Covariant quantization of superstrings based on Becchi-Rouet-Stora invariance

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A manifestly covariant quantization of the superstring is presented in the Neveu-Schwarz-Ramond formalism on the basis of the Becchi-Rouet-Stora (BRS) invariance principle. The critical dimension $D=10$ and the correct Regge intercept are shown to follow from the requirement of the nilpotency $Q_B{}^2=0$ of the BRS charge. A modified form of the subsidiary condition of Kugo and Ojima $Q_B\,|\,\text{phys}\rangle=0$ to define the physical subspace is sufficient to demonstrate the no-ghost theorem.

I. INTRODUCTION

It has recently become clear that superstring theories[1] are promising candidates for consistent theories of quantum gravity. These theories developed from the old Neveu-Schwarz-Ramond (NSR) spinning string theories.[2] However, it has long been known[2–4] that these theories can be consistent only if the space-time dimension D is 10 and the leading Regge intercept equal to $\frac{1}{2}$, although there is some recent work on string compactification.[5] In the light-cone gauge formalism in which unitarity is manifest, this follows from the requirement of Lorentz covariance.[4] In the covariant formulation, on the other hand, the generalized Virasoro conditions do not eliminate negative-norm states unless these conditions are satisfied.[2]

Another (and related) deficiency in the superstring theories is the lack of a covariant formulation of second-quantized superstrings (with manifest space-time supersymmetry). In the original formulation of NSR models, which describe the superstrings in their Majorana fermion and even G-parity sector,[2,6] the D-dimensional space-time supersymmetry is not manifest. The new Lorentz-covariant action of Green and Schwarz[1] has a space-time local supersymmetry built in, but this same local supersymmetry turns out to be the obstacle to a covariant quantization of the superstrings.[7] Obviously covariant quantization is most useful to the study of string interactions and the above compactification problem. It is therefore desirable to find a covariant formulation.

Recently Siegel has succeeded in formulating covariantly the second-quantized *bosonic* string,[8] starting from the covariant quantization based on the Becchi-Rouet-Stora (BRS) invariance[9] of Kato and Ogawa.[10] One naturally expects the extension of this formalism to the superstring case. In view of the above-mentioned difficulty in the action of Green and Schwarz, it is natural to consider the problem in the original NSR models.

In this paper we take the first step in this direction. Namely, we perform the covariant quantization of NSR models in the light of BRS invariance, following the work of Kato and Ogawa. An alternative formulation has been given in Ref. 11, in which the fermion vertex operator is also constructed. We believe that our formulation, which is very close to that in Ref. 10, together with their result, will be quite useful to the covariant second quantization of the superstrings. The second quantization along the line of Siegel's work will be discussed elsewhere.

In order to perform the covariant quantization of a superstring using a BRS symmetry, we start from the action[12,13]

$$\mathscr{L}_0=-\frac{1}{2\kappa}e\,e_a^i e^{aj}\partial_i x^\mu\partial_j x_\mu +\frac{e}{\kappa}\left[-\frac{i}{2}\bar{\phi}^\mu\gamma^i\partial_i\phi_\mu+\frac{i}{2}\bar{\psi}_j\gamma^i\gamma^j\phi^\mu\partial_i x_\mu -\tfrac{1}{16}\bar{\phi}^\mu\phi_\mu\bar{\psi}_j\gamma^i\gamma^j\psi_i\right], \tag{1.1}$$

which is invariant under the general coordinate and local Lorentz transformations. It is a classic result[12] that (1.1) gives the correct equations of motion and constraints in a special gauge for the NSR models, and that it is also invariant under the two-dimensional local supersymmetry transformations

$$\delta e_{ai}=i\bar{\epsilon}\gamma_a\psi_i,\quad \delta\psi_i=2D_i\epsilon\,, \tag{1.2a}$$

$$\delta x^\mu=i\bar{\epsilon}\phi^\mu,\quad \delta\phi^\mu=\gamma^i\left[\partial_i x^\mu-\frac{i}{2}\bar{\psi}_i\phi^\mu\right]\epsilon\,, \tag{1.2b}$$

where $D_i=\partial_i+\frac{1}{2}\omega_i\gamma_5$ with

$$\omega_i=-\frac{1}{e}e_{ai}\epsilon^{jk}\partial_j e_k^a-\frac{i}{2}\bar{\psi}_i\gamma_5\gamma^j\psi_j$$

which satisfies

$$\partial_i e_{aj}-\partial_j e_{ai}-\omega_i\epsilon_{ab}e_j^b+\omega_j\epsilon_{ab}e_i^b=\frac{i}{2}\bar{\psi}_i\gamma_a\psi_j\ .$$

However, as was already noted in Ref. 12, the algebra of supergravity does not close off-shell. In this case it is well known[14] that one cannot obtain the correct BRS transformation which should be nilpotent by simply replacing the transformation parameter $\epsilon(\xi)$ with $\lambda C(\xi)$, where C is the Faddeev-Popov (FP) ghost field and λ is an anticommuting c-number parameter.

To find the correct BRS transformation, one should have transformation rules which close off-shell. This problem in our two-dimensional case was discussed in

Ref. 15. It is shown that we need two-dimensional scalar fields S and F^μ, and (1.2) is modified as

$$\delta e_{ai}=i\bar{\epsilon}\gamma_a\psi_i,\quad \delta\psi_i=2(D_i+\tfrac{1}{2}\gamma_i S)\epsilon\ ,$$
$$\delta S=-\frac{i}{2}S\bar{\epsilon}\gamma^i\psi_i-\frac{1}{2e}\epsilon^{ij}\bar{\epsilon}\gamma_5(D_i\psi_j-D_j\psi_i)\ , \tag{1.3a}$$

$$\delta x^\mu=i\bar{\epsilon}\phi^\mu, \delta\phi^\mu=\left[F^\mu+\gamma^i\left[\partial_i x^\mu-\frac{i}{2}\bar{\psi}_i\phi^\mu\right]\right]\epsilon\ ,$$
$$\delta F^\mu=i\bar{\epsilon}\gamma^i\left[D_i\phi^\mu-\tfrac{1}{2}\gamma^j\left[\partial_j x^\mu-\frac{i}{2}\bar{\psi}_j\phi^\mu\right]\psi_i-\tfrac{1}{2}F^\mu\psi_i\right]\ , \tag{1.3b}$$

and there appears an additional $(F_\mu)^2$ term in the Lagrangian. We note in particular that (1.3a) contains invariance under

$$\delta e_{ai}=0,\quad \delta\psi_i=\gamma_i\epsilon,\quad \delta S=-\frac{i}{2}\bar{\epsilon}\gamma^i\psi_i\ , \tag{1.4}$$

because we are free to choose the constant value of S. This scalar field S will play a crucial role in our formulation.

With these symmetries manifest, we can easily apply the techniques known in field theories such as the Kugo-Ojima formalism of non-Abelian gauge theory,[16] as in the case of bosonic strings.[10] We find that the critical dimension $D=10$ and the correct intercept follow from the consistency that the BRS charge Q_B indeed becomes nilpotent, in much the same way as the bosonic case.

The paper is organized as follows. In Sec. II we perform the covariant quantization of (1.1). We define the BRS transformation such that it is nilpotent and quantize (1.1) covariantly with BRS-invariant gauge fixing $e_{ai}=\eta_{ai}$ $[=\mathrm{diag}(-1,+1)]$ and $\psi_i=0$.

In Sec. III we construct the Fock space and find the BRS charge. We show that the nilpotency condition ${Q_B}^2=0$ leads to the critical dimension. The physical state condition is also specified.

In Sec. IV it is pointed out that the proof of the no-ghost theorem in our formalism goes through as for the bosonic case[10] with minor modifications of transverse operators[3] for the superstrings.

Section V is devoted to conclusions.

II. COVARIANT CANONICAL QUANTIZATION

In this section we perform the covariant quantization using BRS invariance of the theory. Our procedure is as follows. First, we determine the BRS transformation which is nilpotent. Second, we choose the orthogonal gauge in which the zweibein $e^{ai}=\eta^{ai}$ and the Rarita-Schwinger field $\psi_i=0$. These conditions are imposed in a BRS-invariant way. Finally, we give mode expansion of the relevant variables and determine their commutation relations by Dirac's quantization method.

