Color screening and topological index in the classical Yang-Mills theory with sources

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We reexamine the relation between the topological charge M and the occurrence of color screening in the classical Yang-Mills equations with sources. We find that the $M=0$ sector allows partial-screening solutions as well as the total-screening solutions.

Several years ago Mandula[1] showed that for a point source the Abelian Coulomb solution of the SU(2) Yang-Mills (YM) equations is stable only when $gQ<\frac{3}{2}$, where g is the gauge-field coupling constant and Q the external source strength. As the external source strength increases, the color components of the gauge-field potential A_μ^a orthogonal to the source direction in the color space get excited and instability occurs, suggesting the inflow of color charge which may lead to color screening of the external source. Not long after that Sikivie and Weiss and others[2] constructed explicit solutions for an extended external source which exhibit color screening. However, because of the gauge dependence in the definition of total color used, Hughes[3] demonstrated that screening of the extended external source by the YM field does not take place in the so-called physical gauge. In Ref. 4 we argued that, independent of gauge choice, color screening can be realized at the classical level since a gauge-invariant expression for the color charge can be written down.[5] Recently, Mandula, Meiron, and Orszag and Carson, Goldflam, and Wilets[6] obtained axially symmetric static solutions for a spherically symmetric δ-function source specified in the Abelian-gauge frame. Their solutions indicate that color screening is only partial, not complete. This is in contrast with the spherically symmetric solutions[7] for a spherical shell source specified in the radial gauge frame which display complete color screening. This naturally leads one to speculate whether the occurrence of color screening is related to the topological charge value M of the system, since for the complete-screening spherically symmetric solutions[7] $M=1$, whereas for the partial-screening axially symmetric solution[6] $M=0$. In Ref. 8 Carson conjectured that when $M=0$ the external color source can never be totally screened.

The purpose of this note is to examine this conjecture carefully. We find that for $M=1$ all solutions must be totally screening, whereas for $M=0$ one can have partial-screening solutions as well as complete-screening solutions. This later finding is different from that of Ref. 8 and hence Carson's conjecture is untenable. We shall consider only static sources and spherically symmetric gauge fields.

The YM equations in the presence of an external source are

$$D_\mu F^{\mu\nu}=j^\nu\ , \tag{1a}$$

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]\ , \tag{1b}$$

$$A=gA_\mu^a\frac{\sigma^a}{2i}\ , \tag{1c}$$

where σ^a are the Pauli matrices. The external source current is gauge-covariantly conserved,

$$D_\mu j^\mu=0\ ,$$

and for static source $j_i^a=0$ this becomes

$$[A_0,j^0]=0\ . \tag{2}$$

As we shall see, the topological charge is determined by the color direction of j^0, and hence that of A_0.

Introducing an adjoint-representation scalar field[5]

$$\eta(x)=\eta^a(x)\frac{\sigma^a}{2i}\ , \tag{3a}$$

$$\eta^a(x)\eta^a(x)=1\ , \tag{3b}$$

the gauge-invariant total color of the external source is[4]

$$Q_s=\int d^3x\, j_a^0\eta^a\ , \tag{4}$$

and that of the source-field system is

$$Q_T=\int_{\substack{\text{surface}\\ \text{at }\infty}} ds\, n_i(\eta^a F^{ai0})=Q_S+Q_F\ , \tag{5}$$

where n_i is the unit vector x^i/r, $r^2=x^ix^i$, and Q_F is the gauge-invariant total color carried by the YM field,

$$Q_F=\int d^3x\,(D_i\eta)^a F^{ai0}\ . \tag{6}$$

Total color screening of the external source is attained if $Q_T=0$ and $Q_S\neq 0$. Because of the gauge-invariant characterization of the total color charge, it is thus meaningful to discuss color screening of an external extended source.[4] From now on we employ η^a to describe the color structure of the external source[9] and we write

$$j_0^a=q(r)\eta^a\ . \tag{7}$$

The conserved topological charge of the source-field system is defined by the Kronecker index,[10]

