factor $C_B$ is then fixed as $\frac{2}{3}$ for the baryon.

(b) QCD evolution conserves isospin.

(c) The helicity of the quarks is conserved. However, the evolution kernel given by Eq. (3.2) has the $\delta_{h_i \bar{h}_j}$ term, which means the spin evolution operator is not diagonal for spin multiplets classified by total spin while it is diagonal for $S_Z$ components.

(d) For the orbital evolution, the total power $n$ of orbital representation [total power of $\widetilde{\phi}_n(y)$ before gluon exchange and $\widetilde{\phi}_n(x)$ after gluon exchange] is conserved upon integration $\int_0^1 [dy]$. However, many different representations can have the same power $n$. Thus we must allow mixing between different orbital states under evolution. The evolution equation Eq. (3.1) has a general solution of the form

$$\phi(x,Q) = x_1 x_2 x_3 \sum_{n=0}^{\infty} \mathscr{A}_n \widetilde{\phi}_n(x) \left[ \ln \frac{Q^2}{\Lambda^2} \right]^{-\gamma_n} , \tag{3.3}$$

where $\gamma_n$ and $\phi_n$ satisfy

$$x_1 x_2 x_3 \left[ \frac{3C_F}{2\beta} - \gamma_n \right] \widetilde{\phi}_n = \frac{C_B}{\beta} \int_0^1 [dy] V(x,y) \widetilde{\phi}_n(y) . \tag{3.4}$$

The $\gamma_n$ are the anomalous dimensions corresponding to the three-quark eigensolutions $\widetilde{\phi}_n$.

The calculations of the $\gamma_n$ and $\widetilde{\phi}_n$ were already given in Ref. 1 for the proton ($h = \frac{1}{2}$). However, the complete classification of all baryon states was not given. To do this, we use the basis elements in Table II to diagonalize the three-quark evolution equation, Eq. (3.4). In general,

TABLE II. Completely antisymmetric three-quark representations for (A) the $N$ system and (B) the $\Delta$ system. For all representations, the color-singlet representation factor $\epsilon_{ijk}/\sqrt{6}$ given by Eq. (2.1) is abbreviated and the orbital normalization constant given by Table I must be multiplied for the correct normalization. Even though we present the representations for positive $T_Z$ and $S_Z$ values, those for negative $T_Z$ and $S_Z$ values can be simply obtained by the replacement of $u \leftrightarrow d$ and $\uparrow \leftrightarrow \downarrow$, respectively, for the corresponding representations of positive $T_Z$ and $S_Z$ values. For example, $T_Z = -\frac{1}{2}$ representation is obtained by the replacement of $u \leftrightarrow d$ in the $T_Z = \frac{1}{2}$ representation.

**A. $N$ system ($T = 1/2$); $T_Z = 1/2$ is fixed for convenience**

| Index–Power Symmetry | Representation of the Three–Quark System |
|---|---|

**1.** $(S, S_Z) = (3/2, 3/2)$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \frac{1}{3\sqrt{2}} (\uparrow\uparrow\uparrow) \; [\, duu(2y_1 - y_2 - y_3) + udu(-y_1 + 2y_2 - y_3) + uud(-y_1 - y_2 + 2y_3) \,]$$

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \frac{1}{3\sqrt{2}} (\uparrow\uparrow\uparrow) \; [\, duu\{(2y_2y_3 - y_1y_3 - y_1y_2) + \frac{1}{4}(2y_1 - y_2 - y_3)\} + udu\{(-y_2y_3 + 2y_1y_3 - y_1y_2) + \frac{1}{4}(-y_1 + 2y_2 - y_3)\} + uud\{(-y_2y_3 - y_1y_3 + 2y_1y_2) + \frac{1}{4}(-y_1 - y_2 + 2y_3)\} \,]$$

