Static quark-antiquark potential with renormalization-group-improved lattice action

S. Itoh, Y. Iwasaki, and T. Yoshié
Institute of Physics, University of Tsukuba, Ibaraki 305, Japan
(Received 8 July 1985; revised manuscript received 25 November 1985)

The expectation values of Wilson loops are calculated with a renormalization-group (RG)-improved lattice action on a 12^4 lattice for $2.1 \leq \beta \leq 2.7$ with steps $\Delta\beta = 0.1$. Then the quark-antiquark static potential is calculated by a fitting procedure directly to the logarithms of the expectation values of Wilson loops rather than Creutz ratios. The inverse of the lattice spacing a at $\beta = 2.4$ turns out to be $a^{-1} = 1470(30)$ MeV by inputting the string tension $\sigma = (420 \text{ MeV})^2$. This value is in remarkable agreement with the value $a^{-1} = 1460(60)$ MeV which has been obtained from the meson spectrum calculated on an $8^3 \times 16$ lattice at $\beta = 2.4$ with the same RG-improved lattice action. The scaling behavior of the string tension and the problem of universality are also investigated.

I. INTRODUCTION

One of the fundamental physical quantities which can be derived from lattice gauge theories[1] is the string tension. After Creutz[2] calculated the string tension in the SU(2) gauge theory by Monte Carlo (MC) simulations, a lot of works,[3] for the string tension have been done. Recent works,[4–6] among others, have observed the following: (i) When extracting the string tension from the expectation values of Wilson loops, one has to take into account the effect of a Coulomb term in the quark-antiquark static potential; (ii) the string tension does not satisfy asymptotic scaling up to the coupling constants where MC calculations have been done so far.

Because of these facts it is not so easy to derive the numerical value of the string tension which is free from possible systematic errors. In this paper we would like to calculate the string tension as well as the quark-antiquark static potential with an effort to reduce possible systematic errors. In particular, to reduce the systematic errors due to finite-lattice-spacing effects, we take a renormalization-group (RG) -improved lattice action. We also propose a fitting procedure directly to the logarithms of the expectation values of Wilson loops in order to reduce the systematic errors due to the process of extracting the potential from the MC data. (See below for more details.)

The organization of the paper is as follows: After the action is introduced in Sec. II A, the results of the calculation of Wilson loops are presented in Sec. II B. In Sec. II C, a fitting procedure is introduced and finally the potential is derived. Section III is devoted to discussion. An algorithm of the updating of link variables for a vector processor is given in the Appendix.

II. CALCULATIONS

A. Action

We take a RG-improved action

$$S = \frac{1}{g^2}\left[c_0 \sum \text{Tr}(\text{simple plaquette loop}) + c_1 \sum \text{Tr}(1\times 2 \text{ rectangular loop})\right] \quad (2.1)$$

with

$$c_1 = -0.331, \quad c_0 = 1 - 8c_1 , \quad (2.2)$$

the form of which has been determined by a perturbative block-spin RG study[7] and by the analysis of instantons on the lattice.[8] In the sum over loops, each oriented loop appears once. We expect that short-distance lattice artifacts are reduced with this RG-improved action. Our earlier results[9] in the two-dimensional O(3) σ model have indeed confirmed that short-distance lattice artifacts are reduced with a RG-improved action.

B. Calculation of Wilson loops

We take a four-dimensional hypercubic lattice with periodic boundary conditions. The lattice size is 12^4. To generate gauge configurations we use the algorithm by Cabibbo and Marinari,[10] modified slightly for vector processors (see the Appendix for details). The β range ($\beta = 6/g^2$) investigated is $2.1 \leq \beta \leq 2.7$ with steps $\Delta\beta = 0.1$. The MC simulation is done as follows: Starting from a completely disordered configuration, we perform 3000 iterations at $\beta = 2.1$. The final configuration is then used as the initial condition for the simulation of 2500 iterations at $\beta = 2.2$. In a similar way we perform 2500 iterations at each β from $\beta = 2.3$ to $\beta = 2.7$. We disregard the first 1000 iterations at $\beta = 2.1$ and the first 500 iterations at $\beta = 2.2$–2.7 for thermalization.

We measure the Wilson loops of size $I \times J$ with $1 \leq I, J \leq 6$ every ten iterations: The expectation value of the Wilson loop of size $I \times J$ is defined by

$$W(I,J) = \tfrac{1}{3}\left\langle \text{Tr} \prod_C U \right\rangle , \quad (2.3)$$

where $\prod_C U$ means the ordered product of link variables around the loop with size $I \times J$.

The mean value of the expectation value of the Wilson loop and the statistical error for the mean value are estimated by the standard method, after checking the statistical independency of the data for each configuration. We present the mean values and the statistical errors of $W(I,J)$ for $2.1 \leq \beta \leq 2.7$ in Table I.

We make one more independent run on a $12^2 \times 24$ lat-

TABLE I. Mean value and statistical errors of $W(I,J)$ for $2.1 < \beta < 2.7$.

