Quantum mechanics of measurements distributed in time. A path-integral formulation

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Consider measurements that provide information about the position of a nonrelativistic, one-dimensional, quantum-mechanical system. An outstanding question in quantum mechanics asks how to analyze measurements distributed in time—i.e., measurements that provide information about the position at more than one time. I develop a formulation in terms of a path integral and show that it applies to a large class of measurements distributed in time. For measurements in this class, the path-integral formulation provides the joint statistics of a sequence of measurements. Specialized to the case of instantaneous position measurements, the path-integral formulation breaks down into the conventional machinery of nonrelativistic quantum mechanics: a system quantum state evolving in time according to two rules—between measurements, unitary evolution, and at each measurement, "collapse of the wave function" ("reduction of the state vector"). For measurements distributed in time, the path-integral formulation has no similar decomposition; the notion of a system quantum state evolving in time has no place.

I. INTRODUCTION AND OVERVIEW

In this paper I consider measurements that provide information about the position $x(t)$ of a nonrelativistic, one-dimensional, quantum-mechanical system. The standard lore of quantum mechanics provides a description[1–4] of an instantaneous measurement of position—e.g., a snapshot of a pendulum bob—or a sequence of instantaneous measurements. Such instantaneous measurements I call samplings of $x(t)$; each sampling provides information about $x(t)$ at a single instant of time. For a sequence of samplings the standard description ultimately produces—after much clanking of machinery and turning of gears—a joint probability distribution for the possible results, from which one can compute statistical correlations among the samplings. This—and no more—is what quantum mechanics allows. Indeed, one may take the point of view[3,5,6]—I take it here—that the "function of quantum mechanics is to give statistical correlations between the outcomes of successive observations."[5]

How might one best display the advertised joint probability distribution? An especially simple form, with a natural interpretation, emerges from a path-integral[7] formulation. Let $\psi_0(x,t_0)$ be the wave function of the system at some initial time t_0—i.e., the probability amplitude for the system to be at position x at time t_0. Consider a sequence of Q samplings of $x(t)$ at times $t_1,\dots,t_Q$ ($t_0<t_1<\cdots<t_Q$). The samplings are made by some measuring apparatus. Account for its irresolution or imprecision by introducing a conditional probability amplitude $\Upsilon(\bar{x}-x)$, which I call the resolution amplitude; $\Upsilon(\bar{x}-x)$ is the amplitude to obtain the value $\bar{x}$ as the result of a sampling, given that the system is at position x at the time of the sampling. The resolution amplitude is the only way the measuring apparatus enters into the standard description. Its detailed form depends on the properties of the measuring apparatus. Crudely speaking, however, the variance of the associated conditional probability distribution $|\Upsilon(\bar{x}-x)|^2$ is the resolution of the measuring apparatus; the apparatus can resolve positions that are separated by more than the variance. Define now a probability amplitude $\Phi(\bar{x}_1,\dots,\bar{x}_Q;x,t_Q)$, the joint amplitude for the Q samplings to yield the sequence of results $\bar{x}_1,\dots,\bar{x}_Q$ and for the system to be at position x at the time of the Qth sampling. Write this fundamental joint amplitude as a Feynman path integral:[7]

$$\Phi(\bar{x}_1,\dots,\bar{x}_Q;x,t_Q)=\int_{t_0}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{x}_q-x(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\ . \tag{1.1a}$$

Here the integral denotes a sum over all paths $x(t)$ on the interval $[t_0,t_Q]$, with arbitrary initial values $x(t_0)$, but with final value $x(t_Q)=x$; and

$$S[x(t)]=\int_{t_0}^{t_Q}dt\,L(x,\dot{x};t)$$

is the action for the path $x(t)$, $L(x,\dot{x};t)$ being the system's Lagrangian. Finally, derive from Φ the desired joint probability distribution

$$P(\bar{x}_1,\dots,\bar{x}_Q)=\int dx\ |\Phi(\bar{x}_1,\dots,\bar{x}_Q;x,t_Q)|^2\ , \tag{1.1b}$$

the probability distribution to obtain the sequence $\bar{x}_1,\dots,\bar{x}_Q$ as the results of the samplings.

Interpretation comes from Feynman's rules[7] for combining probability amplitudes. The amplitude of A given

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B times the amplitude of B is the joint amplitude of A and B. Depending on whether B is potentially observable, the probability of A is derived from the joint amplitude in one of two ways. If B is unobservable, then the joint amplitude, summed over all values of B, yields the amplitude of A, whose absolute square is the probability of A. If B is observable, then the absolute square of the joint amplitude is the joint probability of A and B, which, summed over B, yields the probability of A.

