Force on a charge in the space-time of a cosmic string

B. Linet
Université Pierre et Marie Curie, Institut Henri Poincaré, 11 rue Pierre et Marie Curie, 75231 Paris Cedex 05, France
(Received 28 October 1985)

We determine the electrostatic self-force acting on a point test charge in the space-time describing a static, cylindrically symmetric cosmic string. We find a repulsive interaction.

When a point charge is held fixed in a static gravitational field, this induces specifically an electrostatic self-force acting on this charged particle. This result was first derived in the case where the gravitational field is weak,[1] then extended to certain space-times in which the global electrostatic potential generated by a point test charge is known: within a spherical shell of matter[2] and in the Schwarzschild space-time.[3]

The purpose of this paper is to determine the electrostatic self-force in the space-time describing a static, cylindrically symmetric cosmic string[4] whose metric has been recently found.[5] It can be written

$$ds^2 = -d\rho^2 - dz^2 - B^2\rho^2 d\phi^2 + c^2 dt^2 \text{ with } 0 < B \leqslant 1 \ , \tag{1}$$

in a coordinate system (t,ρ,z,ϕ) with $\rho \geqslant 0$ and $0 \leqslant \phi \leqslant 2\pi$, the hypersurfaces $\phi = 0$ and $\phi = 2\pi$ being identified. Metric (1) has a conical singularity[6] and therefore it induces on the axis $\rho = 0$ a singular line source of Einstein equations having the following energy-momentum tensor characteristic of a static, cylindrically symmetric cosmic string:

$$T_t^t = T_z^z = \left[\frac{c^2(1-B)}{4G}\right]\frac{\delta(\rho)}{(-\tilde{g})^{1/2}} \text{ and } T_\rho^\rho = T_\phi^\phi = 0 \ , \tag{2}$$

where $\tilde{g}$ is the determinant of the induced metric on the two-surface $t = \text{const}$ and $z = \text{const}$. From expressions (2), we see that the linear mass density μ is given by

$$\mu = \frac{c^2(1-B)}{4G} \text{ with } 0 \leqslant \mu < \frac{c^2}{4G} \ . \tag{3}$$

The space-time described by metric (1) is locally flat but of course it is not globally flat.

We are interested in determining the electrostatic potential V of a point test charge q located at the point $\rho = \rho_0$, $z = 0$, and $\phi = \pi$ in the background metric (1). From the Maxwell equations in curved space, we obtain immediately the potential equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{B^2\rho^2}\frac{\partial^2}{\partial\phi^2}\right]V = -\left[\frac{q}{\epsilon_0}\right]\frac{\delta(\rho-\rho_0)\delta(z)\delta(\phi-\pi)}{B\rho} \ , \tag{4}$$

where ϵ_0 is the permittivity of free space. With the coordinate transformation

$$\theta = B\phi \ , \tag{5}$$

Eq. (4) reduces to the usual potential equation, in the subset of Minkowski space-time covered by the coordinate system (t,ρ,z,θ) with $0 \leqslant \theta \leqslant 2\pi B$, with a point charge located at $\rho = \rho_0$, $z = 0$, and $\theta = B\pi$. However, the potential must satisfy the unusual boundary conditions

$$V(\rho,z,0) = V(\rho,z,2\pi B) \ , \qquad \frac{\partial V}{\partial\theta}(\rho,z,0) = \frac{\partial V}{\partial\theta}(\rho,z,2\pi B) = 0 \ . \tag{6}$$

Such a solution has never been published, so far as we know. By using some results of Macdonald[7] on the electrostatics for a wedge formed from two semi-infinite conducting planes, we have shown that this solution may be written in the form

$$V(\rho,z,\theta) = \frac{q}{(4\pi\epsilon_0)2\pi B(2\rho\rho_0)^{1/2}}\int_\eta^\infty \left[\frac{\sinh(\zeta/2B)}{\cosh(\zeta/2B)+\sin(\theta/2B)} + \frac{\sinh(\zeta/2B)}{\cosh(\zeta/2B)-\sin(\theta/2B)}\right]\frac{d\zeta}{(\cosh\zeta-\cosh\eta)^{1/2}} \ , \tag{7}$$

where η is defined by $\cosh\eta = (\rho^2+\rho_0^2+z^2)/2\rho\rho_0$ $(\eta \geqslant 0)$. Consequently, introducing coordinate ϕ by formula (5), we obtain after some manipulations the following form of electrostatic potential satisfying Eq. (4):

$$V(\rho,z,\phi) = \frac{q}{(4\pi\epsilon_0)\pi B(2\rho\rho_0)^{1/2}}\int_\eta^\infty \frac{\sinh(\zeta/B)\,d\zeta}{[\cosh(\zeta/B)+\cos\phi](\cosh\zeta-\cosh\eta)^{1/2}} \ . \tag{8}$$