The BRS transformation is defined by replacing the transformation parameters with their respective ghosts multiplied by an anticommuting c number λ:

$$\delta e_{ai}=\lambda[C^j\partial_j e_{ai}+(\partial_i C^j)e_{aj}+C_a{}^b e_{bi}-\bar{C}\gamma_a\psi_i]\ , \tag{2.1a}$$

$$\delta\psi_i=\lambda[C^j\partial_j\psi_i+(\partial_i C^j)\psi_j+\tfrac{1}{2}C_{ab}\sigma^{ab}\psi_i+2i(D_i+\tfrac{1}{2}\gamma_i S)C]\ , \tag{2.1b}$$

$$\delta S=\lambda\left[C^j\partial_j S+\tfrac{1}{2}S\bar{C}\gamma^j\psi_j-\frac{i}{2e}\epsilon^{ij}\bar{C}\gamma_5(D_i\psi_j-D_j\psi_i)\right]\ , \tag{2.1c}$$

and

$$\delta x^\mu=\lambda(C^i\partial_i x^\mu-\bar{C}\phi^\mu)\ , \tag{2.2a}$$

$$\delta\phi^\mu=\lambda\left\{C^i\partial_i\phi^\mu+\frac{1}{2}C_{ab}\sigma^{ab}\phi^\mu+i\left[F^\mu+\gamma^i\left[\partial_i x^\mu-\frac{i}{2}\bar{\psi}_i\phi^\mu\right]\right]C\right\}\ , \tag{2.2b}$$

$$\delta F^\mu=\lambda\left\{C^i\partial_i F^\mu-\bar{C}\gamma^i\left[D_i\phi^\mu-\frac{1}{2}\gamma^j\left[\partial_j x^\mu-\frac{i}{2}\bar{\psi}_j\phi^\mu\right]\psi_i-\tfrac{1}{2}F^\mu\psi_i\right]\right\}\ , \tag{2.2c}$$

where C^i and C^{ab} are the anticommuting FP ghosts for the general coordinate and local Lorentz transformations, respectively, and C is the commuting Majorana spinor ghost for the local supersymmetry transformation.

The BRS transformation of these ghost fields is determined by the requirement that the above BRS transformation be nilpotent. The result turns out to be

$$\delta C^i=\lambda(C^j\partial_j C^i+i\bar{C}\gamma^i C)\ , \tag{2.3a}$$

$$\delta C_{ab}=\lambda(C^i\partial_i C_{ab}+C_a{}^c C_{cb}+i\bar{C}\sigma_{ab}CS+\bar{C}\gamma^i C\omega_i\epsilon_{ab})\ , \tag{2.3b}$$

$$\delta C=\lambda(C^i\partial_i C+\tfrac{1}{2}C_{ab}\sigma^{ab}C-\tfrac{1}{2}\psi^i\bar{C}\gamma_i C)\ . \tag{2.3c}$$

In fact one can show that the above transformation is nilpotent, which is to be expected since the algebra closes off-shell.

Under the BRS transformation, the Lagrangian (1.1) transforms into a total divergence[12,15]

$$\delta\mathscr{L}_0=\lambda\partial_i\left[C^i\mathscr{L}_0+\frac{e}{2\kappa}\bar{C}\gamma^i\gamma^j\phi_\mu\partial_j x^\mu-\frac{i}{4\kappa}e\bar{C}\gamma^j\gamma^i\phi_\mu(\bar{\psi}_j\phi^\mu)\right]\ . \tag{2.4}$$

This implies the BRS invariance of the action if we impose the boundary conditions

$$C^1=0\ , \tag{2.5a}$$

$$\bar{C}\gamma_1\gamma^j\phi_\mu\partial_j x^\mu=0\ , \tag{2.5b}$$

at $\sigma=0$ and π, because the last term in Eq. (2.4) vanishes in our gauge choice to be discussed shortly.

Before going into the gauge conditions, we should note that our Lagrangian in fact has an additional invariance to those given above owing to masslessness of the system.[12] This invariance is under Weyl transformations given by

$$\delta e_i^a = \Lambda e_i^a, \quad \delta\psi_i = \tfrac{1}{2}\Lambda\psi_i \,,$$
$$\delta S = \text{arbitrary}, \quad \delta x^\mu = 0 \,, \tag{2.6}$$
$$\delta\phi^\mu = -\tfrac{1}{2}\Lambda\phi^\mu, \quad \delta F^\mu = -\Lambda F^\mu, \quad \delta C = \tfrac{1}{2}\Lambda C \,.$$

This is not included in our BRS transformation but is important in going to the orthogonal gauge in which $e_i^a = \eta_i^a$. However, to simply add this transformation to our BRS transformation invalidates its nilpotency since the algebra then does not close off-shell. Indeed one can easily see that the BRS transformation on the ghosts will not become nilpotent. To include this transformation, one would have to extend the algebra to the superconformal algebra, but this method introduces many additional ghosts and simply complicates the analysis without much new insight into the theory.

This problem is easily avoided by fixing this conformal symmetry from the start. That is, we redefine all fields so that they do not transform under (2.6). This can be achieved by the redefinition

$$\tilde{e}^{ai} = e^{1/2} e^{ai}, \quad \tilde{\psi}_i = e^{-1/4}\psi_i \,,$$
$$\phi^\mu = e^{1/4}\phi^\mu, \quad \tilde{F}^\mu = e^{1/2}F^\mu, \quad \tilde{C} = e^{-1/4}C \,, \tag{2.7}$$

and other fields remain unchanged. [One could also redefine ψ_i as $\tilde{\psi}_i = e^{-1/4}(\psi_i - \frac{1}{2}\gamma_i\gamma^j\psi_j)$ with final results unchanged.] The Lagrangian then takes the form

$$\mathcal{L}_0 = -\frac{1}{2\kappa}\tilde{e}^{\,i}_{\;a}\tilde{e}^{\,ja}\partial_i x^\mu \partial_j x_\mu + \frac{1}{\kappa}\left[-\frac{i}{2}\bar{\tilde{\phi}}^\mu \tilde{\gamma}^{\,i}\partial_i\tilde{\phi}_\mu + \tfrac{1}{2}\tilde{F}_\mu{}^2 + \frac{i}{2}\bar{\tilde{\psi}}_j\tilde{\gamma}^{\,i}\tilde{\gamma}^{\,j}\tilde{\phi}^\mu\partial_i x_\mu - \tfrac{1}{16}\bar{\tilde{\phi}}^\mu\tilde{\phi}_\mu\bar{\tilde{\psi}}_j\tilde{\gamma}^{\,i}\tilde{\gamma}^{\,j}\tilde{\psi}_i \right] + E(\det\tilde{e} - 1) \,, \tag{2.8}$$

where $\tilde{\gamma}^{\,i} \equiv e^{1/2}\gamma^i$ and we have imposed the condition $\det\tilde{e} = 1$ by the multiplier field E because $\tilde{e}^{\,i}_{\;a}$ satisfies it. It is clear that this Lagrangian has no Weyl invariance as long as we regard these fields as fundamental. The BRS transformations of each field are then deduced from (2.1), (2.2), and (2.3). Here we write down only those necessary for later discussions:

$$\delta\tilde{e}^{\,ai} = \lambda[C^j\partial_j\tilde{e}^{\,ai} + \tfrac{1}{2}(\partial_j C^j)\tilde{e}^{\,ai} - \tfrac{1}{2}\bar{\tilde{C}}\tilde{\gamma}^{\,j}\tilde{\psi}_j\tilde{e}^{\,ai} - \tilde{e}^{\,aj}\partial_j C^i + \tilde{e}^{\,bi}C^a{}_b + \tilde{e}^{\,aj}\bar{\tilde{C}}\tilde{\gamma}^{\,i}\tilde{\psi}_j] \,, \tag{2.9a}$$

$$\delta\tilde{\psi}_i = \lambda[C^j\partial_j\tilde{\psi}_i - \tfrac{1}{4}(\partial_j C^j)\tilde{\psi}_i + \tfrac{1}{4}\bar{\tilde{C}}\tilde{\gamma}^{\,j}\tilde{\psi}_j\tilde{\psi}_i + (\partial_i C^j)\tilde{\psi}_j + \tfrac{1}{2}C_{ab}\sigma^{ab}\tilde{\psi}_i + 2i\,e^{-1/4}(D_i + \tfrac{1}{2}\gamma_i S)e^{1/4}C] \,, \tag{2.9b}$$

$$\delta\tilde{\phi}^\mu = \lambda\left\{ C^i\partial_i\tilde{\phi}^\mu + \tfrac{1}{4}(\partial_i C^i)\tilde{\phi}^\mu - \tfrac{1}{4}\bar{\tilde{C}}\tilde{\gamma}^{\,i}\tilde{\psi}_i\tilde{\phi}^\mu + \tfrac{1}{2}C_{ab}\sigma^{ab}\tilde{\phi}^\mu + i\left[\tilde{F}^\mu + \tilde{\gamma}^{\,i}\left[\partial_i x^\mu - \frac{i}{2}\bar{\tilde{\psi}}_i\tilde{\phi}^\mu\right]\tilde{C}\right]\right\} \,, \tag{2.9c}$$

$$\delta\tilde{C} = \lambda[C^i\partial_i\tilde{C} - \tfrac{1}{4}(\partial_i C^i)\tilde{C} + \tfrac{1}{4}\bar{\tilde{C}}\tilde{\gamma}^{\,i}\tilde{\psi}_i\tilde{C} + \tfrac{1}{2}C_{ab}\sigma^{ab}\tilde{C} - \tfrac{1}{2}\tilde{\psi}^{\,i}\bar{\tilde{C}}\tilde{\gamma}_i\tilde{C}] \,, \tag{2.9d}$$

$$\delta E = \lambda\partial_i(C^i E) \,. \tag{2.9e}$$

Notice that by this redefinition "effective" Weyl invariance is introduced from the general coordinate transformations in the second terms in Eqs. (2.9). Note also that, with (2.9a) and (2.9e), the last term in Eq. (2.8) transforms into a total divergence which vanishes at boundaries owing to (2.5a), and hence it is BRS invariant by itself. Clearly these transformations are nilpotent. We are now ready to discuss our gauge choice.