Select a path $x(t)$, and begin with the quantity $e^{(i/\hbar)S[x(t)]}$, the familiar quantum-mechanical amplitude for the path, conditioned on the initial value $x(t_0)$.[8] Multiply by the initial wave function $\psi_0(x(t_0),t_0)$, the amplitude for the path's initial value $x(t_0)$; thereby obtain the unconditioned amplitude

$$e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$$

for the path $x(t)$. For each $q=1,\ldots,Q$, multiply by the resolution amplitude $\Upsilon(\bar{x}_q-x(t_q))$, the amplitude to obtain the value $\bar{x}_q$ in the qth sampling, given the path's value $x(t_q)$—or, more generally, the amplitude to obtain $\bar{x}_q$, given the path $x(t)$; thereby find the joint amplitude

$$\left[\prod_{q=1}^{Q}\Upsilon(\bar{x}_q-x(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$$

for the sequence of results $\bar{x}_1,\ldots,\bar{x}_Q$ and for the path $x(t)$. Apply Feynman's rules to compute the probability amplitude $\Phi(\bar{x}_1,\ldots,\bar{x}_Q;x,t_Q)$: sum over all intermediate unobservable quantities; i.e., sum over all paths such that $x(t_Q)=x$. Why not sum over final values $x(t_Q)$ as well? Because the system's final position is potentially observable by an independent measurement. Hence, first square Φ to obtain a probability distribution; then integrate over final values x to get the joint probability distribution (1.1b).

How does the standard description of instantaneous measurements arrive at the joint probability distribution (1.1b)? By a route circuitous indeed when judged against the simplicity of the path-integral formulation (1.1), but a route well-traveled nonetheless. The standard description constructs the joint probability distribution from parts—conditional probability distributions—manufactured by the conventional machinery of nonrelativistic quantum mechanics. To see this machinery at work, it is sufficient to consider just two samplings. Begin by writing the amplitude (1.1a) for $Q=2$ in the form

$$\Phi(\bar{x}_1,\bar{x}_2;x,t_2)=\Upsilon(\bar{x}_2-x)\int dx_0dx_1\,K(x,t_2\,|\,x_1,t_1)\Upsilon(\bar{x}_1-x_1)K(x_1,t_1\,|\,x_0,t_0)\psi_0(x_0,t_0)\;, \tag{1.2}$$

where the sum over paths has been eliminated in favor of the (nonrelativistic) propagator

$$K(x,t\,|\,x',t')\equiv\int_{(x',t')}^{(x,t)}\mathscr{D}x(t)\,e^{(i/\hbar)S[x(t)]}\;. \tag{1.3}$$

The sum over paths in Eq. (1.3) includes all paths on the interval $[t',t]$ such that $x(t')=x'$ and $x(t)=x$. Feynman's rules imply that the propagator is the conditional probability amplitude for the system to be at x at time t, given that it was at x' at time t'. To describe the first sampling, introduce the quantity

$$\psi_0(x,t_1)=\int dx_0\,K(x,t_1\,|\,x_0,t_0)\psi_0(x_0,t_0)\;. \tag{1.4a}$$

One can derive from Eqs. (1.1b) and (1.2) the probability distribution $P(\bar{x}_1)$ to obtain $\bar{x}_1$ as the result of the first sampling:

$$\begin{aligned}P(\bar{x}_1)&=\int d\bar{x}_2\,P(\bar{x}_1,\bar{x}_2)\\&=\int dx\;|\Upsilon(\bar{x}_1-x)|^2\,|\psi_0(x,t_1)|^2\;.\end{aligned} \tag{1.4b}$$

Equations (1.4) speak a familiar language. According to Feynman's rules, $\psi_0(x,t_1)$ is the amplitude for the system to be at x at time t_1; it is the usual wave function of the system at time t_1. The evolution of the wave function from t_0 to t_1 using the propagator is equivalent to evolution via a Schrödinger equation. The probability distribution $P(\bar{x}_1)$ arises directly from Feynman's rules applied to the joint amplitude $\Upsilon(\bar{x}_1-x)\psi_0(x,t_1)$, taking into account that the position x at the time of the first sampling is potentially observable by an independent measurement. Turn now to the second sampling, and introduce the quantity

$$\psi_{\bar{x}_1}(x,t_2)=\int dx_1\,K(x,t_2\,|\,x_1,t_1)\psi_{\bar{x}_1}(x_1,t_1)\;, \tag{1.5a}$$

where

$$\psi_{\bar{x}_1}(x,t_1)\equiv\Upsilon(\bar{x}_1-x)\psi_0(x,t_1)/[P(\bar{x}_1)]^{1/2}\;. \tag{1.5b}$$