$$M=\frac{1}{4\pi}\int_{r\to\infty} dS\, n_i\epsilon_{ijk}(\partial_j a_k+\tfrac{1}{2}\epsilon^{abc}\eta^a\partial_j\eta^b\partial_k\eta^c)\ , \tag{8}$$

where

$$a_\mu=gA_\mu^a\eta^a\ . \tag{9}$$

When a_μ is regular everywhere, the first term of the integrand gives no contribution and we have

$$M=\frac{1}{8\pi}\epsilon_{ijk}\epsilon^{abc}\int_{r\to\infty} dS\, n^i\eta^a\partial_j\eta^b\partial_k\eta^c\ , \tag{10}$$

which is an element of the second homotopy group $\pi_2[e^a]$. Thus, provided that the YM potential has no singularity, we have $M=0$ for the external source specified in the Abelian-gauge frame $\eta^a=\delta^a_3$ and $M=1$ for the source specified in the radial-gauge frame $\eta^a=x^a/r$. Higher M values will render the external source nonspherically symmetrical.

We now proceed to investigate how the value M controls the asymptotic behavior of $\eta^a F^{aio}$, and hence the value of Q_T. From Eqs. (2) and (10) it follows that, for $M=0$,

$$A^a_0=\delta^a_3 a_0/g \tag{11}$$

and, for $M=1$,

$$A^a_0=\frac{x^a}{r}a_0/g \ , \tag{12}$$

provided a_μ is regular. Thus the value of M determines, via condition (2), the form A^a_0 must assume. But the value of the total charge Q_T of the system is controlled by the asymptotic behavior of $\eta^a F^{aio}$, and hence that of A^a_0; in this way we see that M and Q_T are related.

Consider the case $M=1$. As discussed in Ref. 7, *all* spherically symmetric YM potentials can be written as

$$A^0_a=\frac{1}{gr}n^a f(y) \ , \tag{13a}$$

$$A^i_a=\frac{1}{gr}\epsilon^{aij}n^j[a(y)-1] \ , \tag{13b}$$

$$y=r/r_0 \ , \tag{13c}$$

where r_0 is an arbitrary length scale. The more general Witten's ansatz,[11] which involves four independent functions of y, is equivalent to Eqs. (13).[7] The requirement that the total system has a finite energy necessitates that A^a_μ fall at least as fast as $O(1/r)$ at large distances. Finiteness of Q_S then implies that either A_0 vanishes *at least* as fast as $O(1/r)$ and A_i *at least* as fast as $O(1/r^{1+\epsilon})$, $\epsilon>0$, or A_0 vanishes *at least* as fast as $O(1/r^{1+\epsilon})$ and A_i *at least* as fast as $O(1/r)$, as $r\to\infty$. The non-Abelian Gauss's law and Ampere's law as given by Eqs. (1) warrant only the latter asymptotic behavior. In other words, for $M=1$ at large distances A_0 must fall faster than $O(1/r)$ but A_i can either behave as a pure gauge $O(1/r)$ or vanish faster than $1/r$. In either case one finds that at large distances

$$\eta^a F^{aio}\approx n^i\frac{d}{dy}\left[\frac{f}{y}\right]\approx n^i O\left[\frac{1}{r^{2+\epsilon}}\right] \ . \tag{14}$$

Thus for $M=1$, all solutions are totally screening; partial-screening solutions do not exist. Carson[8] arrived at the same conclusion by using the Abelian-gauge frame for the external source, but then in that frame A^a_i must be singular. Examples of the $M=1$ solutions are the type-I and type-II YM configurations given in Ref. 7.