**2.** $(S, S_Z) = (3/2, 1/2)$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \frac{1}{3\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \; [\, duu(2y_1 - y_2 - y_3) + udu(-y_1 + 2y_2 - y_3) + uud(-y_1 - y_2 + 2y_3) \,]$$

$$\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \frac{1}{3\sqrt{6}} [(\uparrow\uparrow\downarrow) + (\uparrow\downarrow\uparrow) + (\downarrow\uparrow\uparrow)] \; [\, duu\{(2y_2y_3 - y_1y_3 - y_1y_2) + \frac{1}{4}(2y_1 - y_2 - y_3)\} + udu\{(-y_2y_3 + 2y_1y_3 - y_1y_2) + \frac{1}{4}(-y_1 + 2y_2 - y_3)\} + uud\{(-y_2y_3 - y_1y_3 + 2y_1y_2) + \frac{1}{4}(-y_1 - y_2 + 2y_3)\} \,]$$

**3.** $(S, S_Z) = (1/2, 1/2)$

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline\end{array} \qquad \frac{1}{3\sqrt{2}} \; [\, (\downarrow\uparrow\uparrow)(2duu - udu - uud) + (\uparrow\downarrow\uparrow)(-duu + 2udu - uud) + (\uparrow\uparrow\downarrow)(-duu - udu + 2uud) \,] \times 1$$

$$\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 \\ \cline{1-1}\end{array} \qquad \frac{1}{3\sqrt{6}} \; [\, (\downarrow\uparrow\uparrow)\{duu(-2y_1 + y_2 + y_3) + udu(y_1 + y_2 - 2y_3) + uud(y_1 - 2y_2 + y_3)\} + (\uparrow\downarrow\uparrow)\{duu(y_1 + y_2 - 2y_3) + udu(y_1 - 2y_2 + y_3) + uud(-2y_1 + y_2 + y_3)\} + (\uparrow\uparrow\downarrow)\{duu(y_1 - 2y_2 + y_3) + udu(-2y_1 + y_2 + y_3) + uud(y_1 + y_2 - 2y_3)\} \,]$$

$$\begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline\end{array} \qquad \frac{1}{3\sqrt{6}} \; [\, (\downarrow\uparrow\uparrow)(2duu - udu - uud) + (\uparrow\downarrow\uparrow)(-duu + 2udu - uud) + (\uparrow\uparrow\downarrow)(-duu - udu + 2uud) \,] \times \left( y_2y_3 + y_3y_1 + y_1y_2 - \frac{2}{7} \right)$$

TABLE II. (*Continued*).

| Index–Power Symmetry | Representation of the Three–Quark System |
|---|---|

$\begin{array}{|c|c|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array}$

$$\begin{aligned}\frac{1}{3\sqrt{6}}\Big((\downarrow\uparrow\uparrow)\,[\,&duu\{(-2y_2y_3+y_1y_3+y_1y_2)+\frac{1}{4}(-2y_1+y_2+y_3)\}\\ &+udu\{(y_2y_3+y_1y_3-2y_1y_2)+\frac{1}{4}(y_1+y_2-2y_3)\}\\ &+uud\{(y_2y_3-2y_1y_3+y_1y_2)+\frac{1}{4}(y_1-2y_2+y_3)\}\,]\\ +(\uparrow\downarrow\uparrow)\,[\,&duu\{(y_2y_3+y_1y_3-2y_1y_2)+\frac{1}{4}(y_1+y_2-2y_3)\}\\ &+udu\{(y_2y_3-2y_1y_3+y_1y_2)+\frac{1}{4}(y_1-2y_2+y_3)\}\\ &+uud\{(-2y_2y_3+y_1y_3+y_1y_2)+\frac{1}{4}(-2y_1+y_2+y_3)\}\,]\\ +(\uparrow\uparrow\downarrow)\,[\,&duu\{(y_2y_3-2y_1y_3+y_1y_2)+\frac{1}{4}(y_1-2y_2+y_3)\}\\ &+udu\{(-2y_2y_3+y_1y_3+y_1y_2)+\frac{1}{4}(-2y_1+y_2+y_3)\}\\ &+uud\{(y_2y_3+y_1y_3-2y_1y_2)+\frac{1}{4}(y_1+y_2-2y_3)\}\,]\Big)\end{aligned}$$