I \ J	1	2	3	4	5	6
			$\beta=2.1$			
1	0.566 846	0.310 484	0.172 055	0.095 489	0.052 990	0.029 406
	0.000 066	0.000 095	0.000 094	0.000 082	0.000 069	0.000 059
2		0.097 658	0.033 050	0.011 479	0.004 014	0.001 424
		0.000 098	0.000 066	0.000 046	0.000 041	0.000 039
3			0.007 533	0.001 812	0.000 427	0.000 095
			0.000 057	0.000 037	0.000 035	0.000 033
4				0.000 305	0.000 037	0.000 005
				0.000 046	0.000 034	0.000 034
5					−0.000 023	−0.000 083
					0.000 046	0.000 031
6						0.000 076
						0.000 045
			$\beta=2.2$			
1	0.595 769	0.348 759	0.207 423	0.123 602	0.073 696	0.043 921
	0.000 062	0.000 094	0.000 097	0.000 087	0.000 073	0.000 062
2		0.129 713	0.052 377	0.021 617	0.008 953	0.003 657
		0.000 100	0.000 078	0.000 058	0.000 046	0.000 040
3			0.015 878	0.004 991	0.001 561	0.000 512
			0.000 064	0.000 044	0.000 036	0.000 037
4				0.001 306	0.000 381	0.000 036
				0.000 048	0.000 036	0.000 037
5					0.000 010	0.000 022
					0.000 049	0.000 034
6						−0.000 017
						0.000 050
			$\beta=2.3$			
1	0.619 201	0.380 702	0.238 384	0.149 553	0.093 883	0.058 919
	0.000 054	0.000 086	0.000 106	0.000 099	0.000 088	0.000 075
2		0.159 029	0.072 326	0.033 644	0.015 668	0.007 323
		0.000 119	0.000 094	0.000 074	0.000 056	0.000 046
3			0.026 346	0.010 058	0.003 929	0.001 533
			0.000 071	0.000 052	0.000 042	0.000 036
4				0.003 155	0.001 046	0.000 336
				0.000 051	0.000 037	0.000 032
5					0.000 230	0.000 026
					0.000 046	0.000 036
6						−0.000 052
						0.000 048
			$\beta=2.4$			
1	0.638 740	0.407 791	0.265 465	0.173 180	0.113 087	0.073 837
	0.000 054	0.000 086	0.000 092	0.000 089	0.000 082	0.000 075
2		0.185 297	0.091 674	0.046 364	0.023 633	0.012 050
		0.000 112	0.000 101	0.000 077	0.000 059	0.000 049
3			0.037 879	0.016 345	0.007 164	0.003 092
			0.000 086	0.000 061	0.000 049	0.000 038

TABLE I. (*Continued*).

I \ J	1	2	3	4	5	6
4				0.006 237 0.000 063	0.002 455 0.000 041	0.000 923 0.000 037
5					0.000 856 0.000 051	0.000 231 0.000 034
6						0.000 070 0.000 050
			$\beta=2.5$			
1	0.655 698 0.000 053	0.431 599 0.000 084	0.289 619 0.000 093	0.194 763 0.000 093	0.131 116 0.000 089	0.088 307 0.000 083
2		0.209 204 0.000 107	0.110 095 0.000 097	0.059 199 0.000 081	0.032 001 0.000 066	0.017 330 0.000 057
3			0.049 745 0.000 088	0.023 446 0.000 067	0.011 150 0.000 049	0.005 366 0.000 044
4				0.009 918 0.000 064	0.004 333 0.000 045	0.001 963 0.000 037
5					0.001 649 0.000 048	0.000 653 0.000 034
6						0.000 266 0.000 046
			$\beta=2.6$			
1	0.670 777 0.000 046	0.453 049 0.000 081	0.311 707 0.000 099	0.215 050 0.000 104	0.148 465 0.000 099	0.102 520 0.000 090
2		0.231 197 0.000 116	0.127 848 0.000 111	0.072 260 0.000 096	0.041 027 0.000 079	0.023 330 0.000 065
3			0.016 899 0.000 104	0.031 304 0.000 081	0.015 989 0.000 059	0.008 184 0.000 050
4				0.014 548 0.000 066	0.006 899 0.000 049	0.003 263 0.000 039
5					0.003 047 0.000 057	0.001 392 0.000 037
6						0.000 657 0.000 053
			$\beta=2.7$			
1	0.684 318 0.000 045	0.472 619 0.000 075	0.331 930 0.000 103	0.233 945 0.000 105	0.165 017 0.000 102	0.116 413 0.000 095
2		0.251 457 0.000 126	0.144 533 0.000 122	0.084 750 0.000 103	0.049 930 0.000 085	0.029 457 0.000 070
3			0.073 779 0.000 113	0.391 195 0.000 088	0.020 992 0.000 070	0.011 367 0.000 054
4				0.019 323 0.000 072	0.009 656 0.000 052	0.004 900 0.000 046
5					0.004 498 0.000 051	0.002 149 0.000 038
6						0.001 024 0.000 048

tice at $\beta=2.4$ partly in order to see the independence of the result from the initial condition and partly in order to prepare some configurations for a calculation of the hadron spectrum. We are particularly interested in the value of the string tension at $\beta=2.4$, because we have already calculated[11] the hadron spectrum on an $8^3\times 16$ lattice at $\beta=2.4$. Therefore it is possible to test whether the lattice spacings calculated from the hadron spectrum and from the string tension are identical. (Although we have calculated the string tension on a 6^4 lattice with the RG-improved action,[12] the size of the lattice is certainly too small to derive a reliable value for the string tension.) The MC simulations are done as follows: We make 3000 iterations after disregarding 1500 iterations starting from a completely disordered state. The measurement of Wilson loops $W(I,J)$ with $1\leq I,J\leq 6$ is made every ten iterations as previously. We call the direction of the lattice with the linear extension 24 the timelike direction and other three directions spacelike directions. We separately calculate the expectation values and statistical errors of the Wilson loops for time-space (TS) directions and for space-space (SS) directions. The mean values of the Wilson loops both for the SS case and the TS case on the $12^3\times 24$ lattice at $\beta=2.4$ agree with those at $\beta=2.4$ in Table I within statistical errors for large Wilson loops and within 1% for small Wilson loops and therefore are not represented here.