The conditional probability distribution $P(\bar{x}_2\,|\,\bar{x}_1)$ to obtain $\bar{x}_2$ as the result of the second sampling, conditioned on $\bar{x}_1$ as the result of the first sampling, is given by

$$\begin{aligned}P(\bar{x}_2\,|\,\bar{x}_1)&=P(\bar{x}_1,\bar{x}_2)/P(\bar{x}_1)\\&=\int dx\;|\Upsilon(\bar{x}_2-x)|^2\,|\psi_{\bar{x}_1}(x,t_2)|^2\end{aligned} \tag{1.5c}$$

[Eqs. (1.1b) and (1.2)]. Equations (1.5) speak the same language for the second sampling as do Eqs. (1.4) for the first, *if* one interprets $\psi_{\bar{x}_1}(x,t_1)$ as the wave function of the system just after the first sampling, conditioned on the result $\bar{x}_1$ for the first sampling. Thus the standard description posits a sudden change of the wave function from $\psi_0(x,t_1)$ just before the first sampling to $\psi_{\bar{x}_1}(x,t_1)$ just after. This sudden change,[1,4] variously called "collapse of the wave function," "reduction of the wave packet," or "reduction of the state vector," modifies the wave function to be consistent with the known result of the first sampling.[9] The two samplings having now been described in the same way, one has available two probability distributions, $P(\bar{x}_1)$ and $P(\bar{x}_2\,|\,\bar{x}_1)$; the joint probability distribution is then constructed as a product $P(\bar{x}_1,\bar{x}_2)=P(\bar{x}_2\,|\,\bar{x}_1)P(\bar{x}_1)$.

Apparent now are the inner workings of the standard description of instantaneous measurements.[1–4] The basic parts are conditional probability distributions, one for each sampling, each conditioned on the results of previous samplings and each independent of the existence of subsequent samplings. These parts are manufactured by machinery that one usually thinks of as being nonrelativistic quantum mechanics. The fundamental quantity is the quantum state of the system, here represented by a wave function,[10] which evolves in time according to two rules: (i) between samplings the quantum state undergoes unitary evolution, the wave function evolving according to the time-dependent Schrödinger equation; (ii) at each sampling the quantum state suffers an instantaneous, nonunitary change, the collapse of the wave function, which modifies the wave function to be consistent with the result of the sampling. Unitary evolution and wave-function collapse carry the wave function through as many samplings as one desires. The conditional probability distribution for a particular sampling arises from the probabilistic interpretation of the wave function, supplemented by a conditional probability distribution $|\Upsilon(\bar{x}-x)|^2$ that accounts for the irresolution of the samplings. Finally, the joint probability distribution (1.1b) is constructed as the product of the conditional probability distributions for the samplings. A route more circuitous—more like a Rube Goldberg machine[11]—can scarcely be imagined.

Mention collapse of the wave function, and you are likely to encounter vague uneasiness or, in extreme cases, real discomfort.[12] This uneasiness can usually be traced to a feeling that wave-function collapse lies "outside" quantum mechanics: the real quantum mechanics is said to be unitary Schrödinger evolution; wave-function collapse is regarded as an ugly duckling of questionable status, dragged in to interrupt the beautiful flow of Schrödinger evolution. The point of view taken here is more benign: wave-function collapse is just a tool of the standard description—but, within the standard description, an essential tool. Take a snapshot of a pendulum bob; find the bob at a particular position. Take a second snapshot immediately after the first; find the bob with overwhelming probability at the same position, within the resolution of the snapshots—not at some far-away position allowed by the bob's wave function before the first snapshot. If one insists on using the standard description—if one insists on attributing to the bob a wave function—then there must be a sudden change in the wave function after the first snapshot. Collapse of the wave function is simply the formal device within the standard description which modifies the wave function after a measurement to take into account the information acquired in the measurement. It leads naturally to conditional questions: what is the probability for this result of this measurement, given the results of previous measurements?