**B. $\Delta$ system ($T=3/2$); for $T_Z=3/2$, $\boxed{u|u|u}=uuu$, and for $T_Z=1/2$, $\boxed{u|u|d}=[(1/\sqrt{3})\,(uud+udu+duu)]$ are factors for all representations.**

| Index–Power Symmetry | Representation of the Three–Quark System |
|---|---|

1. $(S,S_Z)=(3/2,\,3/2)$

$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline\end{array}$ $\quad(\uparrow\uparrow\uparrow)\times 1$

$\begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array}$ $\quad\dfrac{1}{\sqrt{3}}(\uparrow\uparrow\uparrow)\left(y_2y_3+y_3y_1+y_1y_2-\dfrac{2}{7}\right)$

2. $(S,S_Z)=(3/2,\,1/2)$

$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline\end{array}$ $\quad\dfrac{1}{\sqrt{3}}\,[(\uparrow\uparrow\downarrow)+(\uparrow\downarrow\uparrow)+(\downarrow\uparrow\uparrow)]\times 1$

$\begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array}$ $\quad\dfrac{1}{3}\,[(\uparrow\uparrow\downarrow)+(\uparrow\downarrow\uparrow)+(\downarrow\uparrow\uparrow)]\left(y_2y_3+y_3y_1+y_1y_2-\dfrac{2}{7}\right)$

3. $(S,S_Z)=(1/2,\,1/2)$

$\begin{array}{|c|c|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}$

$$\begin{aligned}\frac{1}{3\sqrt{2}}\,[\,&(\downarrow\uparrow\uparrow)(2y_1-y_2-y_3)\\ &+(\uparrow\downarrow\uparrow)(-y_1+2y_2-y_3)\\ &+(\uparrow\uparrow\downarrow)(-y_1-y_2+2y_3)\,]\end{aligned}$$

$\begin{array}{|c|c|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array}$

$$\begin{aligned}\frac{1}{3\sqrt{2}}\Big[\,&(\downarrow\uparrow\uparrow)\{2y_2y_3-y_1y_3-y_1y_2+\frac{1}{4}(2y_1-y_2-y_3)\}\\ &+(\uparrow\downarrow\uparrow)\{-y_2y_3+2y_1y_3-y_1y_2+\frac{1}{4}(-y_1+2y_2-y_3)\}\\ &+(\uparrow\uparrow\downarrow)\{-y_2y_3-y_1y_3+2y_1y_2+\frac{1}{4}(-y_1-y_2+2y_3)\}\,]\end{aligned}$$

the eigensolutions $\tilde{\phi}_n$ are linear combinations of antisymmetric representations. As a result, we find different $\gamma_n$ for the nucleon and isobar states. We present the general solutions for each baryon in Table III.

For specific calculations, one does not always have to use the full antisymmetric representation since we can use an effective representation in which the helicity configuration is fixed. As an example, we can choose an $\uparrow\downarrow\uparrow$ term for $S_Z=\frac{1}{2}$ state as an effective representation. We can also narrow the effective representation by fixing the isospin configuration: e.g., choose the $uud$ term for $(T,T_Z)=(\frac{1}{2},\frac{1}{2})$ and $S_Z=\frac{1}{2}$. The orbital wave-function coefficient then provides a practical representation. However, we must choose a proper term in order to calculate the mixing coefficients. If we use an improper term (such as the $udu$ term in the above example), then we cannot represent the eigensolution because the mixed states ($S=\frac{3}{2}$ and $\frac{1}{2}$ states) have the same orbital representation.

TABLE III. Eigenvalues and eigensolutions for (A) $N$ system and (B) $\Delta$ system: the anomalous dimensions are related to $b$ by Eq. (4.4), i.e., $\gamma=(2bC_B+3C_F/2)/\beta$. The normalization factor $\sqrt{N}$ for the effective representation is also given.