C. Calculation of the potential

If the asymptotic form of Wilson loops $W(I,J)$ for large I and J is given by

$$W(I,J)=\exp[-\tilde{\sigma}IJ-m(I+J)+c]\ , \tag{2.4}$$

we can derive the string tension in units of a^{-2} (a is the lattice spacing) directly from the ratio[13] (to be referred as the Creutz ratio)

$$\chi(I,J)=-\ln\frac{W(I,J)W(I-1,J-1)}{W(I-1,J)W(I,J-1)} \tag{2.5}$$

by the relation

$$\tilde{\sigma}=\chi(I,I)\ . \tag{2.6}$$

However, $\chi(I,I)$, in fact, decreases as I increases when $I=4$–6, as for the standard model. This indicates that the potential is not purely a linear rising one.

Therefore let us assume that the potential in units of a^{-1} is the sum of a linear rising term and a Coulomb term

$$\tilde{V}(I)=\tilde{\sigma}I+c+d/I\ , \tag{2.7}$$

as generally assumed. This implies that $-\ln W(I,J)$ contains a term proportional to J/I as well as a term proportional to $I\times J$. Because of the symmetry between I and J in $W(I,J)$, this further implies that $-\ln W(I,J)$ contains a term proportional to I/J.

This motivates us to assume that

$$-\ln W(I,J)=V_1(I)J+V_2(I)+V_3(I)\frac{1}{J}\ , \tag{2.8}$$

where

$$V_1(I)=\tilde{V}(I)\ . \tag{2.9}$$

Of course, there are no *a priori* justifications for the assumption of the form (2.7) and therefore of the form (2.8). The last term of Eq. (2.7) might be more complicated than $1/I$ and might contain logarithmic factors. However the numbers of I's and J's which are available are not so large and are not sufficient to determine such a delicate behavior of the potential. Our strategy is rather to assume simple forms for $W(I,J)$ as well as for $\tilde{V}(I)$ which are consistent with the existence of a Coulomb term in the potential. After the analysis of the data with these assumptions, we are able to judge the validity of the assumptions *a posteriori*. These assumptions are essentially the same as those in Ref. 5. However, the procedure of the fitting is different, as shown below.

We fit directly $-\ln W(I,J)$ in terms of the form (2.8) instead of the ratio $\chi(I,J)$ as in Ref. 5, because in this way the ratios of statistical errors to the signals do not increase. The fitting procedure is the standard least-squares method: We determine $V_i(I)$ ($i=1$–3) in such a way to minimize

$$\Delta=\sum_{J_{\min}}^{J_{\max}}\frac{[-\ln W(I,J)-V_1(I)J-V_2(I)-V_3(I)/J]^2}{[\delta\ln W(I,J)]^2} \tag{2.10}$$

with I fixed. Here we assume that

$$\delta\ln W(I,J)=\delta W(I,J)/W(I,J)\ , \tag{2.11}$$

where $\delta W(I,J)$ is the statistical error for $W(I,J)$. The $V_i(I)$ can be determined in the form

$$V_i(I)=\sum_{J_{\min}}^{J_{\max}}C_i(I,J)\ln W(I,J)\ . \tag{2.12}$$

The statistical errors of $V_i(I)$ are estimated from

$$[\Delta V_i(I)]^2=\sum[C_i(I,J)]^2[\delta\ln W(I,J)]^2 \tag{2.13}$$

assuming no correlations between $\ln W(I,J)$ with different J with fixed I. In general, $J_{\min}=1$ and $J_{\max}=6$. However, when the mean value is negative and/or when the ratio of the statistical error to the mean value is larger than 10%, we have excluded the data from the fitting, because such data are not reliable.

The results for the fitting are shown in Fig. 1 for $\beta=2.4$ and 2.7. We see that the fittings to the data are remarkable, especially for $\beta=2.7$, where the statistical errors for $W(I,J)$ themselves are small. This implies that our assumption of the form (2.8) is not unreasonable. Note that if we fit $-\ln W(I,J)$ to the form $\tilde{V}(I)J+V_2(I)$ for $J\simeq 4$–6, we will obtain a substantially different value for $\tilde{V}(I)$.

Then we fit $V_1(I)=\tilde{V}(I)$ to

$$V_1(I)=V_{11}I+V_{12}+V_{12}\frac{1}{I}\ . \tag{2.14}$$

The fitting procedure is the same as for $-\ln W(I,J)$. The results are shown in Fig. 2 for $\beta=2.4$ and 2.7. The fittings are again remarkable. We also fit $V_2(I)$ and $V_3(I)$ to