We are now in a position to determine the electrostatic self-force. We call V_M the solution to Eq. (4), defined outside the hypersurface $\phi = 0$, corresponding to the Coulomb potential in Minkowski space-time. We have

$$V_M(\rho,z,\phi) = \frac{q}{4\pi\epsilon_0[\rho^2+\rho_0^2+z^2-2\rho\rho_0(\sin B\phi\,\sin B\pi+\cos B\phi\,\cos B\pi)]^{1/2}} \ . \tag{9}$$

We need to write potential (9) in integral form:

$$V_M(\rho,z,\phi) = \frac{q}{(4\pi\epsilon_0)\pi(2\rho\rho_0)^{1/2}}\int_\eta^\infty \frac{\sinh\zeta\,d\zeta}{(\cosh\zeta-\sin B\phi\,\sin B\pi-\cos B\phi\,\cos B\pi)(\cosh\zeta-\cosh\eta)^{1/2}} \ . \tag{10}$$

In the neighborhood of the point charge, we can always write the electrostatic potential (8) in the form

$$V(\rho,z,\phi)=V_M(\rho,z,\phi)+H(\rho,z,\phi) \ . \tag{11}$$

The first term V_M is irregular at the position of the charge, whereas the second term H is regular and is a solution of the homogeneous equation (4). We ignore the infinite forces arising from the electrostatic potential V_M because we are locally in the Minkowski space-time. Consequently, the potential H may be considered as an "external" electrostatic potential which exerts a force on the charge q following the familiar Lorentz force. This electrostatic force applied to it can be evaluated directly from the electrostatic energy using the standard procedure. This has the general expression

$$W=\tfrac{1}{2}qH(\rho_0,0,\pi) \ . \tag{12}$$

Taking into account expressions (8) and (10), formula (12) becomes

$$W=\left[\frac{L_B}{4\pi}\right]\left[\frac{q^2}{4\pi\epsilon_0\rho_0}\right] \ , \tag{13}$$

where L_B is a positive constant, depending on parameter B $(0<B\leq 1)$, defined by the integral

$$L_B=\int_0^\infty\left[\frac{\sinh(\zeta/B)}{B[\cosh(\zeta/B)-1]}-\frac{\sinh\zeta}{\cosh\zeta-1}\right]\frac{d\zeta}{\sinh(\zeta/2)} \ . \tag{14}$$

We deduce from (13) that the exerted force is

$$f^\rho=\left[\frac{L_B}{4\pi}\right]\left[\frac{q^2}{4\pi\epsilon_0\rho_0{}^2}\right] \text{ and } f^z=f^\phi=0 \ . \tag{15}$$

Hence electrostatic self-force (15) acts on the charge in the direction away from the cosmic string.

To calculate integral (14) a numerical analysis is necessary. The physically interesting limit is the one $B\rightarrow 1$, where we find approximately

$$f^\rho\sim\left[\frac{2.5}{\pi}\right]\left[\frac{G\mu}{c^2}\right]\left[\frac{q^2}{4\pi\epsilon_0\rho_0{}^2}\right] \text{ as } \mu\rightarrow 0 \ , \tag{16}$$

where we have expressed B in terms of μ using formula (3).

In the space-time of a static, cylindrically symmetric cosmic string, there is no gravitational force acting on a massive test particle. However, we point out that, according to formula (15) which reduces to form (16) in the limit where the linear mass density goes to zero, there is a repulsive interaction between a charged test particle and this cosmic string.

Université Pierre et Marie Curie is "unité associée au Centre National de la Recherche Scientifique No. 769."

[1]C. M. DeWitt and B. S. DeWitt, Physics (N.Y.) 1, 3 (1964); C. H. McGruder III, Nature (London) **272**, 806 (1978); A. Vilenkin, Phys. Rev. D **20**, 373 (1979).

[2]W. G. Unruh, Proc. R. Soc. London **A348**, 447 (1976).

[3]A. G. Smith and C. M. Will, Phys. Rev. D **22**, 1276 (1980); A. J. Zel'nikov and V. P. Frolov, Zh. Eksp. Teor. Fiz. **82**, 321 (1982) [Sov. Phys. JETP **55**, 191 (1982)]; B. Léauté and B. Linet, Int. J. Theor. Phys. **22**, 67 (1983).

[4]A. Vilenkin, Phys. Rep. **121**, 263 (1985), and references therein.

[5]J. R. Gott III, Astrophys. J. **288**, 422 (1985); W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985); B. Linet, Gen. Relativ. Gravit. **17**, 1109 (1985).

[6]A. H. Taub, J. Math. Phys. **21**, 1423 (1980), and references therein.

[7]H. M. Macdonald, Proc. Lond. Math. Soc. **26**, 156 (1895), quoted from L. D. Landau and E. M. Lifshitz, *Electrodynamics of Continuous Media* (Pergamon, New York, 1960), p. 15.