**A. $N$ system ($T = 1/2$); $T_Z = 1/2$ is fixed.**

| Spin Configuration | $b$ | Spin $\times$ Orbital | $\sqrt{N}$ | $\tilde{\phi}(y)$ (Effective Representation) |
|---|---|---|---|---|
| $\uparrow\uparrow\uparrow$ | $\frac{3}{2}$ | $\begin{array}{\|c\|c\|c\|}\hline \uparrow&\uparrow&\uparrow\\\hline\end{array}\times\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}$ | $\sqrt{21}\times\sqrt{5!}$ | $\frac{1}{3\sqrt{2}}\,[\,duu(2y_1-y_2-y_3)+udu(-y_1+2y_2-y_3)+uud(-y_1-y_2+2y_3)\,]$ |
| $\uparrow\uparrow\uparrow$ | $\frac{17}{6}$ | $\begin{array}{\|c\|c\|c\|}\hline \uparrow&\uparrow&\uparrow\\\hline\end{array}\times\begin{array}{\|c\|c\|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array}$ | $6\sqrt{28}\times\sqrt{5!}$ | $\frac{1}{3\sqrt{2}}\,[\,duu\{(2y_2y_3-y_1y_3-y_1y_2)+\frac{1}{4}(2y_1-y_2-y_3)\}+udu\{\,1\leftrightarrow 2\,\}+uud\{\,1\leftrightarrow 3\,\}\,]$ |
| $\uparrow\downarrow\uparrow$ | $-1$ | $\begin{array}{\|c\|c\|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}\times\begin{array}{\|c\|c\|c\|}\hline 0&0&0\\\hline\end{array}$ | $\sqrt{5!}$ | $\frac{1}{\sqrt{6}}(-duu+2udu-uud)\times 1$ |
| $\uparrow\downarrow\uparrow$ | $\frac{2}{3}$ | $\frac{1}{\sqrt{2}}\begin{array}{\|c\|c\|c\|}\hline \uparrow&\downarrow&\uparrow\\\hline\end{array}\times\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}+\frac{1}{\sqrt{2}}\begin{array}{\|c\|c\|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}\times\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}$ | $\sqrt{21}\times\sqrt{5!}$ | $\frac{1}{2}(duu-uud)(y_1-y_3)$ |
| $\uparrow\downarrow\uparrow$ | $1$ | $-\frac{1}{\sqrt{2}}\begin{array}{\|c\|c\|c\|}\hline \uparrow&\downarrow&\uparrow\\\hline\end{array}\times\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}+\frac{1}{\sqrt{2}}\begin{array}{\|c\|c\|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}\times\begin{array}{\|c\|c\|}\hline 0&0\\\hline 1\\\cline{1-1}\end{array}$ | $\sqrt{21}\times\sqrt{5!}$ | $\frac{1}{6}(duu-2udu+uud)(-y_1+2y_2-y_3)$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{3}$ | $+\frac{1}{\sqrt{30}}\begin{array}{\|c\|c\|c\|}\hline \uparrow&\downarrow&\uparrow\\\hline\end{array}\times\begin{array}{\|c\|c\|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array}-\sqrt{\frac{28}{30}}\begin{array}{\|c\|c\|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}\times\begin{array}{\|c\|c\|c\|}\hline 0&1&1\\\hline\end{array}-\sqrt{\frac{1}{30}}\begin{array}{\|c\|c\|}\hline \uparrow&\uparrow\\\hline \downarrow\\\cline{1-1}\end{array}\times\begin{array}{\|c\|c\|}\hline 0&1\\\hline 1\\\cline{1-1}\end{array}$ | $6\sqrt{28}\times\sqrt{5!}$ | $\frac{1}{6\sqrt{15}}(duu-2udu+uud)\times\{-6y_2y_3-9y_1y_3-6y_1y_2+\frac{1}{4}(9y_1+6y_2+9y_3)\}$ |

TABLE III. (*Continued*).