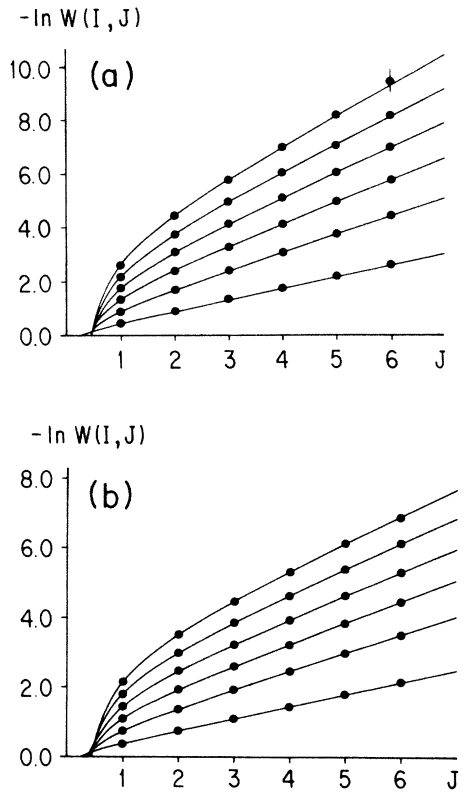

FIG. 1. (a) Fitting curves for $-\ln W(I,J)$ to determine the potential at $\beta=2.4$ in the SS case on the $12^3\times 24$ lattice. (b) Fitting curves for $-\ln W(I,J)$ to determine the potential at $\beta=2.7$.

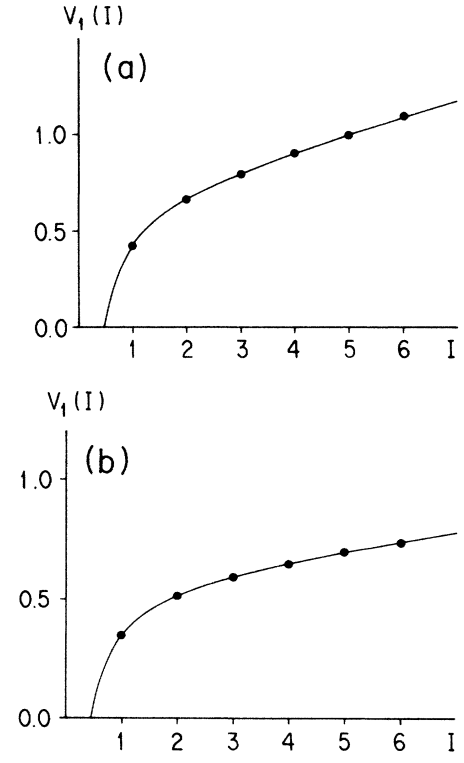

FIG. 2. (a) Fitting curve for $V_1(I)$ to determine the string tension at $\beta=2.4$ in the SS case on the $12^3\times 24$ lattice. (b) Fitting curve for $V_1(I)$ to determine the string tension at $\beta=2.7$.

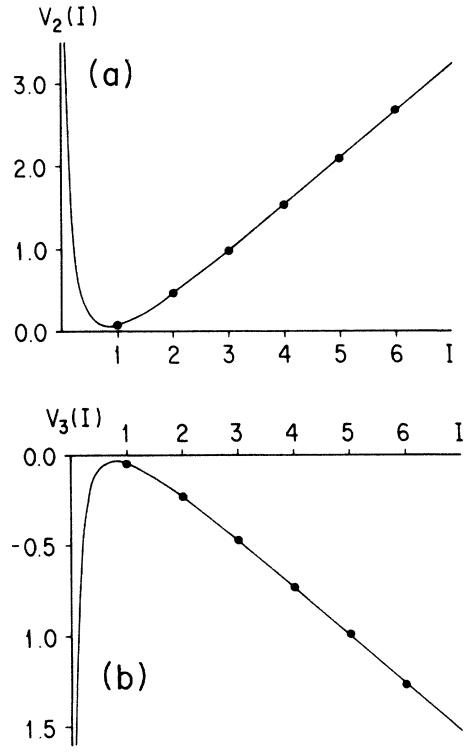

FIG. 3. (a) Fitting curve for $V_2(I)$ at $\beta=2.7$. (b) Fitting curve for $V_3(I)$ at $\beta=2.7$.

$$V_2(I)=V_{21}I+V_{22}+V_{23}\frac{1}{I}\,,$$
$$V_3(I)=V_{31}I+V_{32}+V_{33}\frac{1}{I}\,. \tag{2.15}$$

The results for $V_2(I)$ and $V_3(I)$ for $\beta=2.7$ are shown in Fig. 3. The fittings are again excellent

We reproduce V_{ij} $(i,j=1-3)$ for $\beta=2.4$ in Table II. The above fitting procedure corresponds to the parametrization

$$-\ln W(I,J)=V_{11}IJ+V_{12}J+V_{13}\frac{J}{I}+V_{21}I+V_{22}+V_{23}\frac{1}{I}+V_{31}\frac{I}{J}+V_{32}\frac{1}{J}+V_{33}\frac{1}{IJ}\,. \tag{2.16}$$

TABLE II. Fitting parameters V_{ij} and statistical errors δV_{ij} for $\beta=2.4$.

i \ j	1	2	3
1	0.082 665	0.652 235	−0.312 518
	0.001 826	0.006 364	0.004 618
2	0.652 396	−1.053 294	0.478 458
	0.006 527	0.023 373	0.017 190
3	−0.313 491	0.481 821	−0.219 959
	0.004 778	0.017 346	0.012 842

TABLE III. Numerical values of Wilson loops $W(I,J)$ calculated by the parameters V_{ij} for $\beta=2.4$.