Ask not conditional questions. Ask instead questions about the joint statistics of a sequence of samplings. Pose the questions in terms of the path-integral formulation (1.1). Then all reference to wave-function collapse disappears.[13] Moreover, this is not the only vanishing act; disappearing also is the very notion of a quantum state evolving in time—either by unitary evolution or by wave-function collapse. In the path-integral formulation there are not two sorts of evolution. There is no evolution. Nothing evolves. Mistrust wave-function collapse, and mistrust as well the notion of a quantum state evolving unitarily. These concepts are equally and only tools of the standard description—tools used ultimately to construct the joint probability distribution (1.1b).

The standard description, despite its cumbersome appearance, provides a practical method for analyzing a sequence of instantaneous measurements. Its chief drawback is not unwieldiness, but rather that it does not apply to real measurements—even those intended to be nearly instantaneous. Real measuring apparatuses have nonzero time resolution; the best they can do is to approximate an instantaneous value of position as some sort of average of $x(t)$ over a very short time. More importantly, in real situations one is often interested not at all in instantaneous values of position, but instead in a particular behavior of position as a function of time—i.e., in a signal with a particular time signature; one uses a measuring apparatus carefully designed to pick out the desired signal. In general, then, a real measurement is distributed in time; it provides information about the behavior of the position during some finite time interval in the past. It can be described formally as a sampling (instantaneous readout) at time t of a quantity $y(t)$, which is a real functional of $x(t')$ for times t' in the interval $[t-\Delta_t,t]$, Δ_t being the duration of the interval. A general functional is more than I can handle; throughout this paper, therefore, I specialize $y(t)$ to have the form

$$y(t)=\mathscr{Y}_t[x(t')]\equiv\int_{-\infty}^{\infty}dt'\,Y_{tt'}(x(t'))=\int_{t-\Delta_t}^{t}dt'\,Y_{tt'}(x(t'))\;, \tag{1.6}$$

where $\mathscr{Y}_t$ denotes a real functional of $x(t')$ which depends on t, and $Y_{tt'}$ denotes a real function of position which depends on both t and t' and which vanishes identically for $t-t'<0$ or $t-t'>\Delta_t$. The form (1.6) is "local" in the time t' in the sense that $y(t)$ is an integral of separate contributions from times t'; it corresponds to a measuring apparatus that gathers information about $x(t')$ sequentially during the interval $[t-\Delta_t,t]$ and adds up the information from different times t'. Despite this restriction, the form (1.6) covers a wide class of measurements. An important special case, which is particularly important for linear systems (free particles and harmonic oscillators), occurs when $y(t)$ is related to $x(t')$ by a time-stationary linear filter $[Y_{tt'}(x(t'))=g(t-t')x(t');\ \Delta_t=\Delta$ for all $t]$:

$$y(t)=\int_{-\infty}^{\infty}dt'\,g(t-t')x(t')=\int_{t-\Delta}^{t}dt'\,g(t-t')x(t')\;. \tag{1.7}$$

Here $g(t-t')$ is a "filter function" satisfying $g(t-t')=0$ if $t-t'<0$ or $t-t'>\Delta$. A good example, which often occurs in practice, is the case where one wishes to detect a periodic signal in $x(t')$; then one uses a filter function designed to pick out the desired Fourier component within some bandwidth.

To analyze real measurements—not just the theoretical construct of instantaneous measurements—requires a quantum-mechanical formulation for a sequence of samplings of $y(t)$. The objective of this paper is to develop and interpret such a formulation—at first sight a difficult task indeed. The machinery of the standard description, applied to the system alone, fails when confronted with measurements distributed in time. A sampling of $y(t)$ provides information about $x(t')$ at various times t' in the past—perhaps a continuous interval of times t'. How calculate, from a wave function, probabilities for various results? Use which wave function at which time? Or somehow average over times t'? How collapse the wave function to be consistent with the result of a sampling? Collapse which wave function at which time? Or somehow collapse the wave function a little bit at each time t'—perhaps continuously? How evolve a wave function? Use what combination of unitary evolution and wave-function collapse? These questions are manifestations of the wave function's tie to time; the quantum state of a system at some time contains information—all information—about the system at that time, but only at that time. To use Schrödinger's term, the wave function is an "expectation catalog"[14]—a catalog of possibilities—for all possible measurements on a system at a particular time. The concept of a quantum state is matched to a description of instantaneous measurements—it arises for precisely that description—but it is woefully ill suited to measurements distributed in time. Put the conclusion baldly: the language of a system quantum state just will not work for samplings of $y(t)$.