| | | | | |
|---|---|---|---|---|
| $\uparrow\downarrow\uparrow$ | $\frac{7}{3}$ | $\frac{1}{\sqrt{2}}\boxed{\uparrow}\boxed{\downarrow}\boxed{\uparrow}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}+\frac{1}{\sqrt{2}}\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}$ | $6\sqrt{28}\times\sqrt{5!}$ | $\frac{1}{2}(duu-uud)\times\left\{y_2y_3-y_1y_2+\frac{1}{4}(y_1-y_3)\right\}$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{2}$ | $-\sqrt{\frac{7}{15}}\boxed{\uparrow}\boxed{\downarrow}\boxed{\uparrow}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}-\sqrt{\frac{1}{15}}\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\boxed{0}\boxed{1}\boxed{1}+\sqrt{\frac{7}{15}}\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}$ | $\frac{42}{\sqrt{15}}\sqrt{5!}$ | $\frac{1}{3\sqrt{2}}(duu-2udu+uud)\times(-3y_2y_3+3y_1y_3-3y_1y_2+\frac{2}{7}-\frac{1}{2}y_1+y_2-\frac{1}{2}y_3)$ |

**B. $\Delta$ system ($T=3/2$); for $T_Z=3/2$, $\boxed{u}\boxed{u}\boxed{u}=uuu$, and for $T_Z=1/2$, $\boxed{u}\boxed{u}\boxed{d}=[(1/\sqrt{3})(uud+udu+duu)]$ are factors for the effective representations.**

| Spin Configuration | $b$ | Spin $\times$ Orbital | $\sqrt{N}$ | $\tilde{\phi}(y)$ (Effective Representation) |
|---|---|---|---|---|
| $\uparrow\uparrow\uparrow$ | 0 | $\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}\times\boxed{0}\boxed{0}\boxed{0}$ | $\sqrt{5!}$ | 1 |
| $\uparrow\uparrow\uparrow$ | $\frac{7}{3}$ | $\boxed{\uparrow}\boxed{\uparrow}\boxed{\uparrow}\times\boxed{0}\boxed{1}\boxed{1}$ | $42\sqrt{5!}$ | $\frac{1}{\sqrt{3}}\left(y_2y_3+y_3y_1+y_1y_2-\frac{2}{7}\right)$ |
| $\uparrow\downarrow\uparrow$ | $-1$ | $\boxed{\uparrow}\boxed{\downarrow}\boxed{\uparrow}\times\boxed{0}\boxed{0}\boxed{0}$ | $\sqrt{5!}$ | 1 |
| $\uparrow\downarrow\uparrow$ | 1 | $\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\begin{array}{l}\boxed{0}\boxed{0}\\\boxed{1}\end{array}$ | $\sqrt{21}\times\sqrt{5!}$ | $\frac{1}{\sqrt{6}}(-y_1+2y_2-y_3)$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{3}$ | $\sqrt{\frac{14}{15}}\boxed{\uparrow}\boxed{\downarrow}\boxed{\uparrow}\times\boxed{0}\boxed{1}\boxed{1}+\sqrt{\frac{1}{15}}\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}$ | $\frac{6\sqrt{14}\times\sqrt{5!}}{\sqrt{15}}$ | $\sqrt{3}\left(2y_2y_3+3y_1y_3+2y_1y_2-\frac{3}{4}y_1-\frac{1}{2}y_2-\frac{3}{4}y_3\right)$ |
| $\uparrow\downarrow\uparrow$ | $\frac{5}{2}$ | $-\sqrt{\frac{1}{15}}\boxed{\uparrow}\boxed{\downarrow}\boxed{\uparrow}\times\boxed{0}\boxed{1}\boxed{1}+\sqrt{\frac{14}{15}}\begin{array}{l}\boxed{\uparrow}\boxed{\uparrow}\\\boxed{\downarrow}\end{array}\times\begin{array}{l}\boxed{0}\boxed{1}\\\boxed{1}\end{array}$ | $\frac{42\times\sqrt{5!}}{\sqrt{15}}$ | $\frac{1}{\sqrt{3}}\left(-3y_2y_3+3y_1y_3-3y_1y_2+\frac{2}{7}-\frac{1}{2}y_1+y_2-\frac{1}{2}y_3\right)$ |