I \ J	1	2	3	4	5	6
1	0.638 703	0.407 990	0.265 139	0.173 048	0.113 138	0.074 032
2		0.185 302	0.091 666	0.046 321	0.023 606	0.012 082
3			0.037 846	0.016 332	0.007 174	0.003 179
4				0.006 169	0.002 395	0.000 943
5					0.000 830	0.000 293
6						0.000 093

The symmetry property of $W(I,J)$ between I and J implies the symmetry property of V_{ij} between i and j. Although our fitting procedure is not symmetric between i and j in V_{ij}, if our fitting form is reasonable, the derived V_{ij}'s will satisfy approximate equalities between V_{ij} and V_{ji}. The results satisfy remarkably these equalities. The results for other β's are similar. This *a posteriori* supports our fitting procedure.

We present in Table III the value of the Wilson loops at $\beta=2.4$ for the parameters determined, using Eq. (2.16). They are in excellent agreement with the data. This is typical. If β is larger, the fitting is better. Thus our fitting procedure seems reasonable. We have also fitted $-\ln W(I,J)$ using six parameters, assuming $V_{ij}=V_{ji}$. There are no noticeable changes for the results, as expected.

The fitting by the least-squares method enhances the weight of the data with small statistical errors. For $-\ln W(I,J)$, the errors of the data for small I,J ($\lesssim 2$) are extremely small. However, $W(I,J)$ for small I,J may contain artifacts of lattice theory. Therefore we have tried several fitting procedures in such a way that relative weights for $W(I,J)$ with small I,J decrease: For example, if $\delta W(I,j)/W(I,J)<10^{-3}$, for some I,J set $\delta W(I,J)/W(I,J)=10^{-3}$ for those I,J; or, if $\delta W(I,J)/W(I,J)<10^{-2}$, set $\delta W(I,J)/W(I,J)=10^{-2}$; or neglect $W(I,J)$ with $I,J\lesssim 2$ for the fitting. As far as we have tried, there are no noticeable changes in the result for the potential. The value of the string tension changes only by about 1%. This is due to the fact that the fitting to $-\ln W(I,J)$ is good for all I,J. See the results for some selected cases at $\beta=2.4$ (the SS case) in Table IV.

Our fitting procedure is the same in spirit as that of Barkai, Moriarty, and Rebbi, as noted earlier. However, we think it is better to fit directly the data of $-\ln W(I,J)$ than the data of the Creutz ratios $\chi(I,J)$, because when taking Creutz ratios the ratios of error to the data become large when I,J are large.

TABLE IV. String tensions obtained by several fitting procedures (see text) at $\beta=2.4$ in the SS case on the $12^3\times 24$ lattice: 1, our standard fitting procedure; 2, if $\delta W(I,J)/W(I,J)<10^{-2}$, set $\delta W(I,J)/W(I,J)=10^{-2}$; 3, if $\delta W(I,J)/W(I,J)<10^{-3}$, set $\delta W(I,J)/W(I,J)=10^{-3}$; 4, neglect $W(I,J)$ with $I,J\lesssim 2$.

1	5.660 903±0.095 342
2	5.687 602±0.267 635
3	5.668 031±0.101 056
4	5.634 301±0.134 236

Now some comments on finite-size effects are in order. The deconfining phase transition at $\beta=2.7$ will occur only when the size of a lattice in temporal direction is less than six, according to our crude estimate of the deconfining temperature.[12] Thus the size of the lattice on which we have calculated the Wilson loops is about twice that where the deconfining phase transition occurs. Further, the correlation length $\xi=1/\sqrt{\tilde{\sigma}}$ at $\beta=2.7$ is about 5.46 and less than half of the lattice size. Therefore we believe that finite-size effects are under control.

Let us finally determine the physical potential from $V_1(I)=V(I)$ calculated for $2.1\lesssim\beta\lesssim 2.7$. When one directly deals with $-\ln W(I,J)$ instead of the Creutz ratio, the constant term in the potential is a potentially dangerous term, because it does not approach a constant in the continuum limit. However, this term corresponds to the self-energy of the static sources and therefore we can safely subtract it from the potential, as usual.

The physical potential is defined by

$$V(r)=\frac{1}{a}\tilde{V}(I) \quad \text{with } r=Ia \ , \tag{2.17}$$

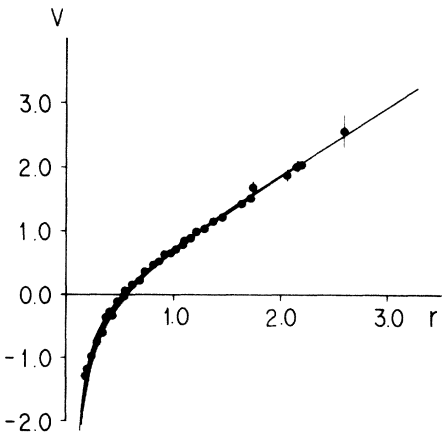

FIG. 4. The potential curve in physical units for $\beta=2.1$–2.7, together with the data in the case of the RG-improved action.

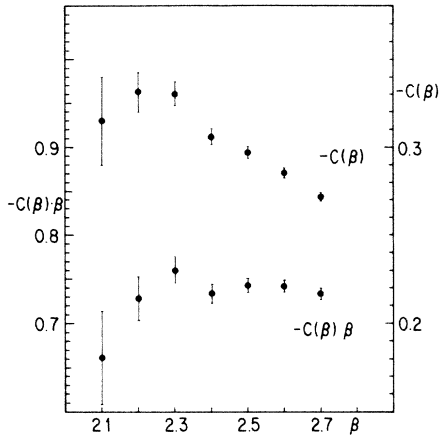

FIG. 5. $c(\beta)$ and $\beta c(\beta)$ vs β, for the RG-improved action.

