Grassmann oscillator

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(Received 18 October 1985)

The quantum mechanics of the oscillator with anticommuting coordinates is discussed. The relation of this oscillator to the super Poincaré group is described. In the analysis one is led to introduce the Grassmann extension of Hermite and of Bargmann-Wigner functions.

I. INTRODUCTION

The familiar annihilation and creation operators of a fermionic field theory are separately the generators of a Grassmann algebra. This algebra is also encountered in the various realizations of supersymmetry by fields and strings and in the description of superspace.[1] DeWitt has given a general description of supermanifolds with some applications to simple quantum-mechanical systems.[2] The use of simple supersymmetric quantum-mechanical models also permits elementary proofs of the geometric index theorems.[3]

There is a very general investigation in which Schwinger showed long ago how bosonic and fermionic systems may be uniformly treated on the basis of an action principle.[4] Starting from this action principle we shall here give a detailed discussion of the fermionic oscillator, first as an illustration of Grassmannian quantum mechanics and second as a useful tool for analyzing the super Poincaré group in d dimensions.[5] Our treatment will closely parallel the corresponding treatment of the familiar bosonic oscillator and will depend on the use of the Grassmannian generalization of the usual Hermite functions. In the discussion of the super Poincaré group, we shall encounter the Grassmannian generalization of the Bargmann-Wigner multispinors.

II. THE SCHWINGER ACTION PRINCIPLE

Grassmannian quantum mechanics may be discussed in just the same way as the usual quantum mechanics. We shall base our treatment on the Schwinger action principle:

$$\delta\langle q_2't_2 \,|\, q_1't_1\rangle = \frac{i}{\hbar}\langle q_2't_2 \,|\, \delta W_{21} \,|\, q_1't_1\rangle \,, \tag{2.1}$$

where $\langle q_2't_2 \,|\, q_1't_1\rangle$ is the transition amplitude between the initial state $|q_1't_1\rangle$ and final state $|q_2't_2\rangle$ and where the action operator is

$$W=\int_{t_1}^{t_2} L(q,\dot{q},t)dt \,. \tag{2.2}$$

This action principle relates variations of the states on the left-hand side to variations of the operators on the right-hand side. If $|q_1't_1\rangle$ and $|q_2't_2\rangle$ are not varied, one has the operator principle of least action

$$\delta\int L\,dt=0 \,. \tag{2.3}$$

If the operator equations of motion are satisfied and only the final state is varied, one has

$$\delta\langle 2' \,|\, 1'\rangle = \frac{i}{\hbar}\langle 2' \,|\, -\delta t_2 H+\delta q_2 p_2 \,|\, 1'\rangle \,. \tag{2.4}$$

From (2.4), one finds the Schrödinger conditions on the states

$$H|\;\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial t}|\;\rangle \,, \tag{2.5a}$$

$$p|\;\rangle = \frac{\hbar}{i}\frac{\partial}{\partial q}|\;\rangle \,, \tag{2.5b}$$

and the Schrödinger equation

$$\left[H\left[\frac{\hbar}{i}\frac{\partial}{\partial q},q\right]+\frac{\hbar}{i}\frac{\partial}{\partial t}\right]|\;\rangle=0 \,. \tag{2.5c}$$

From (2.4) and the right-hand side of (2.1), one finds the operator equations

$$H=-\frac{\partial W}{\partial t} \,, \tag{2.6a}$$

$$p=\frac{\partial W}{\partial q} \,, \tag{2.6b}$$

and the operator Hamilton-Jacobi equation

$$H\left[\frac{\partial W}{\partial q},q\right]+\frac{\partial W}{\partial t}=0 \,. \tag{2.6c}$$

The integration of the action principle leads to the Green's function

$$\langle q_2't_2 \,|\, q_1't_1\rangle = \exp\left[\frac{i}{\hbar}\mathscr{W}_{21}\right] , \tag{2.7}$$

where $\mathscr{W}_{21}$ is the time-ordered form of W_{21}.

The implementation of the formalism depends on the nature of the coordinates q. In particular, the commutation rules between p and q depend on the commutators of the coordinates according to (2.5b). If the coordinates commute, then

$$(q^\alpha,q^\beta)_-=0\ , \tag{2.8a}$$

$$\left[\frac{\partial}{\partial q^\alpha},q^\beta\right]_-=\delta_\alpha{}^\beta\ , \tag{2.8b}$$

$$(q^\alpha,p_\beta)_-=i\hbar\delta^\alpha{}_\beta\ , \tag{2.8c}$$

$$(p_\alpha,p_\beta)_-=0\ . \tag{2.8d}$$

Also

$$p=p^\dagger\ . \tag{2.9}$$

On the other hand, if the coordinates anticommute (are Grassmannian or fermionic)

$$(q^\alpha,q^\beta)_+=0\ , \tag{2.10a}$$

$$\left[\frac{\partial}{\partial q^\alpha},q^\beta\right]_+=\delta_\alpha{}^\beta\ , \tag{2.10b}$$

$$(q^\alpha,p_\beta)_+=-i\hbar\delta_\beta{}^\alpha\ , \tag{2.10c}$$

$$(p_\alpha,p_\beta)_+=0\ , \tag{2.10d}$$

and

$$p=-p^\dagger\ . \tag{2.10e}$$

Equations (2.8c) and (2.10c) are the two implementations of (2.5b). In both cases $p=-i\hbar\partial/\partial q$. In the Bose case p is Hermitian, and in the fermionic case it is anti-Hermitian as required by (2.8c) and (2.10c) and as further discussed in the Appendix. With this single change, the complete quantal formalism goes through in the same way for both kinds of coordinates.