One way out of these difficulties would be to abandon temporarily a description in terms of the system alone. Model an apparatus capable of measuring $y(t)$, and analyze quantum-mechanically the measuring apparatus coupled to the system. Such a measurement model in hand, one might then develop a more abstract formulation in terms of the system alone—in much the same way one might abstract the standard description from a model for samplings of $x(t)$. Should there be a more abstract formulation for samplings of $y(t)$, however, one is perhaps just as likely to guess it directly from fundamental considerations, without going to the trouble of developing and analyzing a measurement model. The purpose of a measurement model would be to check one's guess.

Another approach—a way to generate a guess—is to abandon the language of a system quantum state evolving in time in favor of a more general language—the language of paths. Discard the tools of the standard description; they are no longer useful. Hew closely to the path-integral formulation (1.1); then one sees how to generalize to samplings of $y(t)$.

Consider a sequence of Q samplings of $y(t)$ at times $t_1,\ldots,t_Q$ $(t_1<t_2<\cdots<t_Q)$; the Q sampled quantities are given by[15]

$$y(t_q)=\mathscr{Y}_{t_q}[x(t)]=\int_{t_q-\Delta_{t_q}}^{t_q} dt\, Y_{t_q t}(x(t))\;, \qquad q=1,\ldots,Q \tag{1.8}$$

[cf. Eq. (1.6)]. Let t_0 be a time preceding all times that contribute to the sampled quantities ($t_0<t_q-\Delta_{t_q}$, for $q=1,\ldots,Q$), and let $\psi_0(x,t_0)$ be the wave function of the system at time t_0. Account for the imprecision of the samplings by introducing a resolution amplitude $\Upsilon(\bar{y}-y)$, the amplitude to obtain the value $\bar{y}$ as the result of a sampling of $y(t)$, given that $y(t)$ has the value y. Define a probability amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$, the joint amplitude for the Q samplings to yield the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$ and for the system to have position x at time t_Q. Write this fundamental joint amplitude as a path integral[16]

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int_{t_0}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\;, \tag{1.9a}$$

which has the same form as Eq. (1.1a) and which reduces to Eq. (1.1a) when $y(t)=x(t)$. For each path $x(t)$ on the interval $[t_0,t_Q]$, $y(t_q)$ in Eq. (1.9a) is evaluated using Eq. (1.8). Finally, derive from Φ the joint probability distribution to obtain the sequence $\bar{y}_1,\ldots,\bar{y}_Q$ as the results of the samplings:

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=\int dx\;|\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)|^2 \tag{1.9b}$$

[cf. Eq. (1.1b)]. Interpretation of Eqs. (1.9) is word for word the same as the interpretation given Eqs. (1.1)—except that in the resolution amplitudes quantities characteristic of $y(t)$ replace quantities characteristic of position.

Form the same. Interpretation the same. But consequences profoundly different. A sampling of $x(t)$ provides information about the system at a particular time; the resolution amplitude $\Upsilon(\bar{x}_q-x(t_q))$ depends only on a path's value at the time of the qth sampling. The consequences are an unrestricted sum over paths between samplings, corresponding to unitary evolution, and a restriction of the sum at each sampling time, corresponding to instantaneous wave-function collapse. Thus find the key to decomposing the path integral (1.1a) into the machinery of the standard description—a system quantum state that undergoes unitary evolution between samplings, punctuated by wave-function collapse at each sampling. A sampling of $y(t)$ probes the history of the system's motion; the resolution amplitude $\Upsilon(\bar{y}_q-y(t_q))$ depends on a path's behavior at various times—perhaps a continuous interval of times—leading up to time t_q. There is, in general, no period of unrestricted sum over paths, no demarcation between unitary evolution and wave-function collapse, no decomposition of the path integral (1.9a) into a system quantum state evolving in time. For samplings of $x(t)$, choice between the standard description and the path-integral formulation is a matter of taste. For samplings of $y(t)$, choice disappears: the path-integral formulation cannot be broken down into the conventional machinery of nonrelativistic quantum mechanics.

Why is the language of paths more powerful than the language of quantum states? If one speaks the language of quantum states, one's questions are restricted to information contained in a quantum state, and that information is restricted to instants of time. If one speaks the language of paths, one's questions may deal with the history of a system's motion. One may query each possible history individually: what is the amplitude for this result, given this path's entire history? The amplitude for each path, based on its entire history, is judged separately.

These views have been expressed forcefully by Aharonov and Albert.[17(a)] They note that the language of states is inadequate for measurements distributed in time—what they call measurements of "multiple-time observables"—if one restricts consideration to the system alone, and they point out that the more powerful language of paths allows a description of multiple-time observables which refers only to the system. Recently Aharonov, Albert, and D'Amato[17(b)] have given further consideration to the question of measuring multiple-time observables. Peres and Wootters[18] have also discussed finite-duration measurements in quantum mechanics. Mensky[19,20] has used path integrals to analyze various special cases of measurements distributed in time; although his approach is different from the formulation (1.9), his work stimulated the present discussion.