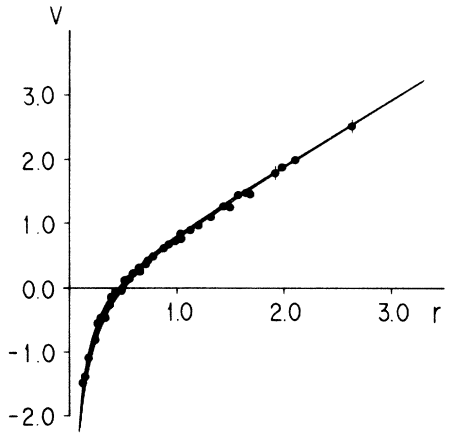

FIG. 7. The potential curve in physical units for $\beta=5.6$–6.6, together with the data in the case of the standard one-plaquette action.

where a is the lattice spacing. Renormalizing $V(r)$ and r by $V(r)/\sqrt{\sigma}$ and $\sqrt{\sigma}r$ ($\sigma=\tilde{\sigma}/a^2$ is the physical string tension), we have

$$V(r)=r+\frac{c(\beta)}{r} \tag{2.18}$$

for $2.1\leq\beta\leq2.7$. Here $c(\beta)$ equals V_{13} in Eq. (2.14). If the potential is given exactly by the form $r+c/r$, and the β range we have examined is in the scaling region, and, furthermore, the data at short distances are free from artifacts of lattice theory; $c(\beta)$ should be a constant independent of β.

We plot in Fig. 4 all results for $V(r)$ in the range $2.1\leq\beta\leq2.7$ together with the data. They are approximately on the universal curve. However, $c(\beta)$ slightly depends on β. We plot $c(\beta)$ and $\beta c(\beta)$ vs β in Fig. 5. We see that $\beta c(\beta)$ is constant rather than $c(\beta)$ itself. One might interpret this result as follows: The potential is of the form $r+\alpha(r)/r$, where $\alpha(r)$ is an effective coupling constant which decreases as r decreases. As β increases, the number of the points at short distance where the data are taken increases and, consequently, $c(\beta)$ effectively decreases when the data are fitted to $r+c(\beta)/r$. Of course, from the limited numbers of I and J for $W(I,J)$ we cannot exclude the possibility that other conditions listed in the preceding paragraph are not satisfied.

We plot $\tilde{\sigma}/\Lambda^2$ vs β in Fig. 6. We see that asymptotic scaling is not satisfied up to $\beta\simeq2.5$, and that there is a possibility that asymptotic scaling sets in at $\beta\simeq2.6$. To verify this we have to do MC calculations on a larger lattice for $\beta\geq2.8$. If we assume asymptotic scaling for the data at $\beta=2.6$ and 2.7, we obtain

$$\frac{\sigma}{\Lambda_{\rm IM}{}^2}=4.47\pm0.03 \text{ or } \frac{\Lambda_{\rm IM}}{\sqrt{\sigma}}=0.473\pm0.003\ , \tag{2.19}$$

where $\Lambda_{\rm IM}$ is the scale parameter for the RG-improved action. This implies

$$\frac{\Lambda_S}{\sqrt{\sigma}}=0.008\pm0.0002 \tag{2.20}$$

because[14]

$$\frac{\Lambda_{\rm IM}}{\Lambda_S}=59.05\pm1.0\ . \tag{2.21}$$

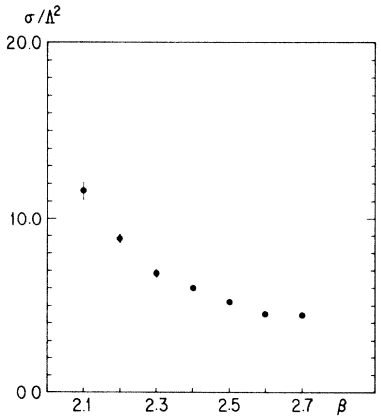

FIG. 6. σ/Λ vs β, for the RG-improved action.

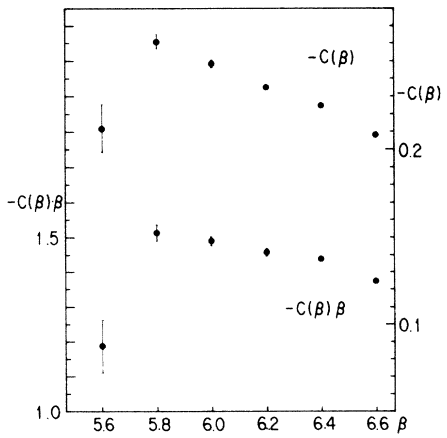

FIG. 8. $c(\beta)$ and $\beta c(\beta)$ vs β, for the standard one-plaquette action.

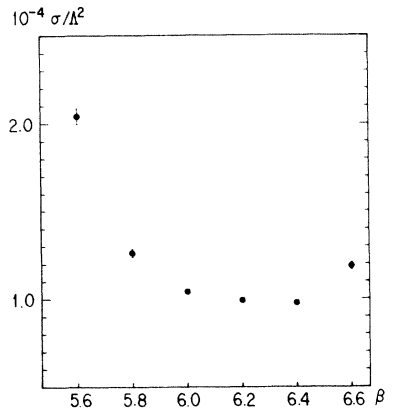

FIG. 9. σ/Λ^2 vs β, for the standard one-plaquette action.