Hamilton's principle in an unsymmetrized form is

$$\delta\int\left[\sum\dot q^\alpha p_\alpha-H\right]dt=0 \tag{2.11}$$

and Hamilton's equations are

$$\dot q^\alpha=\left[\frac{\partial H}{\partial p_\alpha}\right]_r\ , \tag{2.12a}$$

$$\dot p_\alpha=-\left[\frac{\partial H}{\partial q^\alpha}\right]_l\ , \tag{2.12b}$$

where l and r refer to left and right derivatives.

III. THE GRASSMANN OSCILLATOR

The n-dimensional bosonic oscillator is described by the Lagrangian

$$L=\tfrac{1}{2}(\dot x^k g_{kl}\dot x^l-\omega^2 x^k g_{kl}x^l)\ , \tag{3.1a}$$

where

$$g_{kl}=g_{lk},\quad k,l=1,\ldots,n\ . \tag{3.1b}$$

We shall describe the n-dimensional fermionic oscillator by the analogous Lagrangian

$$L=\tfrac{1}{2}(\dot q^\alpha C_{\alpha\beta}\dot q^\beta-\omega^2 q^\alpha C_{\alpha\beta}q^\beta)\ , \tag{3.2a}$$

where

$$C_{\alpha\beta}=-C_{\beta\alpha},\quad \alpha,\beta=1,\ldots,n\ . \tag{3.2b}$$

The anticommuting q^α may be thought of as the fermionic coordinates of superspace.

Define the symplectic adjoint by

$$\bar q_\alpha=q^\beta C_{\beta\alpha} \tag{3.3}$$

and the corresponding scalar product

$$\bar q q=\bar q_\alpha q^\alpha=q^\alpha C_{\alpha\beta}q^\beta\ . \tag{3.4}$$

Our convention for lowering an index is the same as in the usual spinor algebra. Then

$$L=\frac{1}{2}\left[\frac{d\bar q}{dt}\frac{dq}{dt}-\omega^2\bar q q\right]\ . \tag{3.5}$$

Let p_α be defined as the left derivative:

$$p_\alpha=\left[\frac{\partial L}{\partial \dot q^\alpha}\right]_l\ . \tag{3.6}$$

Then

$$p_\alpha=C_{\alpha\beta}\dot q^\beta\ , \tag{3.7}$$

$$\dot q^\alpha=(C^{-1})^{\alpha\beta}p_\beta\ . \tag{3.8}$$

Define the Hamiltonian in accordance with (2.11) as

$$H=\dot q^\alpha p_\alpha-L\ . \tag{3.9}$$

Then

$$H=\tfrac{1}{2}(-pC^{-1}p+\omega^2 qCq)\ . \tag{3.10}$$

The equations of motion by (2.12) are

$$\dot q^\alpha=\left[\frac{\partial H}{\partial p_\alpha}\right]_r=-(pC^{-1})^\alpha=(C^{-1}p)^\alpha\ , \tag{3.11a}$$

$$\dot p_\alpha=-\left[\frac{\partial H}{\partial q^\alpha}\right]_l=-\omega^2(Cq)_\alpha\ , \tag{3.11b}$$

where l and r refer to left and right derivatives. Then

$$\ddot q^\alpha=-\omega^2 q^\alpha\ , \tag{3.12}$$

$$\dot H=0\ . \tag{3.13}$$

The following relations are also preserved:

$$i\hbar\dot q=(q,H)_-\ , \tag{3.14a}$$

$$i\hbar\dot p=(p,H)_-\ , \tag{3.14b}$$

and these are consistent with the temporal independence of $(q^\alpha,p_\beta)_+$:

$$i\hbar\frac{d}{dt}(q^\alpha,p_\beta)_+=((q^\alpha,H)_-,p_\beta)_++(q^\alpha,(p_\beta,H)_-)_+ +(H,(p_\beta,q^\alpha)_+)_-=0 \tag{3.15}$$

by virtue of the super Jacobi identities.

Although p is anti-Hermitian, the Hamiltonian is Hermitian. Since n must be even, we may choose C to be

$$C=\begin{bmatrix} 0 & -i & & & \\ i & 0 & & & \\ & & 0 & -i & \\ & & i & 0 & \\ & & & & \ddots \end{bmatrix} . \tag{3.16}$$

If $C^2=1$, then

$$H=\tfrac{1}{2}(-pCp+\omega^2 qCq) . \tag{3.17}$$

However, we shall keep the general form (3.10) to allow for the possibility that C may depend on external or background variables to which the oscillator is coupled. C must be purely imaginary so that both the Lagrangian and the Hamiltonian are real and Hermitian.