The remainder of the paper is organized as follows. Section II considers samplings of position: Sec. II A reviews the standard description of instantaneous measurements; Sec. II B revives a model for instantaneous position measurements—system coupled to a "measuring apparatus"—and obtains from it the same results as the standard description; Sec. II C shows that the standard description and the model lead to the path-integral formulation (1.1). The purpose of this exercise is to build confidence in the model. Section III considers samplings of $y(t)$: Sec. III A develops a model for samplings of $y(t)$—a straightforward generalization of the model for samplings of $x(t)$—and follows it to the path-integral formulation (1.9); Sec. III B discusses and elucidates the path-integral formulation; Sec. III C specializes to several cases of particular interest. In Sec. III C one discovers that for some examples of samplings of $y(t)$, the model of system coupled to measuring apparatus has a curious consequence: after interaction with the apparatus the system is left uncorrelated with the apparatus, its quantum state the same as if there had been no interaction; what has become of the disturbance that is reputed to accompany a measurement in quantum mechanics? Section III C discusses this curious situation. A brief Sec. IV offers concluding thoughts.

II. SAMPLINGS OF POSITION

A. Standard description of instantaneous measurements

Conventional nonrelativistic quantum mechanics provides a standard description[1–4] of an instantaneous measurement or a sequence of instantaneous measurements.[21] To formalize this description, consider a quantum-mechanical system with an observable A. I refer to the instantaneous measurements of A as samplings of A. Let $\hat{A}$ be the Hermitian operator associated with the observable A (throughout this paper operators are distinguished by a caret), assume for simplicity that $\hat{A}$ is nondegenerate, and let $|A\rangle$ denote the eigenstate of $\hat{A}$ with (real) eigenvalue A ($\hat{A}|A\rangle = A|A\rangle$). The eigenstates $|A\rangle$ form an orthonormal ($\langle A|A'\rangle=\delta_{AA'}$) basis for the Hilbert space of state vectors; completeness of the basis can be expressed as the resolution of the identity

$$\hat{1}=\sum_A |A\rangle\langle A| = \sum_A \hat{P}_A , \tag{2.1}$$

where

$$\hat{P}_A \equiv |A\rangle\langle A| \tag{2.2}$$

is the operator that projects onto $|A\rangle$. Notice that $\hat{P}_A^2=\hat{P}_A$. The operator $\hat{A}$ has the spectral resolution

$$\hat{A}=\sum_A A\,|A\rangle\langle A| = \sum_A A\hat{P}_A . \tag{2.3}$$

The system's quantum state at time t is described by a state vector[10] $|\psi(t)\rangle$, which has the decomposition

$$|\psi(t)\rangle = \sum_A \hat{P}_A|\psi(t)\rangle = \sum_A |A\rangle\langle A|\psi(t)\rangle ; \tag{2.4a}$$

normalization implies

$$1=\langle\psi(t)|\psi(t)\rangle=\sum_A |\langle A|\psi(t)\rangle|^2 . \tag{2.4b}$$

1. Single measurement

Consider first a single, arbitrarily precise sampling of A at time T. The standard description describes such samplings in terms of the projection operators $\hat{P}_A$ by using two suppositions:[1–3] (i) the probability to obtain the eigenvalue A as the result of the sampling is

$$P(A)=\langle\psi(T)|\hat{P}_A|\psi(T)\rangle = |\langle A|\psi(T)\rangle|^2 ; \tag{2.5a}$$

(ii) if the result of the sampling is the eigenvalue A, then the state of the system immediately afterward is

$$|\psi_A(T)\rangle = \frac{\hat{P}_A|\psi(T)\rangle}{[P(A)]^{1/2}} = |A\rangle\frac{\langle A|\psi(T)\rangle}{|\langle A|\psi(T)\rangle|} . \tag{2.5b}$$

Equation (2.5a) represents the probabilistic interpretation of the state vector; the sampling is arbitrarily precise because $P(A)$ depends only on the state vector of the system, with no further uncertainty due to some measuring apparatus. Equation (2.5b) is the usual collapse of the wave function (reduction of the state vector);[1] since the sampling is arbitrarily precise, knowledge of the result A tells one that the system is left immediately after the sampling in the state $|A\rangle$.