## IV. MIXING AND DIAGONALIZATION

The lowest-power orbital state ($0^3$ state) of each baryon is unique. These special states are eigensolutions by themselves, giving the eigenvalues $\gamma_0$ by Eq. (3.4). However, as the orbital power becomes larger, the number of different representations is increased and the mixing between the different representations occurs as described in the last section. Here we describe the mixing of different spin representations for the example given at the end of the last section. We define

$$\phi^{3/2} = \begin{array}{|c|}\hline r\\\hline y\\\hline b\\\hline\end{array} \times \begin{array}{|c|c|}\hline u&u\\\hline \multicolumn{1}{|c|}{d}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|c|}\hline \uparrow&\uparrow&\downarrow\\\hline\end{array} \times \begin{array}{|c|c|}\hline 0&0\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} \qquad (4.1)$$

which is the state $(T,T_Z)=(\frac{1}{2},\frac{1}{2})$, $(S,S_Z)=(\frac{3}{2},\frac{1}{2})$, and $n=1$, and

$$\phi^{1/2} = \begin{array}{|c|}\hline r\\\hline y\\\hline b\\\hline\end{array} \times \begin{array}{|c|c|}\hline u&u\\\hline \multicolumn{1}{|c|}{d}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline \uparrow&\uparrow\\\hline \multicolumn{1}{|c|}{\downarrow}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 0&0\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} \qquad (4.2)$$

which is the state $(T,T_Z)=(\frac{1}{2},\frac{1}{2})$, $(S,S_Z)=(\frac{1}{2},\frac{1}{2})$, and $n=1$. For convenience, we rewrite Eq. (3.4) as

$$-b\tilde{\phi}(x)=\frac{2C_B}{\beta}\tilde{V}(x,y)\tilde{\phi}(y)\ , \qquad (4.3)$$

where $b$ is defined in

$$\gamma_n \equiv (2b_n C_B+\tfrac{3}{2}C_F)/\beta \qquad (4.4)$$

and $\tilde{V}(x,y)=\tilde{V}_\delta(x,y)+\tilde{V}_\Delta(x,y)$ is defined by

$$\begin{aligned}\tilde{V}_\delta(x,y) &\equiv \int_0^1 [dy]\sum_{i\neq j}\theta(y_i-x_i)\delta(x_k-y_k)\\ &\quad\times\frac{y_j}{x_j}\frac{\delta_{h_i\bar{h}_j}}{x_i+x_j}\ ,\\ \tilde{V}_\Delta(x,y) &\equiv \int_0^1 [dy]\sum_{i\neq j}\theta(y_i-x_i)\delta(x_k-y_k)\\ &\quad\times\frac{y_j}{x_j}\frac{\Delta}{y_i+x_i}\ .\end{aligned} \qquad (4.5)$$

Setting the $y$ representations of $\phi^{3/2}$ and $\phi^{1/2}$ [see Eqs. (4.1) and (4.2), and Table II] into the three-quark evolution equation [Eqs. (4.3) and (4.5)], we find

$$\tilde{V}_\delta(x,y)\begin{bmatrix}\tilde{\phi}^{3/2}(y)\\ \tilde{\phi}^{1/2}(y)\end{bmatrix}=\begin{bmatrix}\frac{2}{3}&\frac{1}{6}\\ \frac{1}{6}&\frac{2}{3}\end{bmatrix}\begin{bmatrix}\tilde{\phi}^{3/2}(x)\\ \tilde{\phi}^{1/2}(x)\end{bmatrix}$$

and $\qquad (4.6)$

$$\tilde{V}_\Delta(x,y)\begin{bmatrix}\tilde{\phi}^{3/2}(y)\\ \tilde{\phi}^{1/2}(y)\end{bmatrix}=\begin{bmatrix}-\frac{3}{2}&0\\ 0&-\frac{3}{2}\end{bmatrix}\begin{bmatrix}\tilde{\phi}^{3/2}(x)\\ \tilde{\phi}^{1/2}(x)\end{bmatrix}\ ,$$

which gives the eigenvalues and eigensolutions after diagonalization as

$$\begin{aligned} b&=\tfrac{2}{3}\ \text{ for }\ \tilde{\phi}=\frac{1}{\sqrt{2}}\tilde{\phi}^{3/2}+\frac{1}{\sqrt{2}}\tilde{\phi}^{1/2}\ ,\\ b&=1\ \text{ for }\ \tilde{\phi}=-\frac{1}{\sqrt{2}}\tilde{\phi}^{3/2}+\frac{1}{\sqrt{2}}\tilde{\phi}^{1/2}\ .\end{aligned} \qquad (4.7)$$