IV. SPECTRUM AND STATES OF HAMILTONIAN

Define

$$a_-=\frac{1}{\sqrt{2}}(\omega q+iC^{-1}p) , \tag{4.1}$$

$$a_+=\frac{1}{\sqrt{2}}(\omega q-iC^{-1}p) . \tag{4.1'}$$

The operators a_+ and a_- are Hermitian conjugates of each other because C must be imaginary. Then

$$(a_-^\alpha, a_-^\beta)_+=0 , \tag{4.2}$$

and

$$(a_-^\alpha, \bar{a}_\beta^+)_+=\omega\delta^\alpha{}_\beta , \tag{4.3a}$$

$$(a_-, \bar{a}^+)_+=n\omega , \tag{4.3b}$$

where $\bar{a}_\beta^+=a_+^\alpha C_{\alpha\beta}$ and the Hamiltonian is

$$H=\tfrac{1}{2}(a_-Ca_++a_+Ca_-)$$
$$=\tfrac{1}{2}(\bar{a}^+a_--a_-\bar{a}^+) \tag{4.4}$$
$$=\bar{a}^+a_--\tfrac{1}{2}n\omega \tag{4.4a}$$
$$=-a_-\bar{a}^++\tfrac{1}{2}n\omega . \tag{4.4b}$$

The commutation rules for H with a_- and a_+ are the same as for the bosonic oscillator,

$$(H,a_-^\alpha)=-\omega a_-^\alpha , \tag{4.5}$$

$$(H,a_+^\alpha)_-=\omega a_+^\alpha , \tag{4.5'}$$

and the eigenstates are generated in the famililar way by raising and lowering operators according to the equations

$$Ha_-^\alpha\,|\,n\,\rangle=(\epsilon_n-\omega)a_-^\alpha\,|\,n\,\rangle , \tag{4.6}$$

$$Ha_+^\alpha\,|\,n\,\rangle=(\epsilon_n+\omega)a_+^\alpha\,|\,n\,\rangle . \tag{4.6'}$$

It is convenient to employ the following representations of a_-^α and a_+^α:

$$a_-^\alpha=\frac{1}{\sqrt{2}}\exp(-\tfrac{1}{2}\omega\bar{q}q)\frac{\partial}{\partial\bar{q}_\alpha}\exp(\tfrac{1}{2}\omega\bar{q}q) , \tag{4.7a}$$

$$a_+^\alpha=-\frac{1}{\sqrt{2}}\exp(\tfrac{1}{2}\omega\bar{q}q)\frac{\partial}{\partial\bar{q}_\alpha}\exp(-\tfrac{1}{2}\omega\bar{q}q) . \tag{4.7b}$$

In contrast with the bosonic case, the fermionic oscillator has only a finite number of states. The lowest is determined by

$$a_-^\alpha\exp(-\tfrac{1}{2}\omega\bar{q}q)=0 \tag{4.8a}$$

and the highest by

$$a_+^\alpha\exp(\tfrac{1}{2}\omega\bar{q}q)=0 . \tag{4.8b}$$

Then

$$H\exp(-\tfrac{1}{2}\omega\bar{q}q)=-\tfrac{1}{2}n\omega\exp(-\tfrac{1}{2}\omega\bar{q}q) , \tag{4.9a}$$

$$H\exp(\tfrac{1}{2}\omega\bar{q}q)=\tfrac{1}{2}n\omega\exp(\tfrac{1}{2}\omega\bar{q}q) . \tag{4.9b}$$

Let

$$\psi_0=\exp(-\tfrac{1}{2}\omega\bar{q}q)$$
$$=\exp(-\tfrac{1}{2}\omega qCq) . \tag{4.10}$$

Then ψ_0 is the Grassmann Gaussian and the general eigenstate is

$$\psi^{\alpha_1\cdots\alpha_m}=a_+^{\alpha_1}\cdots a_+^{\alpha_m}\psi_0$$
$$=(-1/\sqrt{2})^m\left[\exp(\tfrac{1}{2}\omega\bar{q}q)\frac{\partial}{\partial\bar{q}_{\alpha_1}}\cdots\frac{\partial}{\partial\bar{q}_{\alpha_m}}\times\exp(-\tfrac{1}{2}\omega\bar{q}q)\right]\psi_0 \tag{4.11}$$

and

$$H\psi^{\alpha_1\cdots\alpha_m}=E_m\psi^{\alpha_1\cdots\alpha_m} , \tag{4.12a}$$

$$E_m=m\omega-\tfrac{1}{2}n\omega . \tag{4.12b}$$

The levels and states of the Hamiltonian are E_m and $\psi^{\alpha_1\cdots\alpha_m}$, $m=0,\ldots,n$. The number of levels is just $n+1$, the degeneracy of each level is C_m^n, and the states may be written

$$\psi^{\alpha_1\cdots\alpha_m}=\exp(-\tfrac{1}{2}\omega\bar{q}q)H^{\alpha_1\cdots\alpha_m}(q) , \tag{4.13}$$

where

$$H^{\alpha_1\cdots\alpha_m}(q)=\frac{(-)^m}{(\sqrt{2})^m}e^{\omega\bar{q}q}\frac{\partial}{\partial\bar{q}_{\alpha_1}}\cdots\frac{\partial}{\partial\bar{q}_{\alpha_m}}e^{-\omega\bar{q}q} . \tag{4.14}$$

We refer to the completely antisymmetric functions $H^{\alpha_1\cdots\alpha_m}(q)$ as the Grassmann-Hermite multinomials. The state of the physical system may also be labeled by $|n_1n_2\cdots\rangle$ instead of $\psi^{\alpha_1\cdots\alpha_m}$ where $n_s=0,1$ and $\sum n_s=m$. The n_s are the population numbers for the component oscillators.

Define

$$\bar{H}_{\alpha_1\cdots\alpha_m}(q)=H^{\beta_1\cdots\beta_m}C_{\beta_1\alpha_1}\cdots C_{\beta_m\alpha_m} . \tag{4.15}$$

The following orthogonality relations are satisfied:

$$\int(dq)e^{-\omega\bar{q}q}H^{\alpha_1\cdots\alpha_m}(q)\bar{H}_{\beta_1\cdots\beta_l}(q)=\delta_l^mB_nA_m\epsilon_{\beta_1\cdots\beta_m}^{\alpha_1\cdots\alpha_m} , \tag{4.16a}$$

where

$$B_n = (-2\omega)^{n/2}\mathrm{Pf}C \ , \qquad (4.16b)$$

$$A_m = \omega^m \epsilon(m), \quad \epsilon(m) = (-)^{m(m-1)/2} \ , \qquad (4.16c)$$

where Pf means Pfaffian and where a Berezin integration is to be understood.