Turn now to a generalization of the above, which arises naturally to describe imprecise measurements. The generalization uses "effects" and "operations" in place of projection operators.[22] It is particularly appropriate for observables with a continuous spectrum of eigenvalues (see Sec. II B), for which an arbitrarily precise measurement is an unrealizable idealization. To deal with this generalization, let $\bar{A}$ label the possible results of a sampling of A. For each $\bar{A}$ posit an operator

$$\hat{\Upsilon}_{\bar{A}} = \sum_A \Upsilon_{\bar{A}A} | A \rangle\langle A | = \sum_A \Upsilon_{\bar{A}A} \hat{P}_A , \tag{2.6}$$

which is diagonal in the basis of eigenstates $| A \rangle$. The Hermitian operators

$$\hat{F}_{\bar{A}} \equiv \hat{\Upsilon}^\dagger_{\bar{A}} \hat{\Upsilon}_{\bar{A}} = \sum_A | \Upsilon_{\bar{A}A} |^2 \hat{P}_A \tag{2.7}$$

are assumed to satisfy a completeness condition

$$\hat{1} = \sum_{\bar{A}} \hat{F}_{\bar{A}} , \tag{2.8a}$$

which is equivalent to

$$1 = \sum_{\bar{A}} | \Upsilon_{\bar{A}A} |^2 . \tag{2.8b}$$

Each operator $\hat{F}_{\bar{A}}$ is an example of an effect.[4,22] For a sampling of A at time T, the standard description now relies on the following two suppositions:[4,22,23] (i) the probability to obtain the value $\bar{A}$ as the result of the sampling is

$$\begin{aligned} P(\bar{A}) &= \langle \psi(T) | \hat{F}_{\bar{A}} | \psi(T) \rangle \\ &= \sum_A | \Upsilon_{\bar{A}A} |^2 | \langle A | \psi(T) \rangle |^2 ; \end{aligned} \tag{2.9a}$$

(ii) if the result of the sampling is the value $\bar{A}$, then the state of the system immediately afterward is

$$| \psi_{\bar{A}}(T) \rangle = \frac{\hat{\Upsilon}_{\bar{A}} | \psi(T) \rangle}{[P(\bar{A})]^{1/2}} = \sum_A | A \rangle \frac{\Upsilon_{\bar{A}A} \langle A | \psi(T) \rangle}{[P(\bar{A})]^{1/2}} . \tag{2.9b}$$

Equation (2.9a) generalizes the usual probability (2.5a). The quantity $| \Upsilon_{\bar{A}A} |^2$ can be interpreted as a conditional probability, normalized by Eq. (2.8b); it is the probability to obtain $\bar{A}$ as the result of the sampling, given that the system has the eigenvalue A. It introduces imprecision into the sampling; one cannot predict the result with certainty even if $| \psi(T) \rangle = | A \rangle$. Equation (2.9b) generalizes the usual collapse of the wave function [Eq. (2.5b)]. Because of the imprecision of the sampling, the system is not left in an eigenstate of $\hat{A}$ immediately after the sampling. The projection-operator description of precise measurements is recovered by choosing $\hat{\Upsilon}_{\bar{A}} = \hat{P}_{\bar{A}}$, i.e., $\Upsilon_{\bar{A}A} = \delta_{\bar{A}A}$.

Even though the collapse (2.9b) takes pure states to pure states, it is useful to write it in terms of density operators. Let $\hat{\rho}(T) \equiv | \psi(T) \rangle\langle \psi(T) |$ be the system density operator just before the sampling, and let

$$\begin{aligned} \hat{\rho}_{\bar{A}}(T) &\equiv | \psi_{\bar{A}}(T) \rangle\langle \psi_{\bar{A}}(T) | \\ &= \hat{\Upsilon}_{\bar{A}} \hat{\rho}(T) \hat{\Upsilon}^\dagger_{\bar{A}} / P(\bar{A}) \end{aligned}$$

be the density operator just after. Define for each $\bar{A}$ a linear mapping on density operators:

$$\mathcal{F}_{\bar{A}}(\hat{\rho}) \equiv \hat{\Upsilon}_{\bar{A}} \hat{\rho} \hat{\Upsilon}^\dagger_{\bar{A}} = \sum_{A,A'} | A \rangle \Upsilon_{\bar{A}A} \langle A | \hat{\rho} | A' \rangle \Upsilon^*_{\bar{A}A'} \langle A' | . \tag{2.10}$$