By using the invariances under (2.9b), it is possible to choose the gauge in which $\tilde{\psi}_i = 0$ (Ref. 12). For the zweibein, the general coordinate and local Lorentz transformations allow us to set three components of $\tilde{e}^{\,ai}$ to zero. To impose these gauge conditions, let us decompose the zweibein $\tilde{e}^{\,ai}$ as

$$\tilde{e}^{\,ai} = \begin{bmatrix} A_1 - A_2 & A_4 \\ A_3 & A_1 + A_2 \end{bmatrix} . \tag{2.10}$$

Our gauge conditions are then

$$A_1 = A_3 = A_4 = 0, \quad \tilde{\psi}_i = 0 \,. \tag{2.11}$$

Together with the equation of motion $\det\tilde{e} = +1$, this means that $\tilde{e}^{\,i}_{\;a} = \eta^i_a$.

Following the general prescription in Ref. 17, the gauge-fixing and FP ghost terms are given concisely by

$$\lambda(\mathcal{L}_{\rm GF} + \mathcal{L}_{\rm FP}) = -i\delta^B(\bar{C}_1 A_1 + \bar{C}_3 A_3 + \bar{C}_4 A_4 + \bar{C}^{\,i}_{\,*}\tilde{\psi}_i) \,, \tag{2.12}$$

where the BRS transformation of antighost fields $\bar{C}_a$ and $\bar{C}^{\,i}_{\,*}$, and multiplier fields B_a and $\bar{B}^{\,i}_{\,*}$ are defined by

$$\delta^B\bar{C}_a = \lambda i B_a, \quad \delta^B B_a = 0 \quad (a = 1,3,4) \,, \tag{2.13a}$$

$$\delta^B\bar{C}^{\,i}_{\,*} = \lambda i\bar{B}^{\,i}_{\,*}, \quad \delta^B\bar{B}^{\,i}_{\,*} = 0 \,. \tag{2.13b}$$

As usual, these fields are all real or Majorana. Explicitly (2.12) has the form

$$\mathscr{L}_{\rm GF}+\mathscr{L}_{\rm FP}=B_1A_1+B_3A_3+B_4A_4+\bar{B}^{\,i}_{*}\psi_i+i\bar{C}_1[C^i\partial_iA_1+\tfrac{1}{2}A_2(\partial_0C^0-\partial_1C^1+i\bar{C}\gamma^0\psi_0-i\bar{C}\gamma^1\psi_1)$$
$$-\tfrac{1}{2}A_3(\partial_1C^0-C_{10}+i\bar{C}\gamma^0\psi_1)-\tfrac{1}{2}A_4(\partial_0C^1-C_{10}+i\bar{C}\gamma^1\psi_0)]$$
$$+i\bar{C}_3[C^i\partial_iA_3+\tfrac{1}{2}A_3(\partial_0C^0-\partial_1C^1+i\bar{C}\gamma^0\psi_0-i\bar{C}\gamma^1\psi_1)-A_1(\partial_0C^1-C_{01}+i\bar{C}\gamma^1\psi_0)$$
$$+A_2(\partial_0C^1+C_{10}+i\bar{C}\gamma_1\psi_0)]$$
$$+i\bar{C}_4[C^i\partial_iA_4-\tfrac{1}{2}A_4(\partial_0C^0-\partial_1C^1+i\bar{C}\gamma^0\psi_0-i\bar{C}\gamma^1\psi_1)-A_1(\partial_1C^0+C_{01}+i\bar{C}\gamma^0\psi_1)$$
$$-A_2(\partial_1C^0+C_{10}+i\bar{C}\gamma^0\psi_1)]$$
$$-i\bar{C}^{\,i}_{*}[C^j\partial_j\psi_i+(\partial_iC^j)\psi_j+\tfrac{1}{2}C_{ab}\sigma^{ab}\psi_i-\tfrac{1}{4}(\partial_jC^j)\psi_i+\tfrac{1}{4}\bar{C}\gamma^j\psi_j\psi_i+2i\,e^{-1/4}(D_i+\tfrac{1}{2}\gamma_iS)e^{1/4}C]\ . \tag{2.14}$$

Here and hereafter we drop the tilde on each field for simplicity. By construction, (2.14) is manifestly BRS invariant owing to the nilpotency of the BRS transformation. This implies that the total action is BRS invariant if we impose the boundary conditions (2.5). We also note that the Lagrangian is Hermitian with our Hermiticity assignment.

The Lagrangian is greatly simplified by the shift of auxiliary fields B_a, $\bar{B}^{\,i}_{*}$, and E; we can eliminate all terms containing the fields $A_1\sim A_4$ and ψ_i. This amounts to going to the orthogonal gauge (2.11), and the Lagrangian now takes the form

$$\mathscr{L}=-\frac{1}{2\kappa}\eta^{ij}\partial_ix^\mu\partial_jx_\mu-\frac{i}{2\kappa}\eta^{ij}\bar{\phi}^\mu\gamma_i\partial_j\phi_\mu$$
$$+\frac{1}{2\kappa}F^\mu F_\mu+E(1-A_2{}^2)+B_1A_1+B_3A_3+B_4A_4$$
$$+\bar{B}^{\,i}_{*}\psi_i+i\bar{C}_1(\partial_0C^0-\partial_1C^1)+i\bar{C}_3(\partial_0C^1-C_{01})$$
$$-i\bar{C}_4(\partial_1C^0-C_{01})+\bar{C}^{\,i}_{*}(\partial_i+\tfrac{1}{2}\gamma_iS)C\ , \tag{2.15}$$

where we have also rescaled the fields as $\bar{C}_1\to2\bar{C}_1$ and $\bar{C}^{\,i}_{*}\to\frac{1}{2}\bar{C}^{\,i}_{*}$.

The field equations and the boundary conditions to be imposed on each field are then derived from the variation of the action. The field equations are given by

$$(-\partial_0{}^2+\partial_1{}^2)x^\mu=0,\ \ \gamma^i\partial_i\phi^\mu=0\ , \tag{2.16}$$

$$\partial_0C^0-\partial_1C^1=0,\ \ \partial_0\bar{C}_1-\partial_1\bar{C}_4=0\ , \tag{2.17a}$$

$$\partial_0C^1-C_{01}=0,\ \ \partial_1\bar{C}_1-\partial_0\bar{C}_3=0\ , \tag{2.17b}$$

$$\partial_1C^0-C_{01}=0,\ \ \bar{C}_3=\bar{C}_4\ , \tag{2.17c}$$

$$(\partial_i+\tfrac{1}{2}\gamma_iS)C=0\ , \tag{2.18a}$$

$$\partial_i\bar{C}^{\,i}_{*}=\bar{C}^{\,i}_{*}\gamma_i=0\ , \tag{2.18b}$$

$$A_1=A_2-1=A_3=A_4=\psi_i=0\ , \tag{2.19a}$$

$$F^\mu=E=B_1=B_3=B_4=\bar{B}^{\,i}_{*}=0\ . \tag{2.19b}$$

The constraint in Eq. (2.18b) arises from the variation with respect to S and reduces the number of independent modes. In fact, it leads to

$$\bar{C}^{\,1}_{*}=\bar{C}^{\,0}_{*}\gamma_5\ . \tag{2.20}$$

This is to be expected because, roughly speaking, eventually this ghost field $\bar{C}^{\,i}_{*}$, together with the other ghost C and two modes out of ϕ^μ, falls into the "quartet" representation of Kugo and Ojima[16] and all of them will decouple from the physical subspace, but this would be impossible if both $\bar{C}^{\,1}_{*}$ and $\bar{C}^{\,0}_{*}$ were independent modes. Thus the introduction of this scalar field S is quite essential in our formalism.