Let

$$u^{\alpha_1\cdots\alpha_m}(q) = (B_n A_m)^{-1/2}\exp(-\tfrac{1}{2}\omega\bar{q}q)H^{\alpha_1\cdots\alpha_m}(q) \qquad (4.17)$$

then

$$\int (dq) u^{\alpha_1\cdots\alpha_m}(q)\bar{u}_{\beta_1\cdots\beta_p}(q) = \delta_p^m \epsilon_{\beta_1\cdots\beta_m}^{\alpha_1\cdots\alpha_m} \ . \qquad (4.18)$$

V. TIME-DEPENDENT STATES AND GREEN'S FUNCTIONS

Let the solution of the time-dependent equation belonging to the energy E_m be

$$\psi^{\alpha_1\cdots\alpha_m}(q,t) = \exp\left[\frac{1}{i\hbar}E_m t\right] u^{\alpha_1\cdots\alpha_m}(q) \ . \qquad (5.1)$$

An arbitrary time-dependent state satisfies the integral equation

$$\psi(q,t) = \int G(q,t;q_0,t_0)\psi(q_0,t_0)(dq_0) \ . \qquad (5.2)$$

Then the particular solutions (5.1) satisfy

$$\lambda_m u^{\alpha_1\cdots\alpha_m}(q) = \int G(q,t;q_0,t_0) u^{\alpha_1\cdots\alpha_m}(q_0)(dq_0) \ , \qquad (5.3)$$

where

$$\lambda_m = \exp\left[\frac{1}{i\hbar}(t-t_0)E_m\right] \ . \qquad (5.4)$$

Let the $u^{\alpha_1\cdots\alpha_m}(q)$ be normalized according to (4.17) so that (4.18) is satisfied. Then

$$G(q,t;q_0,t_0) = \exp\left[\frac{i}{2}n\omega\tau\right]\sum_{m=0}^{n}\frac{\lambda^m}{m!}\bar{u}_{\alpha_1\cdots\alpha_m}(q) \times u^{\alpha_1\cdots\alpha_m}(q_0) \ , \qquad (5.5)$$

where

$$\lambda = e^{-i\omega\tau}, \quad \tau = t - t_0 \ . \qquad (5.6)$$

By (4.16), (4.17), and (5.5)

$$G(q,t;q_0,t_0) = K\sum\left[\frac{\lambda}{\omega}\right]^m \frac{1}{m!}\epsilon(m)\exp[-\tfrac{1}{2}\omega(\bar{q}q + \bar{q}_0 q_0)] \times \bar{H}_{\alpha_1\cdots\alpha_m}(q)H^{\alpha_1\cdots\alpha_m}(q_0) \ , \qquad (5.7a)$$

where

$$K = [(\mathrm{Pf}C)(-2\omega)^{n/2}]^{-1}\exp(i\tfrac{1}{2}n\omega\tau) \ . \qquad (5.7b)$$

VI. CALCULATION OF THE GREEN'S FUNCTION

The Fourier transform of a Grassmann Gaussian is given by the following equation:

$$e^{-\omega\bar{\phi}\phi} = k\int e^{2i\omega\bar{\phi}\theta}e^{-\omega\bar{\theta}\theta}(d\theta) \ , \qquad (6.1a)$$

where a Berezin integration is always to be understood and

$$k = [(2\omega)^{n/2}\mathrm{Pf}(C)]^{-1} \ . \qquad (6.1b)$$

By (4.14)

$$H^{\alpha_1\cdots\alpha_m}(q) = \frac{(-)^m}{(\sqrt{2})^m}ke^{\omega\bar{q}q}\frac{\partial}{\partial\bar{q}_{\alpha_1}}\cdots\frac{\partial}{\partial\bar{q}_{\alpha_m}}\int e^{2i\omega\bar{q}Q}e^{-\omega\bar{Q}Q}(dQ)$$

or

$$H^{\alpha_1\cdots\alpha_m}(q) = \frac{(-)^m}{(\sqrt{2})^m}ke^{\omega\bar{q}q}\int (2i\omega)Q^{\alpha_1}\cdots(2i\omega)Q^{\alpha_m}e^{-\omega\bar{Q}Q + 2i\omega\bar{q}Q}(dQ) \ . \qquad (6.2)$$

The Green's-function series may now be rewritten with the aid of the Fourier representation of $H^{\alpha_1\cdots\alpha_m}(q)$

$$G(q,t;q_0,t_0) = Kk^2\exp[\tfrac{1}{2}\omega(\bar{q}q + \bar{q}_0 q_0)]\sum_m \frac{(-2\lambda\omega)^m}{m!}\epsilon(m)^2\int\int(\bar{Q}Q')^m e^{-\omega(\bar{Q}Q - 2i\bar{q}Q)}e^{-\omega(\bar{Q}'Q' - 2i\bar{q}_0 Q')}(dQ)(dQ')$$

$$= Kk^2\exp[\tfrac{1}{2}\omega(\bar{q}q + \bar{q}_0 q_0)]J \ , \qquad (6.3a)$$

where

$$J = \int (dQ)e^{\omega(-\bar{Q}Q + 2i\bar{q}Q)}J(Q) \qquad (6.3b)$$

and

$$J(Q) = \int (dQ')e^{\omega(-\bar{Q}'Q' + 2i\bar{q}_0 Q' - 2\lambda\bar{Q}Q')} \qquad (6.3c)$$