This gives the expected mixing between $\phi^{3/2}$ and $\phi^{1/2}$ through the QCD evolution.

For the antisymmetric representations given by Table II, we find the mixing between the following states: for the nucleon case, $n=1$ mixes $\phi^{3/2}$ and $\phi^{1/2}$ (the previous example), and $n=2$ mixes

$$\begin{array}{|c|c|c|}\hline \uparrow&\downarrow&\uparrow\\\hline\end{array} \times \begin{array}{|c|c|}\hline 0&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}\ ,\quad \begin{array}{|c|c|}\hline \uparrow&\uparrow\\\hline \multicolumn{1}{|c|}{\downarrow}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline \uparrow&\uparrow\\\hline \multicolumn{1}{|c|}{\downarrow}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 0&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}\ .$$

For the $\Delta$ case, $n=2$ has mixing between

$$\begin{array}{|c|c|c|}\hline \uparrow&\downarrow&\uparrow\\\hline\end{array} \times \begin{array}{|c|c|c|}\hline 0&1&1\\\hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline \uparrow&\uparrow\\\hline \multicolumn{1}{|c|}{\downarrow}\\\cline{1-1}\end{array} \times \begin{array}{|c|c|}\hline 0&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}\ .$$

As in the previous example ($\phi^{3/2}$ and $\phi^{1/2}$), we can diagonalize the mixing matrix $\tilde{V}=\tilde{V}_\delta+\tilde{V}_\Delta$ and find the eigenvalues $b$ and the eigensolutions $\tilde{\phi}$ for all antisymmetric representations as given in Table II. The results for $b$ and $\tilde{\phi}$ are given in Table III.

## V. RESULTS AND DISCUSSION

We have presented the antisymmetric representations for the three-quark system and shown the action of the three-quark evolution equation. We have used the following properties of the evolution kernel.

(1) The unique color-singlet state of baryon is preserved.

(2) The isospin state is preserved.

(3) $S_Z$ components of the spin state is conserved but different spin multiplets ($S=\frac{3}{2}$ and $S=\frac{1}{2}$ states) can mix with each other.

(4) $L_Z=0$ is fixed on the light cone and $n$ value is conserved, but different orbital multiplets (corresponding to different Young diagrams) can mix with each other.

After the diagonalization of the mixing matrix $\widetilde{V}=\widetilde{V}_\delta+\widetilde{V}_\Delta$, we find the eigenvalues and the eigensolutions as summarized in Table III.

The method by which we obtain the above results is sufficiently straightforward that we can find the basis of the eigensolutions before solving the evolution equation and see the evolution of each state explicitly. Furthermore, we can check every step of the calculation explicitly. For example, the symmetry of the evolution potential given by Eq. (3.2) can be checked by obtaining the matrix representation such as Eq. (4.6) for the mixing of $\widetilde{\phi}^{\,3/2}$ and $\widetilde{\phi}^{\,1/2}$. (Note that the matrices $\widetilde{V}_\delta$ and $\widetilde{V}_\Delta$ are symmetric separately.)