The scalar field S in Eq. (2.18a) may be eliminated, yielding the field equation for C:

$$(\gamma_1\partial_0+\gamma_0\partial_1)C=0\ . \tag{2.21}$$

Also Eqs. (2.18b) and (2.20) give the equation for $\bar{C}^{\,0}_{*}$:

$$\bar{C}^{\,0}_{*}(\overleftarrow{\partial}_0+\gamma_5\overleftarrow{\partial}_1)=0\ . \tag{2.22}$$

The boundary conditions for bosonic fields are

$$\partial_1x^\mu=0,\ \ \bar{C}_4=0,\ \ \text{and}\ C^1=0\ , \tag{2.23}$$

at $\sigma=0$ and π. The reason why we do not have conditions on $\bar{C}_1$ is because we must take the variation δC^1 in accordance with (2.5a), i.e., $\delta C^1=0$. For fermionic fields, we can impose two different types of boundary conditions, Ramond (periodic) and Neveu-Schwarz (antiperiodic):

$$\phi^\mu_{(1)}(\tau,0)=\phi^\mu_{(2)}(\tau,0)\ , \tag{2.24a}$$

$$\phi^\mu_{(1)}(\tau,\pi)=\epsilon\phi^\mu_{(2)}(\tau,\pi)\ , \tag{2.24b}$$

where the first and second components of a spinor are denoted by (1) and (2), respectively, and $\epsilon=+1(-1)$ corresponds to the Ramond (Neveu-Schwarz) model.[2]

The boundary conditions on ghosts need careful examination. The requirement of the BRS invariance of the action (2.5b), together with (2.23) and (2.24), tells us that we must impose

$$C_{(1)}(\tau,0)=-C_{(2)}(\tau,0)\ , \tag{2.25a}$$

$$C_{(1)}(\tau,\pi)=-\epsilon C_{(2)}(\tau,\pi)\ . \tag{2.25b}$$

Since the variation of C must be made in accordance with (2.25), we find from the variation of the action (2.15) the boundary conditions on $\bar{C}^{\,1}_{*}$

$$\bar{C}^{\,1}_{*(1)}(\tau,0)=\bar{C}^{\,1}_{*(2)}(\tau,0)\ , \tag{2.26a}$$

$$\bar{C}^{\,1}_{*(1)}(\tau,\pi)=\epsilon\bar{C}^{\,1}_{*(2)}(\tau,\pi)\ , \tag{2.26b}$$

which in turn lead to

$$\bar{C}^{\,0}_{*(1)}(\tau,0)=-\bar{C}^{\,0}_{*(2)}(\tau,0)\ , \tag{2.27a}$$

$$\bar{C}^{0}_{*(1)}(\tau,\pi)=-\epsilon\bar{C}^{0}_{*(2)}(\tau,\pi)\ . \tag{2.27b}$$

The consistency of these boundary conditions with the BRS transformation further requires

$$\partial_0 C^1=0 \ \text{ at } \sigma=0 \text{ and } \pi\ . \tag{2.28}$$

We can now derive the mode expansion of relevant variables. For bosonic variables, they are

$$x^{\mu}(\tau,\sigma)=q_0^{\mu}+\frac{\kappa}{\pi}p_0^{\mu}\tau+\left[\frac{\kappa}{\pi}\right]^{1/2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(a_n^{\mu}e^{-in\tau}+a_n^{\mu\dagger}e^{in\tau})\cos n\sigma\ , \tag{2.29a}$$

$$C^0(\tau,\sigma)=\frac{1}{\sqrt{\pi}}c_0+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}(c_ne^{-in\tau}+c_n^{\dagger}e^{in\tau})\cos n\sigma\ , \tag{2.29b}$$

$$C^1(\tau,\sigma)=-\frac{i}{\sqrt{\pi}}\sum_{n=1}^{\infty}(c_ne^{-in\tau}-c_n^{\dagger}e^{in\tau})\sin n\sigma\ , \tag{2.29c}$$

$$C_{01}(\tau,\sigma)=-\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}n\,(c_ne^{-in\tau}+c_n^{\dagger}e^{in\tau})\sin n\sigma\ , \tag{2.29d}$$

$$\bar{C}_1(\tau,\sigma)=\frac{1}{\sqrt{\pi}}\bar{c}_0+\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}(\bar{c}_ne^{-in\tau}+\bar{c}_n^{\dagger}e^{in\tau})\cos n\sigma\ , \tag{2.29e}$$

$$\bar{C}_3(\tau,\sigma)=\bar{C}_4(\tau,\sigma)=-\frac{i}{\sqrt{\pi}}\sum_{n=1}^{\infty}(\bar{c}_ne^{-in\tau}-\bar{c}_n^{\dagger}e^{in\tau})\sin n\sigma\ . \tag{2.29f}$$

The mode expansion for fermionic fields are different for $\epsilon=\pm1$. For the Ramond model ($\epsilon=+1$), the first components are

$$\phi^{\mu}_{(1)}(\tau,\sigma)=\left[\frac{\kappa}{2\pi}\right]^{1/2}\left[\frac{1}{\sqrt{2}}\Gamma^{\mu}-\Gamma_5\sum_{n=1}^{\infty}(d_n^{\mu}e^{-in(\tau+\sigma)}+d_n^{\mu\dagger}e^{in(\tau+\sigma)})\right]\ , \tag{2.30a}$$

$$C_{(1)}(\tau,\sigma)=\frac{1}{\sqrt{2\pi}}\left[f_0+\sum_{n=1}^{\infty}(f_ne^{-in(\tau-\sigma)}+f_n^{\dagger}e^{in(\tau-\sigma)})\right]\ , \tag{2.30b}$$

$$\bar{C}^{0}_{*(1)}(\tau,\sigma)=\frac{1}{\sqrt{2\pi}}\left[\bar{f}_0+\sum_{n=1}^{\infty}(\bar{f}_ne^{-in(\tau-\sigma)}+\bar{f}_n^{\dagger}e^{in(\tau-\sigma)})\right]\ , \tag{2.30c}$$

and their second components are obtained from (2.30) by

$$\phi^{\mu}_{(2)}(\tau,\sigma)=\phi^{\mu}_{(1)}(\tau,-\sigma)\ , \tag{2.31a}$$

$$C_{(2)}(\tau,\sigma)=-C_{(1)}(\tau,-\sigma)\ , \tag{2.31b}$$

$$\bar{C}^{0}_{*(2)}(\tau,\sigma)=-\bar{C}^{0}_{*(1)}(\tau,-\sigma)\ . \tag{2.31c}$$

In Eq. (2.30a), Γ^{μ} and Γ_5 are the D-dimensional γ matrices and γ_5 analogue, respectively. Because ϕ^{μ} is a Majorana spinor, Γ^{μ} should be taken to anticommute with fermionic variables c_n but commute with other mode operators. On the other hand, Γ_5 is here taken to anticommute only with Γ^{μ} but d_n^{μ} and $d_n^{\mu\dagger}$ to anticommute with fermionic mode variables.

For the Neveu-Schwarz model ($\epsilon=-1$), the mode expansion is

$$\phi^{\mu}_{(1)}(\tau,\sigma)=\left[\frac{\kappa}{2\pi}\right]^{1/2}\sum_{m=1/2}^{\infty}(b_m^{\mu}e^{-im(\tau+\sigma)}+b_m^{\mu\dagger}e^{im(\tau+\sigma)})\ , \tag{2.32a}$$

$$C_{(1)}(\tau,\sigma)=\frac{1}{\sqrt{2\pi}}\sum_{m=1/2}^{\infty}(g_me^{-im(\tau-\sigma)}+g_m^{\dagger}e^{im(\tau-\sigma)})\ , \tag{2.32b}$$

$$\bar{C}^{0}_{*(1)}(\tau,\sigma)=\frac{1}{\sqrt{2\pi}}\sum_{m=1/2}^{\infty}(\bar{g}_me^{im(\tau-\sigma)}+\bar{g}_m^{\dagger}e^{im(\tau-\sigma)})\ , \tag{2.32c}$$

and their second components are again obtained from (2.32) by the rule (2.31). Here the sum is taken over all positive half-integers.

Note that there are constant modes in the expansion of FP ghosts. Since they are Hermitian, so are the constant modes: $c_0=c_0^{\dagger}$, $\bar{c}_0=\bar{c}_0^{\dagger}$, $f_0=f_0^{\dagger}$, and $\bar{f}_0=\bar{f}_0^{\dagger}$. These modes play special roles in the following discussions.