Then

$$J(Q) = k^{-1}e^{\omega(\lambda^2\bar{Q}Q - \bar{q}_0 q - 2i\lambda q_0 Q)} \qquad (6.4)$$

and

$$J = k^{-1}e^{-\omega\bar{q}_0 q}\int (dQ)e^{\omega[(\lambda^2-1)\bar{Q}Q + 2i(\bar{q} - \lambda\bar{q}_0)Q]} \ .$$

After the Q integration, one finds

$$J = k^{-2}(1-\lambda^2)^{n/2}e^{-\omega\bar{q}_0 q_0}e^{-\bar{R}R} \ , \qquad (6.5)$$

where

$$R = \frac{q - \lambda q_0}{(1-\lambda^2)^{1/2}} \ .$$

Since the Green's function is a Berezin integral, the factor $(1-\lambda^2)^{n/2}$ appears in the numerator instead of the denominator. The Green's function itself now becomes

$$G(q,t;q_0,t_0)=K(1-\lambda^2)^{n/2}\exp\left[\tfrac{1}{2}\omega\left[(\bar{q}q+\bar{q}_0q_0)\frac{\lambda^2+1}{\lambda^2-1}-\frac{4\lambda\bar{q}q_0}{\lambda^2-1}\right]\right] \quad (6.6a)$$

or

$$G(q,t;q_0,t_0)=\frac{i^{n/2}}{\mathrm{Pf}C}\left[\frac{\sin(\omega\tau)}{\omega}\right]^{n/2}e^{iS}, \quad (6.6b)$$

where S is just the classical action

$$S=\frac{1}{2}\frac{\omega}{\sin(\omega\tau)}[(\bar{q}q+\bar{q}_0q_0)\cos(\omega\tau)-2\bar{q}q_0]\ . \quad (6.6c)$$

The phase is the same as the corresponding result for the bosonic Green's function but the amplitudes are now inverted. Since the square of the amplitude corresponds to a density in phase space in the (WKB) limit, we see that the density just found is limited as required by the exclusion principle while the corresponding inverted amplitude permits an arbitrarily high density in the bosonic case.

The Green's function for the Grassmannian free particle corresponds to the limit $\omega=0$:

$$G(q,t;q_0,t_0)=\frac{i^{n/2}}{\mathrm{Pf}C}\tau^{n/2}\exp\left[\frac{i}{2\tau}(\bar{q}-\bar{q}_0)(q-q_0)\right]. \quad (6.7)$$

The limit of G as $\tau\to 0$ is a Berezin δ function:

$$\lim_{\tau\to 0}G(q,t;q_0,t_0)=\lim_{\tau\to 0}\frac{i^{n/2}}{\mathrm{Pf}C}\tau^{n/2}\sum_{m=0}^{n/2}\frac{1}{m!}\left[\frac{1}{2}\frac{i}{\tau}\theta^\alpha C_{\alpha\beta}\theta^\beta\right]^m$$
$$=\frac{i^n}{\mathrm{Pf}C}\frac{1}{2^{n/2}}\frac{1}{(n/2)!}C_{\alpha_1\beta_1}\cdots C_{\alpha_{n/2}\beta_{n/2}}\theta^{\alpha_1}\theta^{\beta_1}\cdots\theta^{\alpha_{n/2}}\theta^{\beta_{n/2}},$$

where $\theta=q-q_0$, or

$$\lim_{\tau\to 0}G(q,t;q_0,t_0)=\delta(q-q_0)\ , \quad (6.8)$$

where we have used

$$\delta(\theta)=i^n\theta^1\theta^2\cdots\theta^n=\epsilon(n)\theta^1\theta^2\cdots\theta^n \quad (6.9)$$

and

$$\mathrm{Pf}C=\frac{1}{2^{n/2}(n/2)!}\epsilon^{\alpha_1\cdots\alpha_n}C_{\alpha_1\alpha_2}\cdots C_{\alpha_{n-1}\alpha_n}=\sqrt{\det C}\ . \quad (6.10)$$

VII. THE OPERATOR HAMILTON-JACOBI EQUATION

A second way of determining the Green's function is by solving the operator Hamilton-Jacobi equation (2.6c). If the solution is $\mathscr{W}_{21}$ then the Green's function is given by (2.7)

$$G(q_2t_2;q_1t_1)=\langle q_2t_2\,|\,q_1t_1\rangle=\exp\left[\frac{i}{\hbar}\mathscr{W}_{21}\right], \quad (7.1)$$

where $\mathscr{W}_{21}$ is the matrix element of the time-ordered action between initial and final states.