We have also analyzed the data for the standard model in Ref. 5 by the same method as above. The results are shown in Figs. 7–9. The values for $\tilde{\sigma}$ are similar to those obtained in Ref. 5, but the $\tilde{\sigma}$'s for $\beta=6.4$ and 6.6 become slightly smaller than those in Ref. 5.

Let us discuss universality. The value quoted in Ref. 5 for $\Lambda_S/\sqrt{\sigma}$ is larger than that obtained above for the RG-improved action by 20%. However, the value for $\Lambda_S/\sqrt{\sigma}$ is decreasing from $\beta=6.4$ to $\beta=6.6$. If this tendency continues still for $\beta>6.6$, the results may become consistent with each other. To see what really happens, it is very interesting to calculate $\tilde{\sigma}$ for $\beta>2.7$ with the RG-improved action and for $\beta>6.6$ with the standard action.

III. DISCUSSION

The quark-antiquark static potential we have obtained is

$$V(r)=\sigma r-c/r \tag{3.1}$$

with

$$c\simeq 0.3\pm 0.3\ . \tag{3.2}$$

When we compare this result with previous phenomenological analyses of $c\bar{c}$ and $b\bar{b}$ bound states, a numerical value $c\simeq 0.3$ is not unreasonable.

Concerning artifacts of lattice theory at short distance, the difference between the standard action and the RG-improved action is not manifested clearly when we compare Fig. 4 with Fig. 7. This is quite different from what we have observed in the two-point function $G(n,0)$ of the two-dimensional O(3) σ model.[8] We think this is due to the fact that one is able to calculate $G(n,0)$, e.g., up to $n\simeq 20$, while one is able to calculate $W(I,J)$ only up to $I,J\simeq 6$–8. If one could calculate $W(I,J)$ up to $I,J\simeq 20$, the situation would become clear. Because we cannot do such a calculation at the present time, we have to proceed differently.

One way is to check the consistency between various calculations done for various physical quantities. We have calculated the hadron spectrum in the quenched approximation at $\beta=2.4$ on an $8^3\times 16$ lattice, using the same RG-improved action.[11] Thus let us check whether the lattice spacings determined from the string tension and from the hadron spectrum at $\beta=2.4$ are identical. (We have briefly reported the following in Ref. 15.)

If we input $\sqrt{\sigma}=420$ MeV, we obtain

$$a^{-1}=1460(40)\ \text{MeV} \tag{3.3}$$

at $\beta=2.4$. We also have

$$a^{-1}=1490(40)\ \text{MeV} \tag{3.4}$$

for the SS case and

$$a^{-1}=1460(40)\ \text{MeV} \tag{3.5}$$

for the TS case. All of the results for a^{-1} agree with each other within statistical errors. This verifies the independence of the result from the initial condition. Combining three values we have

$$a^{-1}=1470(30)\ \text{MeV}\ . \tag{3.6}$$

Now let us turn to the calculation of the hadron spectrum.

By fitting the pion mass and the ρ mass, we obtain

$$a^{-1}=1460(60)\ \text{MeV}\ . \tag{3.7}$$

Thus the values for a^{-1} are in remarkable agreement within statistical errors. This means that we can fit both the meson spectrum and the string tension by one *unique* gauge coupling constant. This is a necessary condition that the theory be self-consistent. Of course, for both calculations we have used the quenched approximation. Therefore it is not necessary that the values for a^{-1} agree with each other completely. If we include the effect of virtual quark loops, both numbers will change. However, the notion of the string tension and the quark-antiquark static potential is heavily based on the valence-quark model of hadrons and further both of them characterize mesons. Therefore it is very likely that both of them change in the same manner. Thus we think that it is nice that the values for a^{-1} agree with each other in the quenched approximation.

In the case of the standard model Bowler *et al.* have obtained[16]

$$a^{-1}=1.45(9)\ \text{GeV} \tag{3.8}$$

fitting the ρ mass and the pion mass at $\beta=5.7$ on an $8^3\times 16$ lattice. From the result for the string tension at $\beta=5.7$ by Hasenfratz, Hasenfratz, Heller and Karsch[4] one obtains

$$a^{-1}=950(30)\ \text{MeV} \tag{3.9}$$

if one inputs $\sqrt{\sigma}=420$ MeV. The results for a^{-1} in Eqs. (3.8) and (3.9) do not agree with each other by about 50%. Although it is not necessary that they agree with each other completely in the quenched approximation as mentioned earlier, we feel that the discrepancy by about 50% is larger than expected in the quenched approximation.

We see that there is a clear difference between the RG-improved lattice action and the standard action concerning identity of the values for a^{-1}. We may attribute this

to the fact that systematic errors due to finite lattice spacing effects at small distances are considerably small when one takes a RG-improved lattice action.

ACKNOWLEDGMENTS

The numerical calculation has been performed with a HITAC S810 at Central Research Laboratory, Hitachi, Ltd. We are deeply indebted to Dr. Y. Takeda, General Manager of this laboratory, and Professor M. Namiki of Waseda University, for their kind arrangement which made this work possible. We would also like to thank K. Shimoi of Hitachi, Ltd. for his kind help in computer operation and Professor Y. Oyanagi for advice on programming. Finally, two of us (S.I. and T.Y.) would like to thank the Iwanami Fuju-kai for financial support.