We shall now discuss the $M=0$ situation. Here the YM potentials A^a_0 are given by expression (11). In the event the YM potential is static, Ampere's law from Eqs. (1) can be written as

$$[A_0,[A_0,A_j]]=D_iF_{ij} \ , \tag{15}$$

while Gauss's law is

$$-\nabla^2A_0+[A_i,[A_0,A_i]]+2[\partial_iA_0,A_i]+[A_0,\partial_iA_i]=j_0 \ . \tag{16}$$

As before, the finite-energy requirement implies A_μ must vanish at least as fast as $O(1/r)$ at large distances. We consider two cases: (1) $A_i=O(1/r)$ and (2) $A_i=O(1/r^{1+\epsilon})$. In the case (1), Gauss's law requires A_0 to vanish as $O(1/r^{1+\epsilon})$ although Ampere's law permits A_0 to decrease as $O(1/r)$. Thus to be consistent in case (1) A_0 must tend to $O(1/r^{1+\epsilon})$ and D_iF_{ij} must vanish up to $O(1/r^3)$. The latter can easily be accommodated by requiring that A_i behave as a pure gauge up to order $1/r$. Thus one has $\eta^a F^{aio}$ behaving as $O(1/r^{2+\epsilon})$ for $r\to\infty$ and all solutions are totally screening. For the case (2), Ampere's law (15) can be simplified as

$$[A_0,[A_0,A_j]]=\partial_i\partial_iA_j-\partial_i\partial_jA_i \tag{17}$$

at large distances. The behavior that $A_0=O(1/r)$ at large r is consistent with Ampere's law as well as Gauss's law. This means the electric field strength $\eta^a F^{aio}$ can behave as $O(1/r^2)$, resulting in a nonvanishing total color of the system, $Q_T\neq 0$. The partial-screening solutions found in Ref. 6 precisely possess this asymptotic behavior. However, in the case (2) under consideration, it does not preclude the possibility that A_0 vanishes as $O(1/r^{1+\epsilon})$ at large r. Gauss's law certainly allows this behavior and so does Ampere's law. For example, if $A_\mu=O(1/r^2)$, then Ampere's law is satisfied if the right-hand side of Eq. (17) vanishes up to order $1/r^5$. In this situation, the solution is again completely screening. The total-screening solutions of Ref. 2, although they are time dependent, belong to this subclass. For these solutions, we found that $Q_T=0$ and $Q_S=-Q_F\neq 0$.[4]

To summarize, we find that for the vanishing topological charge $M=0$, there exist partial-screening solutions as well as total-screening solutions, whereas for $M=1$, all the solutions are totally screening. We conclude with two remarks.

(a) The $M=0$ partial-screening solutions given in Ref. 6 are only for a spherically symmetric δ-function source. For an extended spherically symmetric source, a partial-screening solution can also be found. When the topological charge M vanishes, the external source density admits the Hopf index $h[\eta^a]$. For $h[\eta^a]=\pm 1$, one has the non-Abelian Coulomb solution specified in the Abelian-gauge frame.[9] This solution can be derived perturbatively in a non-Albelian gauge frame,[9]

$$A_0=ZA_0^{(1)}+Z^3A_0^{(3)}+\cdots \ , \tag{18a}$$

$$A_i=Z^2A_0^{(2)}+Z^4A_i^{(4)}+\cdots \ . \tag{18b}$$

When the parameter Z vanishes, A_0 tends to zero and A_i becomes a pure gauge in the Abelian gauge frame. For the solution given in Ref. 9 and up to the order Z^2, we find, using expressions (4) and (5),

$$Q_T=(4\pi d/\mu) \ , \tag{19a}$$

$$Q_S=\tfrac{35}{3}(4\pi d/\mu) \ , \tag{19b}$$

where d and μ are constants. Thus we have $Q_S>Q_T\neq 0$, and partial color screening occurs also for an extended source.

(b) Examining the argument of Ref. 8 carefully, the assumption that when $M=0$, A_i can only behave as $O(1/r^{1+\epsilon})$ as $r\to\infty$ is not completely valid. While it is certainly true the behavior $O(1/r^{1+\epsilon})$ will render M vanishing, it is equally true that as long as A_i has no singularity, M vanishes when the external source is specified in the Abelian-gauge frame. Hence a complete-screening solution can exist even though $M=0$. The total-screening solutions of Ref. 2, where $M=0$, thus counter the arguments of Ref. 8. As mentioned earlier, the objection raised by Hughes[3] for the extended source is overcome once the gauge-invariant total color is given.[4]

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