From Table III we find that the eigenvalues and the eigensolutions are consistent with the previous results, but we have the classification in terms of physical baryons. The eigensolutions of the proton with the $S_Z=\frac{1}{2}$ case coincide with the result of Ref. 1 for the distribution amplitude for the proton ($h=\frac{1}{2}$):

$$\begin{aligned}\phi_p^{1/2}=\Bigg[&\left[\left(\frac{d_\uparrow(1)u_\uparrow(3)+u_\uparrow(1)d_\uparrow(3)}{\sqrt{6}}u_\downarrow(2)\right.\right.\\ &\left.-\left(\tfrac{2}{3}\right)^{1/2}u_\uparrow(1)d_\downarrow(2)u_\uparrow(3)\right]\phi^S(x_i,Q)\\ &+\left[\frac{d_\uparrow(1)u_\uparrow(3)-u_\uparrow(1)d_\uparrow(3)}{\sqrt{2}}u_\downarrow(2)\right]\phi^A(x_i,Q)\Bigg]\\ &+(1\leftrightarrow 2)+(2\leftrightarrow 3)\,, \end{aligned} \tag{5.1}$$

where $\phi^S$ and $\phi^A$ are symmetric and antisymmetric under the interchange $x_1\leftrightarrow x_3$ and the color-singlet representation $\epsilon_{ijk}/\sqrt{6}$ is understood. The representation inside the large square brackets turns out to be the same as our effective antisymmetric representation given by Table III. Furthermore, we can give the general distribution amplitudes for the other baryons such as the excited proton $\phi_p^{3/2}$, and isobars $\phi_\Delta^{1/2},\phi_\Delta^{3/2}$:

$$\begin{aligned}\phi_p^{3/2}=&\tfrac{1}{2}(d_\uparrow u_\uparrow u_\uparrow-u_\uparrow d_\uparrow u_\uparrow)\phi^\alpha(x_i,Q)\\ &+\frac{1}{2\sqrt{3}}(d_\uparrow u_\uparrow u_\uparrow+u_\uparrow d_\uparrow u_\uparrow-2u_\uparrow u_\uparrow d_\uparrow)\phi^\beta(x_i,Q)\,,\end{aligned} \tag{5.2}$$

where $\phi^\alpha$ and $\phi^\beta$ have the symmetry represented by the Young diagrams

$$\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\quad\text{and}\quad\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\;,$$

respectively,

$$\begin{aligned}&\phi_\Delta^{1/2}(T_Z=\tfrac{3}{2})=u_\uparrow u_\downarrow u_\uparrow\phi^S(x_i,Q)+(1\leftrightarrow 2)+(2\leftrightarrow 3)\,,\\ &\phi_\Delta^{1/2}(T_Z=\tfrac{1}{2})=\frac{1}{\sqrt{3}}(u_\uparrow u_\downarrow d_\uparrow+u_\uparrow d_\downarrow u_\uparrow+d_\uparrow u_\downarrow u_\uparrow)\phi^S(x_i,Q)\\ &\qquad\qquad+(1\leftrightarrow 2)+(2\leftrightarrow 3)\,,\end{aligned} \tag{5.3}$$

where $\phi^S$ is symmetric under the interchange $x_1\leftrightarrow x_3$, and

$$\begin{aligned}&\phi_\Delta^{3/2}(T_Z=\tfrac{3}{2})=u_\uparrow u_\uparrow u_\uparrow\phi^\gamma(x_i,Q)\,,\\ &\phi_\Delta^{3/2}(T_Z=\tfrac{1}{2})=\frac{1}{\sqrt{3}}(u_\uparrow u_\uparrow d_\uparrow+u_\uparrow d_\uparrow u_\uparrow\\ &\qquad\qquad+d_\uparrow u_\uparrow u_\uparrow)\phi^\gamma(x_i,Q)\,,\end{aligned} \tag{5.4}$$

where the $\phi^\gamma$ are totally symmetric under any interchange between $x_1$, $x_2$, and $x_3$.

As stated in the Introduction, we can apply the above method to multiquark systems which have several color-singlet representations.[3] In this case a much richer phenomenology of QCD states exists including hidden-color configurations. We can also combine this approach with the fractional parentage technique[5] to predict the effective interaction between baryonic clusters within a multiquark system.

In conclusion, we have presented a general technique which combined with evolution equations predicts the short-distance behavior and classifies the spectrum of relativistic many-fermion systems. This approach thus provides a fundamental method for studying short-distance dynamics even in the domain of the multiquark systems of nuclear physics.

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