We now proceed to the determination of the canonical commutation relations. The canonical conjugate momentum for each variable is given by

$$\Pi_{x_{\mu}}=\frac{1}{\kappa}\dot{x}^{\mu}\ , \tag{2.33a}$$

$$\Pi_{\phi_{\mu}}=-\frac{i}{2\kappa}\phi^{\mu}\ , \tag{2.33b}$$

$$\Pi_{C^1}=-i\bar{C}_3,\ \ \Pi_{C^0}=-i\bar{C}_1,\ \ \Pi_C=\bar{C}_*^0\ , \tag{2.34a}$$

$$\Pi_{C_{01}}=\Pi_{\bar{C}_a}=\Pi_{\bar{C}_*^i}=\Pi_{B_a}=\Pi_{\bar{B}_*^i}=\Pi_{A_a}=\Pi_{\psi_i}=\Pi_{F_\mu}=\Pi_E=0\ . \tag{2.34b}$$

Equations (2.33b) and (2.34) are the primary constraints. Applying Dirac's method of quantization, we see that all constraints are of second class and that the variables A_a, $\bar{C}_a$, C_{01}, B_a, $\bar{B}_*^i$, ψ_i, $\bar{C}_*^i$, and E can be eliminated from the theory. The Hamiltonian and the commutation relations are then found to be

$$\mathscr{H}=\frac{\kappa}{2}\Pi_{x^\mu}\Pi_{x_\mu}+\frac{1}{2\kappa}\partial_1 x^\mu\partial_1 x_\mu+\frac{i}{2\kappa}\bar{\phi}^\mu\gamma^1\partial_1\phi_\mu-\Pi_{C^0}\partial_1 C^1-\Pi_{C^1}\partial_1 C^0-\Pi_C\gamma_5\partial^1 C\ , \tag{2.35}$$

$$[\pi_{x^\mu}(\tau,\sigma),x^\nu(\tau,\sigma')]=-i\eta_\mu{}^\nu\delta(\sigma-\sigma'),\ \ \{\phi_\mu(\tau,\sigma),\phi^\nu(\tau,\sigma')\}=\kappa\eta_\mu{}^\nu\delta(\sigma-\sigma')\ ,$$
$$\{\Pi_{C^a}(\tau,\sigma),C^b(\tau,\sigma')\}=-i\delta^{ab}\delta(\sigma-\sigma'),\ \ [\Pi_C(\tau,\sigma),C(\tau,\sigma')]=-i\delta(\sigma-\sigma')\ , \tag{2.36}$$

and other (anti)commutators vanish. In deriving (2.35), use has been made of Eq. (2.20).

Substituting the mode expansion operators (2.29)–(2.32) into (2.36), we obtain the canonical commutation relations of mode variables:

$$[p_0^\mu,q_0^\nu]=-i\eta^{\mu\nu},\ \ [a_n^\mu,a_m^{\nu\dagger}]=\eta^{\mu\nu}\delta_{mn},\ \ \{b_n^\mu,b_m^{\nu\dagger}\}=\eta^{\mu\nu}\delta_{nm},\ \ \{d_n^\mu,d_m^{\nu\dagger}\}=\eta^{\mu\nu}\delta_{nm}\ ,$$
$$\{\bar{c}_0,c_0\}=1,\ \ [\bar{f}_0,f_0]=-i,\ \ \{\bar{c}_n^\dagger,c_m\}=\{\bar{c}_n,c_m^\dagger\}=\delta_{nm}\ , \tag{2.37}$$
$$[\bar{f}_n,f_m^\dagger]=[\bar{f}_n^\dagger,f_m]=-i\delta_{nm},\ \ [\bar{g}_n,g_m^\dagger]=[\bar{g}_n^\dagger,g_m]=-i\delta_{nm}\ ,$$

and other (anti)commutators vanish. We can also derive the Hamiltonian in terms of mode variables. For the Ramond model, it is

$$H=\frac{\kappa}{2\pi}(p_0^\mu)^2+\sum_{n=1}^\infty n a_n^{\mu\dagger}a_{n\mu}+\sum_{n=1}^\infty n d_n^{\mu\dagger}d_{n\mu}+\sum_{n=1}^\infty n(c_n^\dagger\bar{c}_n+\bar{c}_n^\dagger c_n)+i\sum_{n=1}^\infty n(f_n^\dagger\bar{f}_n-\bar{f}_n^\dagger f_n)\ , \tag{2.38}$$

and for the Neveu-Schwarz model,

$$H=\frac{\kappa}{2\pi}(p_0^\mu)^2+\sum_{n=1}^\infty n a_n^{\mu\dagger}a_{n\mu}+\sum_{m=1/2}^\infty m b_m^{\mu\dagger}b_{m\mu}+\sum_{n=1}^\infty n(c_n^\dagger\bar{c}_n+\bar{c}_n^\dagger c_n)+i\sum_{m=1/2}^\infty m(g_m^\dagger\bar{g}_m-\bar{g}_m^\dagger g_m)\ . \tag{2.39}$$

In the Neveu-Schwarz model, one can define the conserved G parity

$$G=(-1)^{\sum b_m^\dagger b_m+i\sum(g_m^\dagger\bar{g}_m-\bar{g}_m^\dagger g_m)-1}\ , \tag{2.40}$$

which will be preserved by the interactions.[2,6]

III. CRITICAL DIMENSION AND PHYSICAL SUBSPACE

In this section we wish to first exhibit the interplay between the nilpotency of the BRS charge $Q_B{}^2=0$ and the critical dimension, then discuss the Fock space in our superstring theories and specify the physical subspace.

The BRS charge Q_B is found from the conserved Noether current $J_{\rm BRS}^i$:

$$\begin{aligned}Q_B&=\int_0^\pi d\sigma J_{\rm BRS}^0\\ &=\int_0^\pi d\sigma\left[-\frac{1}{2\kappa}C^0(\partial_0 x^\mu\partial_0 x_\mu+\partial_1 x^\mu\partial_1 x_\mu)-\frac{1}{2\kappa}C^1(\partial_0 x^\mu\partial_1 x_\mu+\partial_1 x^\mu\partial_0 x_\mu)-\frac{i}{2\kappa}(C^1\bar{\phi}^\mu\gamma^0\partial_1\phi_\mu+C^0\bar{\phi}^\mu\gamma_1\partial_1\phi_\mu)\right.\\ &\qquad+\frac{1}{\kappa}\bar{C}\gamma^i\gamma_0\phi^\mu\partial_i x_\mu+i\bar{C}_1(C^i\partial_i C^0+i\bar{C}\gamma^0 C)+i\bar{C}_3(C^i\partial_i C^1+i\bar{C}\gamma^1 C)\\ &\qquad\left.-\bar{C}_*^0(C^i\partial_i C+\tfrac{1}{2}C_{01}\gamma_5 C-\tfrac{1}{4}\partial_i C^i C)\right]\ .\end{aligned} \tag{3.1}$$

This can be expressed in terms of the mode operators. We find, for the Ramond model,

$$Q_B=c_0 L+\bar{c}_0 M+\frac{1}{\sqrt{\pi}}f_0 F+\frac{1}{\sqrt{\pi}}\bar{f}_0\bar{F}+\left[\frac{\kappa}{\pi}\right]^{1/2}\tilde{Q}_B\ , \tag{3.2}$$

where

$$L=-\frac{1}{\sqrt{\pi}}c_0\left[\frac{\kappa}{2\pi}(p_{0\mu})^2+\tfrac{1}{2}\sum_{n=1}^{\infty}n(a_n^{\mu\dagger}a_{n\mu}+a_n^{\mu}a_{n\mu}^{\dagger})+\tfrac{1}{2}\sum_{n=1}^{\infty}n(d_n^{\mu\dagger}d_{n\mu}-d_n^{\mu}d_{n\mu}^{\dagger})\right.$$
$$\left.+\sum_{n=1}^{\infty}n(c_n^{\dagger}\bar{c}_n-c_n\bar{c}_n^{\dagger}+i\bar{f}_nf_n^{\dagger}-i\bar{f}_n^{\dagger}f_n)\right], \quad (3.3)$$

$$M=\frac{2}{\sqrt{\pi}}\left[\sum_{n=1}^{\infty}nc_n^{\dagger}c_n+\tfrac{1}{2}f_0{}^2+\sum_{n=1}^{\infty}f_n^{\dagger}f_n\right], \quad (3.4)$$

$$F=\left[\frac{\kappa}{2\pi}\right]^{1/2}p_{0\mu}\Gamma^{\mu}+i\Gamma_5\sum_{n=1}^{\infty}\sqrt{n}(a_n^{\mu}d_{n\mu}^{\dagger}-a_n^{\mu\dagger}d_{n\mu})+2\sum_{n=1}^{\infty}(\bar{c}_nf_n^{\dagger}+\bar{c}_n^{\dagger}f_n)+\frac{i}{2}\sum_{n=1}^{\infty}n(c_n^{\dagger}\bar{f}_n-c_n\bar{f}_n^{\dagger}), \quad (3.5)$$