The operator Hamilton-Jacobi equation is by (3.10)

$$-\frac{1}{2}\frac{\partial\mathscr{W}}{\partial q^\alpha}(C^{-1})^{\alpha\beta}\frac{\partial\mathscr{W}}{\partial q^\beta}+\tfrac{1}{2}\omega^2q^\alpha C_{\alpha\beta}q^\beta+\frac{\partial\mathscr{W}}{\partial t}=0\ . \quad (7.2)$$

To solve (7.2) try the ansatz

$$\mathscr{W}=\frac{1}{2}\frac{\omega}{\sin(\omega t)}[(\bar{q}q+\bar{q}_0q_0)\cos(\omega t)-2\bar{q}q_0]+\Phi(\omega,t)\ . \quad (7.3)$$

Then

$$p_\alpha=\frac{\partial\mathscr{W}}{\partial q^\alpha}=\frac{\omega}{\sin(\omega t)}[\bar{q}_{0\alpha}-\bar{q}_\alpha\cos(\omega t)]\ . \quad (7.4)$$

The commutation rule for q^α and p_β is

$$(q^\alpha,p_\beta)_+=-i\hbar\delta^\alpha{}_\beta\ . \quad (7.5)$$

By (7.4) the rule (7.5) implies

$$(q^\alpha,q_0^\beta)_+=-i\hbar(C^{-1})^{\alpha\beta}\left[\frac{\sin(\omega t)}{\omega}\right]. \quad (7.6)$$

We have

$$\frac{\partial\mathscr{W}}{\partial t}=-\frac{1}{2}\left[\frac{\omega}{\sin(\omega t)}\right]^2[(\bar{q}q+\bar{q}_0q_0)-2\bar{q}q_0\cos(\omega t)]+\frac{\partial\Phi}{\partial t}\ . \quad (7.7)$$

According to (7.2), (7.4), (7.6), and (7.7)

$$\frac{d\Phi}{dt}=-(\tfrac{1}{2}in\hbar)\frac{d}{dt}\ln\sin(\omega t)\ ,$$
$$\Phi=-\tfrac{1}{2}in\hbar\ln\sin(\omega t)+\mathrm{const}\ .$$

Then

$$\langle qt \mid q_0t_0 \rangle \sim [\sin(\omega t)]^{n/2} \exp\left[i\frac{S}{\hbar} \right], \tag{7.8a}$$

where

$$S = \frac{1}{2}\frac{\omega}{\sin(\omega t)}[(\bar{q}q + \bar{q}_0 q_0)\cos(\omega t) - 2\bar{q}q_0] \tag{7.8b}$$

in agreement with (6.6). The constant of integration in the solution of the Hamilton-Jacobi equation may be taken from (6.8).

VIII. FEYNMAN PATH INTEGRALS

Let us introduce the eigenstates of q and p that satisfy

$$\begin{aligned} &\langle q' \mid q \rangle = \delta(q'-q), \\ &\langle p' \mid p \rangle = \delta(p'-p), \\ &\langle q \mid p \rangle = \langle p \mid q \rangle^* = e^{ipq} = e^{ip_\alpha q^\alpha}. \end{aligned} \tag{8.1}$$

We now have

$$\begin{aligned} \langle p' \mid p \rangle &= \int [dq] \langle p' \mid q \rangle \langle q \mid p \rangle = \int [dq] e^{i(p-p')q} \\ &= \frac{(-i)^n}{n!}\epsilon^{\alpha_1 \cdots \alpha_n}(p-p')_{\alpha_1} \cdots (p-p')_{\alpha_n} = \delta(p-p'). \end{aligned} \tag{8.2}$$

To find the path-integral formula let us first compute

$$\begin{aligned} \langle q_2(t+\Delta t) \mid q_1(t) \rangle &= \langle q_2 \mid e^{-i\Delta t H} \mid q_1 \rangle \\ &\simeq \langle q_2 \mid e^{(i/2)\Delta t p C^{-1} p} e^{-i\Delta t V(q)} \mid q_1 \rangle \\ &= \int (dp) \langle q_2 \mid e^{(i/2)\Delta t p C^{-1} p} \mid p \rangle \langle p \mid e^{-i\Delta t V(q)} \mid q_1 \rangle. \end{aligned} \tag{8.3}$$

After completing the square we find for the Berezin integral

$$\langle q_2(t+\Delta t) \mid q_1(t) \rangle = \frac{(-i\Delta t)^{n/2}}{\mathrm{Pf}C} \exp\left[\frac{i}{2}(\bar{q}_2 - \bar{q}_1)(q_2 - q_1)/\Delta t - i\Delta t V(q_1) \right], \tag{8.4}$$

where $\mathrm{Pf}C$ is given by (6.10). For a finite interval we have

$$\begin{aligned} \langle q_f t_f \mid q_i t_i \rangle &= \lim_{N\to\infty} \int (dq_{N-1}) \cdots (dq_1) \langle q_f t_f \mid q_{N-1} t_{N-1} \rangle \cdots \langle q_1 t_1 \mid q_i t_i \rangle \\ &= \lim_{N\to\infty} \int \prod_{k=1}^{N-1} (dq_k) \left[\left[\frac{i(t_f - t_i)}{N} \right]^{n/2} \frac{1}{\mathrm{Pf}C} \right]^N \exp\left[i \int_{t_i}^{t_f} dt\, L(q,\dot{q}) \right]. \end{aligned} \tag{8.5}$$

For a free particle this becomes[6]

$$\begin{aligned} \langle q_f t_f \mid q_i t_i \rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} (dq_k) &\left[\left[\frac{i(t_f - t_i)}{N} \right]^{n/2} \frac{1}{\mathrm{Pf}C} \right]^N \\ &\times \exp\left[\frac{iN}{2(t_f - t_i)} \left[(\bar{q}_f q_f + \bar{q}_i q_i) - (\bar{q}_i K_{11}^{-1} q_i - 2\bar{q}_i K_{1N-1}^{-1} q_f + \bar{q}_f K_{N-1,N-1}^{-1} q_f) + \sum_{j,l=1}^{N-1} \bar{q}_j K_{jl} q_l \right] \right], \end{aligned} \tag{8.6}$$

where the $(N-1)$-dimensional matrix K is

$$K = \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}.$$

Therefore one gets for the free particle

$$\langle qt \mid q_0 0 \rangle = \frac{(it)^{n/2}}{\mathrm{Pf}C} \exp\left[\frac{i}{2t}(\bar{q} - \bar{q}_0)(q - q_0) \right], \tag{8.7}$$

where we have used

$$\mathrm{Pf}(C \times K) = (\mathrm{Pf}C)^{N-1} (\det K)^{n/2} = N^{n/2} (\mathrm{Pf}C)^{N-1}. \tag{8.8}$$