APPENDIX: ALGORITHM OF THE UPDATING OF LINK VARIABLES

We use the well-known algorithm of Cabibbo and Marinari[10] to generate gauge-field configurations. However, in order to speed up the updating on a vector processor, we have to slightly modify the algorithm which is used for a usual scalar processor.

First, we have to divide the updating process into independent parts. Since we use the old link variable when calculating the mean field, we can update $U_{n,\mu}$ and $U_{m,\nu}$ independently only in the case that $U_{m,\nu}$ does not appear in the mean field $M_{n,\mu}$ for $U_{n,\mu}$ and $U_{n,\mu}$ not in the mean field $M_{m,\nu}$ for $U_{m,\nu}$. To satisfy this condition, we classify the whole links $U_{n,\mu}$ into 24 classes

$$C_{\mu ab}=\left\{ U_{n,\mu} \;\middle|\; \begin{array}{l} n_\mu \equiv a \pmod 2 \\ n_{\mu_1}+n_{\mu_2}+n_{\mu_3}\equiv b \pmod 3 \end{array} \right\}, \tag{A1}$$

where μ, μ_1, μ_2, and μ_3 are all different and $n=(n_1,n_2,n_3,n_4)$. Thus we can update the links in the same class on a vector processor.

We have to take care of another point for the simulations on a vector processor. The number of independent instructions which are executed for each link in the same class should be equal. Therefore we have to slightly modify the accept-reject process explained below in the modified heat-bath method by Creutz[2] to generate SU(2) submatrices as follows: As is well known, in the modified heat-bath method, to generate a variable z with the probability

$$P(z)\propto \exp(\beta v z)(1-z^2)^{1/2}dz\ , \tag{A2}$$

where v is a constant which is determined from the previous configuration, one generates a trial z with the measure

$$P(z)\propto \exp(\beta v z)dz \tag{A3}$$

and a random number r $(0\le r<1)$ successively until the pair (z,r) satisfies the condition $r\le(1-z^2)^{1/2}$. One accepts the last trial z. This is the accept-reject process. However, we can generate only the same number of trial pairs (z,r) on a vector processor. Thus we modify this accept-reject process in the following way. If all the trial z's do not satisfy the condition for a link, we accept the old link variable as the new one. On the other hand, if some of trial z's satisfy the condition for a link, we take the last z which satisfies it as the new variable. This modified accept-reject process works well. The proof that the system approaches the thermal equilibrium can be done similarly as for the Metropolis algorithm. We have decided to generate eight trial (z,r) pairs for each link. In this case, about 93% of the links are updated in each iteration at $\beta=2.4$.

[1]K. G. Wilson, Phys. Rev. D **10**, 2455 (1974).
[2]M. Cruetz, Phys. Rev D **21**, 2308 (1980).
[3]M. Creutz, Phys. Rev. Lett. **45**, 313 (1980); E. Pietarinen, Nucl. Phys. **B190**, 349 (1981); M. Creutz and K. J. M. Moriarty, Phys. Rev. D **26**, 2166 (1982); R. W. B. Ardill, M. Creutz, and K. J. M. Moriarty, *ibid.* **27**, 1956 (1983); F. Gutbrod, P. Hasenfratz, Z. Kunszt, and I. Montvay, Phys. Lett. **128B**, 415 (1983); M. Fukugita, T. Kaneko, and A. Ukawa, Phys. Rev. D **28**, 2696 (1983); G. Parisi, R. Petronzio, and F. Rapuano, Phys. Lett. **128B**, 413 (1983); D. Barkai, M. Creutz, and K. J. M. Moriarty, Phys. Rev. D **29**, 1207 (1984); J. D. Stack, *ibid.* **29**, 1213 (1984).
[4]A. Hasenfratz, P. Hasenfratz, U. Heller, and F. Karsch, Z. Phys. C **25**, 191 (1984).
[5]D. Barkai, K. J. M. Moriarty, and C. Rebbi, Phys. Rev. D **30**, 1293 (1984).
[6]S. Otto and J. S. Stack, Phys. Rev. Lett. **52**, 2328 (1984).
[7]Y. Iwasaki, University of Tsukuba Report No. UTHEP-118 (unpublished).
[8]Y. Iwasaki and T. Yoshié, Phys. Lett. **131B**, 159 (1984); S. Itoh, Y. Iwasaki, and T. Yoshié, *ibid.* **147B**, 141 (1984).
[9]S. Itoh, Y. Iwasaki, and T. Yoshié, Nucl. Phys. **B250**, 312 (1985); Phys. Lett. **158B**, 140 (1985).
[10]N. Cabibbo and E. Marinari, Phys. Lett. **119B**, 387 (1982); M. Okawa, Phys. Rev. Lett. **49**, 353 (1982).
[11]S. Itoh, Y. Iwasaki, Y. Oyanagi, and T. Yoshié, Phys. Lett. **148B**, 153 (1984).
[12]Y. Iwasaki and T. Yoshié, Phys. Lett. **143B**, 449 (1984).
[13]M. Creutz, Phys. Rev. Lett. **43**, 206 (1979).
[14]Y. Iwasaki and S. Sakai, Nucl. Phys. **B148**, 441 (1984).
[15]S. Itoh, Y. Iwasaki and T. Yoshié, Phys. Lett. **165B**, 131 (1985).
[16]K. C. Bowler, D. L. Chalmbers, A. Kenway, R. D. Kenway, G. S. Pawley, and D. J. Wallace, Nucl. Phys. **B240** [FS12], 213 (1984).