$$\bar{F}=\frac{3i}{2}\sum_{n=1}^{\infty}n(c_n^{\dagger}f_n-c_nf_n^{\dagger}), \quad (3.6)$$

$$\tilde{Q}_B=-\frac{i}{\sqrt{\pi}}p_{0\mu}\left[\sum_{n=1}^{\infty}\sqrt{n}(c_na_n^{\mu\dagger}-c_n^{\dagger}a_n^{\mu})-i\Gamma_5\sum_{n=1}^{\infty}(f_nd_n^{\mu\dagger}+f_n^{\dagger}d_n^{\mu})\right]$$
$$-\frac{1}{2\sqrt{\kappa}}\sum_{n,m}\{2[(n+m)m]^{1/2}(c_na_{n+m}^{\mu\dagger}a_{m\mu}+c_n^{\dagger}a_m^{\mu\dagger}a_{n+m\mu})-\sqrt{nm}(c_{m+n}a_m^{\mu\dagger}a_{n\mu}^{\dagger}+c_{m+n}^{\dagger}a_m^{\mu}a_{n\mu})\}$$
$$+\frac{1}{\sqrt{\kappa}}\sum_{m,n}[m(\bar{c}_{n+m}^{\dagger}c_nc_m-c_n^{\dagger}c_m^{\dagger}\bar{c}_{n+m})+(n+2m)(\bar{c}_n^{\dagger}c_m^{\dagger}c_{n+m}+c_{n+m}^{\dagger}c_m\bar{c}_n)]$$
$$+\frac{1}{\sqrt{2\kappa}}\Gamma_{\mu}\left[i\sum_{n=1}^{\infty}\sqrt{n}(f_na_n^{\mu\dagger}-f_n^{\dagger}a_n^{\mu})-\tfrac{1}{2}\Gamma_5\sum_{n=1}^{\infty}n(c_nd_n^{\mu\dagger}-c_n^{\dagger}d_n^{\mu})\right]$$
$$-\frac{1}{2\sqrt{\kappa}}\sum_{n,m}[n(c_{m+n}^{\dagger}d_m^{\mu}d_{n\mu}-c_{m+n}d_m^{\mu\dagger}d_{n\mu}^{\dagger})+(n+2m)(c_nd_{m+n}^{\mu\dagger}d_{m\mu}+c_n^{\dagger}d_m^{\mu\dagger}d_{n+m\mu})]$$
$$+\frac{i}{\sqrt{\kappa}}\Gamma_5\sum_{n,m}[\sqrt{n+m}(f_n^{\dagger}a_{n+m}^{\mu}d_{m\mu}^{\dagger}-f_na_{n+m}^{\mu\dagger}d_{m\mu})$$
$$+\sqrt{m}(f_na_m^{\mu}d_{m+n\mu}^{\dagger}+f_{m+n}^{\dagger}a_m^{\mu}d_{n\mu}-f_n^{\dagger}a_m^{\mu\dagger}d_{m+n\mu}-f_{m+n}a_m^{\mu\dagger}d_{n\mu}^{\dagger})]$$
$$+\frac{i}{\sqrt{\kappa}}\sum_{n,m}\left[(m+\tfrac{3}{2}n)(c_n^{\dagger}\bar{f}_m^{\dagger}f_{n+m}-c_n\bar{f}_mf_{n+m}^{\dagger})+\tfrac{1}{2}(n+3m)(c_{n+m}^{\dagger}\bar{f}_nf_m-c_{n+m}\bar{f}_n^{\dagger}f_m^{\dagger})\right.$$
$$\left.+\left[m-\frac{n}{2}\right](c_n\bar{f}_{m+n}^{\dagger}f_m-c_n^{\dagger}\bar{f}_{m+n}f_m^{\dagger})\right]$$
$$+\frac{1}{\sqrt{\kappa}}\sum_{n,m}(2\bar{c}_nf_{m+n}^{\dagger}f_m+2\bar{c}_n^{\dagger}f_m^{\dagger}f_{n+m}+\bar{c}_{m+n}f_m^{\dagger}f_n^{\dagger}+\bar{c}_{m+n}^{\dagger}f_mf_n), \quad (3.7)$$

where all the sums are taken over positive integers. We note that F is a generalized Dirac operator in the sense that

$$F^2=H, \quad (3.8)$$

which can be proved by using the commutation relations (2.37). Also one finds

$$\bar{F}^2=0. \quad (3.9)$$

The BRS charge for the Neveu-Schwarz model is given by

$$Q_B=c_0L+\bar{c}_0M+\left[\frac{\kappa}{\pi}\right]^{1/2}\tilde{Q}_B, \quad (3.10)$$

where

$$L=-\frac{1}{\sqrt{\pi}}\left[\frac{1}{2\pi}(p_{0\mu})^2+\tfrac{1}{2}\sum_{n=1}^{\infty}n(a_n^{\mu\dagger}a_{n\mu}+a_n^{\mu}a_{n\mu}^{\dagger})+\tfrac{1}{2}\sum_{m=1/2}^{\infty}m(b_m^{\mu\dagger}b_{m\mu}-b_m^{\mu}b_{m\mu}^{\dagger})\right.$$
$$\left.+\sum_{n=1}^{\infty}n(c_n^{\dagger}\bar{c}_n-c_n\bar{c}_n^{\dagger})+i\sum_{m=1/2}^{\infty}m(\bar{g}_mg_m^{\dagger}-\bar{g}_m^{\dagger}g_m)\right], \quad (3.11)$$

$$M=\frac{2}{\sqrt{\pi}}\left[\sum_{n=1}^{\infty} n c_n^\dagger c_n+\sum_{m=1/2}^{\infty} g_m^\dagger g_m\right], \tag{3.12}$$

$$\begin{aligned}
\tilde{Q}_B={}&-\frac{i}{\sqrt{\pi}}p_{0\mu}\left[\sum_{n=1}^{\infty}\sqrt{n}\,(c_n a_n^{\mu\dagger}-c_n^\dagger a_n^\mu)+i\sum_{m=1/2}^{\infty}(g_m b_m^{\mu\dagger}+g_m^\dagger b_m^\mu)\right]\\
&-\frac{1}{2\sqrt{\kappa}}\sum_{n,m}\{2[(n+m)m]^{1/2}(c_n a_{n+m}^{\mu\dagger}a_{m\mu}+c_n^\dagger a_m^{\mu\dagger}a_{n+m\mu})-\sqrt{nm}\,(c_{m+n}a_m^{\mu\dagger}a_{n\mu}^\dagger+c_{m+n}^\dagger a_m^\mu a_{n\mu})\}\\
&-\frac{1}{2\sqrt{\kappa}}\sum_{n,m}[(n+2m)(c_n^\dagger b_m^{\mu\dagger}b_{n+m\mu}+c_n b_{m+n}^{\mu\dagger}b_{m\mu})-n\,(c_{m+n}b_m^{\mu\dagger}b_{n\mu}^\dagger-c_{m+n}^\dagger b_m^\mu b_{n\mu})]\\
&+\frac{1}{\sqrt{\kappa}}\sum_{m,n}[m\,(\bar{c}^{\,\dagger}_{n+m}c_n c_m-c_n^\dagger c_m^\dagger \bar{c}_{n+m})+(n+2m)(\bar{c}^{\,\dagger}_{n}c_m^\dagger c_{n+m}+c_{n+m}^\dagger c_m \bar{c}_n)]\\
&-\frac{i}{\sqrt{\kappa}}\sum_{m,n}[-\sqrt{n+m}\,(g_n b_m^\mu a_{n+m\mu}^\dagger-g_n^\dagger b_m^{\mu\dagger}a_{n+m\mu})\\
&\qquad\qquad+\sqrt{n}\,(g_m b_{n+m}^{\mu\dagger}a_{n\mu}-g_m^\dagger b_{n+m}^\mu a_{n\mu}^\dagger+g_{m+n}^\dagger b_m^\mu a_{n\mu}-g_{m+n}b_m^{\mu\dagger}a_{n\mu}^\dagger)]\\
&+\frac{i}{\sqrt{\kappa}}\sum_{n,m}\Bigg[-(m+\tfrac{3}{2}n)(c_n g_{n+m}^\dagger \bar{g}_m-c_n^\dagger \bar{g}^{\,\dagger}_{m}g_{n+m})+\tfrac{1}{2}(n+3m)(c_{n+m}^\dagger \bar{g}_n g_m-c_{n+m}\bar{g}^{\,\dagger}_{n}g_m^\dagger)\\
&\qquad\qquad+\left(m-\frac{n}{2}\right)(c_n \bar{g}^{\,\dagger}_{m+n}g_m-c_n^\dagger g_m^\dagger \bar{g}_{m+n})\Bigg]\\
&+\frac{1}{\sqrt{\kappa}}\sum_{n,m}(2\bar{c}_n g_m g_{n+m}^\dagger+2\bar{c}^{\,\dagger}_{n}g_m^\dagger g_{n+m}+\bar{c}_{m+n}g_m^\dagger g_n^\dagger+\bar{c}^{\,\dagger}_{m+n}g_m g_n),
\end{aligned} \tag{3.13}$$

where the sums should be taken over positive integers or half-integers such that the indices on a_n^μ and c_n are integers and those on b_m^μ and g_m are half-integers.