The sum over paths for the fermionic oscillator may be found in the same way as for the free particle, and also in the same way as for the bosonic oscillator except for the Berezin integration. One obtains

$$\langle qt \mid q_0 0 \rangle = C(t) \left[\frac{\sin(\omega t)}{\omega t} \right]^{n/2} e^{iS(q,t)}. \tag{8.9}$$

After $C(t)$ is taken from (8.7), the final result containing the inverted amplitude is

$$\langle qt \mid q_0 0\rangle = \frac{1}{\mathrm{Pf}C}\left[\frac{\sin(\omega t)}{\omega}\right]^{n/2} e^{iS(q,t)}$$

in agreement with (6.6b).

IX. REPRESENTATION OF LIE ALGEBRAS

The familiar representation of the generators of a Lie algebra as bilinears in the absorption and emission operators of a bosonic oscillator holds also for the fermionic case. Let X_a be the generators of a Lie algebra

$$(X_a,X_b)_- = if_{ab}{}^c X_c \ . \tag{9.1}$$

Then

$$(\bar{a}X_b a,\bar{a}X_c a)_- = if_{bc}{}^d \bar{a}X_d a \tag{9.2}$$

if

$$(\bar{a}_b,a^c)_+ = \delta_b{}^c \ . \tag{9.3}$$

We shall next be interested in the case that the X_a are the generators of the $(d-1)$-dimensional rotation group. This application arises in the study of the d-dimensional super Poincaré group and the associated rotational little group.

X. RELATION TO THE SUPER POINCARÉ GROUP

The generators of the super Poincaré group in d dimensions include the Majorana spinor Q^α in addition to the momentum P_A and angular momentum J_{AB}. The corresponding algebra may be enlarged by the addition of the covariant derivatives D^α to these generators. We are interested in the irreducible representations of this algebra on the space of the superfields. Then P^2 is still a Casimir operator and its eigenvalues M^2 label irreducible representations of this superalgebra. Projection operators for positive and negative energy may be defined just as for the simple Poincaré case:

$$\Lambda_\pm = \frac{1}{2M}(M \pm P) \ . \tag{10.1}$$

Then Q^α and D^α may be split by these projection operators and they may also be given a representation as differential operators in superspace as follows:[5]

$$Q^\alpha_\pm = i\Lambda_\pm Q^\alpha = i\Lambda_\pm\left[\exp(\pm\tfrac{1}{4}M\bar{\theta}\theta)\frac{\partial}{\partial\bar{\theta}_\alpha}\exp(\mp\tfrac{1}{4}M\bar{\theta}\theta)\right] , \tag{10.2}$$

$$D^\alpha_\pm = i\Lambda_\pm D^\alpha = i\Lambda_\pm\left[\exp(\mp\tfrac{1}{4}M\bar{\theta}\theta)\frac{\partial}{\partial\bar{\theta}_\alpha}\exp(\pm\tfrac{1}{4}M\bar{\theta}\theta)\right] , \tag{10.3}$$

where the θ^α are the Grassmann coordinates of superspace:

$$(\theta^\alpha,\theta^\beta)_+ = 0 \ , \tag{10.4}$$

$$\bar{\theta}_\alpha = \theta^\beta C_{\beta\alpha} \ . \tag{10.5}$$

The complete set of Casimir operators is constructed in terms of a modified angular momentum

$$U_{AB} = J_{AB} + (P^E/P^2)(J_{EA}P_B - J_{EB}P_A) - \tfrac{1}{4}i(P^E/P^2)(\bar{Q}\Gamma_{EAB}Q) \tag{10.6}$$

as discussed in Ref. 5. There it is shown how the Casimir operators simplify in the subspace corresponding to zero intrinsic spin, since in this case U_{AB} reduces to

$$U_{AB} = -\tfrac{1}{4}i(P^E/P^2)(\bar{D}\Gamma_{EAB}D) \ . \tag{10.7}$$

This form of U_{AB} is analogous to the $\bar{a}Xa$ in Sec. IX because the a and the D are also analogues while the X correspond to the generators of a $(d-1)$-dimensional rotation group in the rest frame. For vanishing intrinsic spin, it is shown in Ref. 5 that the quadratic Casimir operator reduces to a function of $\bar{D}D$ only and that a large class of representations of the super Poincaré group is entirely fixed by this single Casimir operator. The problem of determining these special representations is then a question of finding the eigenvalues and eigenfunctions of $\bar{D}D$:

$$\bar{D}D\psi_m = \epsilon_m\psi_m \ . \tag{10.8}$$

At this point one comes back to the oscillator problem. The highest eigenvalues correspond to the Grassmann Gaussian functions:

$$\psi_0(\theta,x) \simeq \exp(\tfrac{1}{4}M\bar{\theta}\theta)F(x) \ , \tag{10.9}$$