The BRS charges formally derived above from the Noether current contain divergences in L because of the ordering problem. It can be made well defined, however, by taking the normal ordering of L which is just $-H/\sqrt{\pi}$. We thus redefine L by

$$L=-\frac{1}{\sqrt{\pi}}(H-\alpha_0), \tag{3.14}$$

where α_0 is a suitable constant.

Although we defined the BRS transformation to be nilpotent at the start, the BRS charges Q_B in their well-defined forms do not necessarily satisfy the nilpotency condition

$$Q_B{}^2=0, \tag{3.15}$$

because of the ordering problem. In fact, using (3.14) in the definition (3.2) of Q_B, we find for the Ramond model

$$Q_B{}^2=\frac{1}{\pi}\left[\left[\frac{D-10}{8}\right]\sum_{n=1}^{\infty}n^3 c_n^\dagger c_n+2\alpha_0\sum_{n=1}^{\infty}n c_n^\dagger c_n+\left[\frac{D-10}{2}\right]\sum_{n=1}^{\infty}n^2 f_n^\dagger f_n+2\alpha_0\left[\sum_{n=1}^{\infty}f_n^\dagger f_n+\frac{1}{2}f_0{}^2\right]\right]. \tag{3.16}$$

For the Neveu-Schwarz model (3.10), the same procedure gives

$$\begin{aligned}
Q_B{}^2=\frac{1}{\pi}\Bigg[&\left[\frac{D-10}{8}\right]\sum_{n=1}^{\infty}n^3 c_n^\dagger c_n-\left[\frac{D-2}{8}-2\alpha_0\right]\sum_{n=1}^{\infty}n c_n^\dagger c_n\\
&+\left[\frac{D-10}{2}\right]\sum_{m=1/2}^{\infty}m^2 g_m^\dagger g_m-\left[\frac{D-2}{8}-2\alpha_0\right]\sum_{m=1/2}^{\infty}g_m^\dagger g_m\Bigg].
\end{aligned} \tag{3.17}$$

Equation (3.16) shows that the nilpotency of the BRS transformation holds if and only if

$$D=10,\ \alpha_0=0\ , \tag{3.18}$$

for the Ramond model, while Eq. (3.17) tells us that this is true for the Neveu-Schwarz model if

$$D=10,\ \alpha_0=\tfrac{1}{2}\ , \tag{3.19}$$

in agreement with the well-known result.[2,4] We shall see that these intercepts are not directly related with the masses of our ground states in the superstrings, but Eq. (3.19) implies that the ground state in the Neveu-Schwarz model is a tachyon, which can be eliminated by restricting the theory to the even G-parity sector.[6]

It may be interesting to note that (3.17) has a form quite similar to the commutator $[M^{i-},M^{j-}]$ in the light-cone gauge formalism[4] in which the no-ghost theorem is manifest. There the Lorentz covariance of the theory requires its vanishing, leading to the same conclusion. The author could not find a similar computation of the commutator of angular moment in the light-cone gauge in the Ramond model, but our results suggest that it has a similar form and the theory is Lorentz covariant only for (3.18), which was deduced by another reasoning by Schwarz.[18,3]

Since the D-dimensional γ matrices are necessarily introduced as the zero mode of the $\phi^\mu(\tau,\sigma)$ in (2.30a) in the Ramond model, it is clear that the states in this model involve spinors, and hence this model describes fermions. The ground state is indeed given by[2,6]

$$|0,p\rangle u(p)\ , \tag{3.20}$$

where $u(p)$ is a commuting spinor satisfying the Dirac equation

$$p^\mu\Gamma_\mu u(p)=0\ , \tag{3.21}$$

and $|0,p\rangle$ is, as usual, characterized by

$$\begin{aligned} &p_0^\mu|0,p\rangle=p^\mu|0,p\rangle\ ,\\ &a_n^\mu|0,p\rangle=c_n|0,p\rangle=\bar{c}_n|0,p\rangle=0 \quad \text{for } n\geq 1\ ,\\ &d_n^\mu|0,p\rangle=f_n|0,p\rangle=\bar{f}_n|0,p\rangle=0 \quad \text{for } n\geq 1\ . \end{aligned} \tag{3.22}$$

As for the zero modes f_0 and $\bar{f}_0$, we choose $\bar{f}_0|0,p\rangle=0$. On the other hand, the ground state of the Neveu-Schwarz model

$$|0,p\rangle\ , \tag{3.23}$$

is defined by

$$\begin{aligned} &p_0^\mu|0,p\rangle=p^\mu|0,p\rangle\ ,\\ &a_n^\mu|0,p\rangle=c_n|0,p\rangle=\bar{c}_n|0,p\rangle=0 \quad \text{for } n\geq 1\ ,\\ &b_m^\mu|0,p\rangle=g_m|0,p\rangle=\bar{g}_m|0,p\rangle=0 \quad \text{for } m\geq \tfrac{1}{2}\ . \end{aligned} \tag{3.24}$$

In both models, the constant modes c_0 and $\bar{c}_0$ require a special consideration. As discussed in detail in Ref. 10, they have a doublet representation; that is, the state $|+\rangle$ annihilated by c_0

$$c_0|+\rangle=0\ , \tag{3.25}$$

gives another state $|-\rangle\equiv\bar{c}_0|+\rangle$ which is annihilated by $\bar{c}_0$, because of the anticommutation relation $\{c_0,\bar{c}_0\}=1$. The total Fock space is then spanned by the direct products of these doubly degenerate states and those constructed from (3.20) or (3.23) by the standard procedure of applying creation operators.

The fact that the ghost constant modes have a doublet representation causes a slight complication in defining the inner product. It should be defined by using the metric operator $\eta=c_0+\bar{c}_0$ as[10]

$$_\pm\langle\phi|\eta|\phi\rangle_\pm={}_\pm\langle\phi|\phi\rangle_\mp=\langle\phi|\phi\rangle\ , \tag{3.26}$$

where we have used the notation $|\phi\rangle_\pm=|\phi\rangle\times|\pm\rangle$ with $|\phi\rangle$ being the state other than the ghost constant mode. Also one can construct the ghost number charge Q_c, which has fractional eigenvalues on any state in the Fock space because of the ghost constant modes.

It is well known[6] that our theory describes the superstring if we restrict the Ramond model to the Majorana-Weyl sector and the Neveu-Schwarz model to the even G-parity sector, the fermions being contained in the Ramond model and the bosons in the Neveu-Schwarz model. The ground states of the theory are the massless fermion coming from the Ramond model and the massless spin-1 boson from the Neveu-Schwarz model. There is no tachyon in this theory, as we mentioned previously. Still, the no-ghost theorem is true without this restriction.

Finally, our definition of the physical subspace is, as usual,[10] given by

$$Q_B|\text{phys}\rangle=0\ , \tag{3.27}$$

which is equivalent, for the Ramond model, to

$$L\,|\text{phys}\rangle=0\ , \tag{3.28}$$

$$M\,|\text{phys}\rangle=0\ , \tag{3.29}$$

$$F\,|\text{phys}\rangle=0\ , \tag{3.30}$$

$$\bar{F}\,|\text{phys}\rangle=0\ , \tag{3.31}$$

$$\tilde{Q}_B\,|\text{phys}\rangle=0\ , \tag{3.32}$$

and, for the Neveu-Schwarz model, to

$$L\,|\text{phys}\rangle=0\ , \tag{3.33}$$

$$M\,|\text{phys}\rangle=0\ , \tag{3.34}$$

$$\tilde{Q}_B\,|\text{phys}\rangle=0\ . \tag{3.35}$$

Because Q_B is nilpotent, one can show in both models that $\tilde{Q}_B$ is also nilpotent on the physical subspace specified by (3.27).

We note that Eq. (3.30) is a generalization of the Dirac equation, as can be seen from (3.8). Thus Eq. (3.28) is not an independent condition but can be obtained from Eq. (3.30). This is the mass-shell condition whereas (3.29) and (3.31) restrict the excitation of ghosts. In the Neveu-Schwarz model, Eq. (3.33) is the mass-shell condition and Eq. (3.34) is the restriction on ghosts. In both models, $\tilde{Q}_B$ contains terms of generalized Virasoro operators multiplied by the ghost modes, and hence Eq. (3.32) or (3.35) is an analogue of the infinite number of generalized Virasoro conditions[19]

$$L_n \mid \text{phys}\rangle = 0 \quad \text{for } n \geq 1 , \tag{3.36a}$$

$$F_n \mid \text{phys}\rangle = 0 \quad \text{or } G_m \mid \text{phys}\rangle = 0 , \tag{3.36b}$$

in the usual formalism. Here we have a simple condition (3.27) owing to the presence of ghosts. In the next section we briefly discuss that (3.27) is sufficient to eliminate all ghosts.

IV. NO-GHOST THEOREM

In this section we wish to discuss the no-ghost theorem in our formalism. We restrict ourselves to the Neveu-Schwarz model because the discussion on the Ramond model should go through in much the same way by using the physical particle operators similar to those in the Neveu-Schwarz model.[18] The machinery to prove this theorem has been well developed for the bosonic case by Kato and Ogawa.[10] In fact our discussion up to here is quite similar to theirs with some modifications. Here one also easily recognizes the similarity. Therefore we only briefly outline how its proof proceeds and refer the reader to Ref. 10 for a more detailed discussion.

We choose a Lorentz frame in which

$$p_i = 0 \quad \text{for } i = 1, \ldots, D-2 , \tag{4.1}$$

and define the light-cone coordinate

$$u_{\pm} = (1/\sqrt{2})(u_0 \pm u_{D-1}) .$$

Let

$$k^+ = k_i = 0 \quad \text{for } i = 1, \ldots, D-2 , \qquad k^- = 1/p^+ , \tag{4.2}$$

so that $k_\mu p^\mu = -1$.

We can construct the subspace $\mathscr{V}_L$ as a Fock space spanned by redefined mode operators

$$\psi_n \to e^{i(\pi n/\kappa)k^- q_0^+}\psi_n, \quad \psi_n^\dagger \to e^{-i(\pi n/\kappa)k^- q_0^+}\psi_n^\dagger , \tag{4.3}$$

where ψ_n denotes a_n^μ, b_n^μ, c_n, $\bar{c}_n$, g_n, and $\bar{g}_n$. L, M, and $\tilde{Q}_B$ remain unchanged by this redefinition. Since these new variables (4.3) commute with L, any states in $\mathscr{V}_L$ automatically satisfy (3.33) as far as the vacuum satisfies it.

By making the rescaling

$$\kappa \to \frac{1}{\beta^2}\kappa, \quad p_0^- \to \beta^2 p_0^- , \quad q_0^+ \to \frac{1}{\beta^2} q_0^+ , \tag{4.4}$$

Q_B is put into the form

$$\tilde{Q}_B = A + \beta B + \beta^2 C . \tag{4.5}$$

The lengthy explicit form of the operators A, B, and C is not given here since it has a similar form to that given in Ref. 10 with additional fermionic terms and may be obtained easily from Eq. (3.13). The only point to be noticed is that A has the same bilinear form as the usual asymptotic form of the BRS charge in ordinary gauge theories.[10,16] Owing to the nilpotency of Q_B, we have

$$\tilde{Q}_B(\beta)^2 = -\beta^2 \frac{\pi}{\kappa} ML . \tag{4.6}$$

Combining this with (4.5) and comparing terms in the same order in β, we get

$$\begin{aligned} &A^2 = 0, \quad AB + BA = 0 , \\ &AC + CA + B^2 = -\frac{\pi}{\kappa} ML , \\ &BC + CB = 0, \quad C^2 = 0 . \end{aligned} \tag{4.7}$$

Now one can prove, by the same method as Kato and Ogawa,[10] that any state $\mid \psi\rangle$ satisfying $A \mid \psi\rangle = 0$ can be written as

$$\mid \psi \rangle = P^{(0)} \mid \psi \rangle + A \mid \phi \rangle , \tag{4.8}$$

where $P^{(0)}$ is the projection operator into the subspace generated by $a_n^{i\dagger}$ and $b_m^{i\dagger}$:

$$P^{(0)} = \sum_{m,n} \frac{1}{m!n!} a^\dagger_{k_1 i_1} \cdots a^\dagger_{k_m i_m} b^\dagger_{l_1 j_1} \cdots b^\dagger_{l_n j_n} \mid 0 \rangle \times \langle 0 \mid b_{l_n j_n} \cdots b_{l_1 j_1} a_{k_m i_m} \cdots a_{k_1 i_1} . \tag{4.9}$$

Using these relations and assuming everything may be expanded in β, one can then prove by mathematical induction that any state $\mid \psi(\beta)\rangle$ in $\mathscr{V}_L$ satisfying $\tilde{Q}_B(\beta) \mid \psi(\beta)\rangle = 0$ can be written as[10]

$$\mid \psi(\beta)\rangle = P(\beta) \mid \psi(\beta)\rangle + \tilde{Q}_B(\beta) \mid J(\beta)\rangle , \tag{4.10}$$

where $P(\beta)$ is the projection operator onto the transverse sector:

$$P(\beta) = \sum_{m,n} \frac{1}{m!n!} A^\dagger_{k_1 i_1} \cdots A^\dagger_{k_m i_m} B^\dagger_{l_1 j_1} \cdots B^\dagger_{l_n j_n} \mid 0 \rangle \times \langle 0 \mid B_{l_n j_n} \cdots B_{l_1 j_1} A_{k_m i_m} \cdots A_{k_1 i_1} , \tag{4.11}$$

with A_{ni} and B_{mj} being the transverse operators constructed by Brower and Friedman and by Schwarz[3] and suitably rescaled by β. Equation (4.8) is the $n=1$ case of Eq. (4.10) in the expansion in β.

Now that we have found the complete structure (4.10) of the physical subspace satisfying (3.33) and (3.35), we can prove the no-ghost theorem by simply putting $\beta=1$ in Eq. (4.10) and taking its norm

$$\langle \psi \mid \psi \rangle = \langle \psi \mid P \mid \psi \rangle \geq 0 , \tag{4.12}$$

where the first equality follows from the fact that

$$\tilde{Q}_B P = 0, \quad P\tilde{Q}_B = 0 , \tag{4.13}$$

because the transverse operators (anti)commute with $\tilde{Q}_B$. The last inequality is due to the positive definiteness of the transverse state space.[3] This completes the proof of the no-ghost theorem.

V. CONCLUSIONS

We have thus performed the covariant quantization of the superstring in the NSR formalism in the light of the BRS invariance of the theory. The procedure is quite similar to that of bosonic strings due to Kato and Ogawa.[10] The critical dimension $D=10$ follows from the nilpotency of the BRS charge in spite of the fact that we started with nilpotent BRS transformation.

Although we have performed quantization by fixing the Weyl transformation from the outset, it would be an interesting problem to examine whether the theory can be consistent outside the critical dimension if we keep the conformal mode, as suggested by Polyakov.[20] We believe that our operator formalism is more suitable to investigate this problem than the path-integral method.

A natural problem to be discussed is the extension of our formalism to the covariant second quantization of the superstring[21] along the lines of Siegel's work on bosonic strings,[8] which is based on the formulation of Kato and Ogawa.[10] With such a covariant formalism, it would be easier to reconsider the compactification of the superstrings to four dimensions.[5]

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[13]Our notations and conventions are as follows. The early (late) latin letters denote the two-dimensional tangent (curved) parameter-space indices, while bosonic and fermionic coordinates are $x^\mu(\xi)$ and $\phi^\mu(\xi)$. In Eq. (1.1), $e \equiv \det e_i{}^a$, and $e_i{}^a(\xi)$, and $\psi^i(\xi)$ are the zweibein and Rarita-Schwinger fields in two dimensions, respectively. The flat values of the two- and D-dimensional metrics are $\eta_{ij}=\eta_{ab}=\mathrm{diag}(-+)$ and $\eta_{\mu\nu}=\mathrm{diag}(-+\cdots+)$, with $\xi^0=\tau$ and $\xi^1=\sigma$ being parameters on the world sheet $(-\infty<\tau<\infty,\ 0\le\sigma\le\pi)$. The two-dimensional γ matrices are defined as $\gamma^0=i\sigma^2$, $\gamma^1=\sigma^1$, $\gamma^5=\sigma^3$, and $\sigma^{ab}=(\frac{1}{4})(\gamma^a\gamma^b-\gamma^b\gamma^a)$, and $\bar{\psi}=\psi^T\gamma^0$.

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