where $F(x)$ satisfies the d-dimensional Klein-Gordon equation

$$(P^2 - M^2)F(x) = 0 \ . \tag{10.10}$$

Then

$$\bar{D}D\psi_0(x,\theta) = \tfrac{1}{2}nM\psi(x,\theta) \ , \tag{10.11}$$

where $n = 2^{[d/2]}$ and the most general solution to (10.8) is

$$\epsilon_m = (\tfrac{1}{2}n - 2m)M, \quad m = 1,\ldots,n/2 \tag{10.12a}$$

and

$$\psi_m(x,\theta) = \sum_{j=0}^{n/2} Q_+^{\alpha_1}\cdots Q_+^{\alpha_j} D_-^{\beta_1}\cdots D_-^{\beta_m}\exp(\tfrac{1}{4}M\bar{\theta}\theta) \times F_{\alpha_1\cdots\alpha_j;\beta_1\cdots\beta_m}(x) \ . \tag{10.12b}$$

In this expression each D_- lowers the eigenvalue by $2M$. Since the Q and D anticommute, multiplication by Q does not change the eigenvalue and the Q product describes the degeneracy. The total degeneracy of one eigenvalue is then

$$C_m^{n/2}\sum_{j=0}^{n/2} C_j^{n/2} = 2^{n/2}C_m^{n/2}$$

and the total number of eigenfunctions is then

$$2^{n/2}\sum_{m=0}^{n/2} C_m^{n/2} = 2^n \ .$$

By introducing (10.2) and (10.3) the states (10.12b) may be rewritten

$$\psi_m(x,\theta)=\exp(\tfrac{1}{4}M\bar{\theta}\theta)\sum_{p=0}^{n/2}H^{\alpha_1\cdots\alpha_p\beta_1\cdots\beta_m}(\theta)\times\bar{F}_{\alpha_1\cdots\alpha_p;\beta_1\cdots\beta_m}(x)\ , \tag{10.13}$$

where

$$H^{\alpha_1\cdots\alpha_p}(\theta)=\exp(-\tfrac{1}{4}M\bar{\theta}\theta)\frac{\partial}{\partial\bar{\theta}_{\alpha_1}}\cdots\frac{\partial}{\partial\bar{\theta}_{\alpha_p}}\exp(\tfrac{1}{4}M\bar{\theta}\theta)\ . \tag{10.14}$$

These are Grassmann-Hermite multinomials. The $\bar{F}_{\alpha_1\cdots\alpha_b;\beta_1\cdots\beta_m}(x)$ satisfy

$$\Lambda_{-\gamma}^{\alpha_s}\bar{F}_{\alpha_1\cdots\alpha_s\cdots\alpha_p;\beta_1\cdots\beta_m}=0\ , \tag{10.15a}$$

$$\Lambda_{+\gamma}^{\beta_s}\bar{F}_{\alpha_1\cdots\alpha_p;\beta_1\cdots\beta_s\cdots\beta_m}=0\ . \tag{10.15b}$$

These are like Bargmann-Wigner multispinors but completely antisymmetric in the sets $\alpha_1\cdots\alpha_p$ and $\beta_1\cdots\beta_m$ separately. Only their completely antisymmetric projection contributes to $\psi(x,\theta)$. Thus the eigenvalues of the quadratic Casimir operator of the super Poincaré group are essentially the energy levels of the Grassmann oscillator. The eigenfunctions of $\bar{D}D$ are given by the superfield expansion (10.13) in the eigenfunctions of the oscillator. As mentioned earlier and explained in detail in Ref. 5, there is a large class of irreducible representations determined by this single Casimir operator.

ACKNOWLEDGMENT

We thank R. Jackiw for useful comments.

APPENDIX

When the q are anticommuting the consistency of the Hermitian property of $\partial/\partial q$ may be checked by verifying the Berezin equality,

$$\int\psi^*\left[\frac{\partial}{\partial q^1}\phi\right](dq)=\int\left[\frac{\partial\psi}{\partial q^1}\right]^*\phi(dq)\ , \tag{A1}$$

where ψ and ϕ are superfunctions. Let $q^1\cdots q^m$ be a typical term of ψ. Then the only terms of ϕ that multiply the foregoing term of ψ and also contribute to the integral will be of the form $q^1q^{m+1}\cdots q^n$. Without loss of generality we have chosen to differentiate with respect to q^1 and also assumed that the factors $q^1\cdots q^m$ and $q^{m+1}\cdots q^n$ are arranged in their natural order. Then, since the asterisk implies reversal of order,

$$\int(dq)\psi^*\frac{\partial\phi}{\partial q^1}=\int(dq)(q^m\cdots q^1)(q^{m+1}\cdots q^n)=\epsilon(m)\int(dq)(q^1\cdots q^m)(q^{m+1}\cdots q^n)\ . \tag{A2}$$

On the other hand,

$$\int(dq)\left[\frac{\partial\psi}{\partial q^1}\right]^*\phi=\int(dq)\left[\frac{\partial}{\partial q^1}(q^1\cdots q^m)\right]^*(q^1q^{m+1}\cdots q^n)=\int(dq)(q^m\cdots q^2)(q^1q^{m+1}\cdots q^n)=(-)^{m-1}\epsilon(m-1)\int(dq)(q^1\cdots q^m)(q^{m+1}\cdots q^n)\ . \tag{A3}$$

The right-hand sides of (A2) and (A3) are the same except possibly for the numerical factors. But

$$(-)^{m-1}\epsilon(m-1)=\epsilon(m)\ . \tag{A4}$$

Hence (A1) and the corresponding anti-Hermitian property of p are checked.

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