Then Eqs. (2.9) can be rewritten as

$$P(\bar{A}) = \mathrm{tr}[\hat{\rho}(T) \hat{F}_{\bar{A}}] = \mathrm{tr}[\mathcal{F}_{\bar{A}}(\hat{\rho}(T))] , \tag{2.11a}$$

$$\hat{\rho}_{\bar{A}}(T) = \frac{\mathcal{F}_{\bar{A}}(\hat{\rho}(T))}{\mathrm{tr}[\mathcal{F}_{\bar{A}}(\hat{\rho}(T))]} . \tag{2.11b}$$

The mapping $\mathcal{F}_{\bar{A}}$ takes the density operator before the sampling to the density operator afterward (within normalization), given that $\bar{A}$ is the result of the sampling. Each map $\mathcal{F}_{\bar{A}}$ is an example of an operation.[4,22] The effect $\hat{F}_{\bar{A}}$ is associated with the operation $\mathcal{F}_{\bar{A}}$ by the relation $\mathrm{tr}(\hat{\rho}\hat{F}_{\bar{A}}) = \mathrm{tr}[\mathcal{F}_{\bar{A}}(\hat{\rho})]$. The operations considered here are of a very simple sort, called pure operations,[22] because they map pure states to (unnormalized) pure states. The discussion above can be generalized to more complicated operations—mixtures of pure operations—which produce mixed states from pure states,[22] but this generalization is not needed here.[24] Even in the case of pure operations, there are many operations that correspond to the same effect; the effect $\hat{F}_{\bar{A}}$ is determined by the magnitudes of the complex numbers $\Upsilon_{\bar{A}A}$ [Eq. (2.7)], whereas the operation $\mathcal{F}_{\bar{A}}$ depends on the phases of the $\Upsilon_{\bar{A}A}$ as well [Eq. (2.10)]. Put another way, the probability $P(\bar{A})$ is determined by the effect $\hat{F}_{\bar{A}}$ and, hence, depends only on the magnitudes $| \Upsilon_{\bar{A}A} |$, whereas the post-measurement state $\hat{\rho}_{\bar{A}}(T)$ is determined by the operation $\mathcal{F}_{\bar{A}}$ and, hence, depends on the amplitudes $\Upsilon_{\bar{A}A}$. A complete characterization of the imprecision of the sampling requires the amplitudes $\Upsilon_{\bar{A}A}$, not just the probabilities $| \Upsilon_{\bar{A}A} |^2$.

How might one best understand the role of wave-function collapse in the standard description (2.9)? A starting point for understanding is a quantum-mechanical version of the usual relation between conditional probabilities:

$$| \langle A | \psi_{\bar{A}}(T) \rangle |^2 P(\bar{A}) = | \Upsilon_{\bar{A}A} |^2 | \langle A | \psi(T) \rangle |^2 \tag{2.12}$$

[Eq. (2.9b)]. Both sides of Eq. (2.12) are unquestionably the joint probability for the system to have eigenvalue A and for the sampling to yield the result $\bar{A}$, yet the relation is not trivial. The left-hand side refers to the situation just after the sampling; it expresses the joint probability as the product of a conditional probability $| \langle A | \psi_{\bar{A}}(T) \rangle |^2$—the probability that the system has eigenvalue A just after the sampling, given the result $\bar{A}$—and the probability $P(\bar{A})$ for the result $\bar{A}$. The right-hand side refers to the situation just before the sampling; it expresses the joint probability as the product of a conditional probability $| \Upsilon_{\bar{A}A} |^2$—the probability to get the result $\bar{A}$, given that the system has eigenvalue A—and the probability $| \langle A | \psi(T) \rangle |^2$ that the system has eigenvalue A just before the sampling. From Eq. (2.12) comes a quantum-mechanical version of Bayes's theorem:

$$\frac{|\langle A|\psi_{\bar{A}}(T)\rangle|^2}{|\langle A'|\psi_{\bar{A}}(T)\rangle|^2}=\frac{|\langle A|\psi(T)\rangle|^2|\Upsilon_{\bar{A}A}|^2}{|\langle A'|\psi(T)\rangle|^2|\Upsilon_{\bar{A}A'}|^2}\ . \tag{2.13}$$

When one uses the collapse (2.9b), one declares an interest in the consequences of a particular result $\bar{A}$—and not other results; questions after the sampling are to be conditional questions, conditioned on the result $\bar{A}$. Bayes's theorem (2.13) gives the answer to conditional questions about the relative probabilities of different eigenvalues just after the sampling. It says that the probability of A, given the result $\bar{A}$, is proportional to (normalization supplied later) the *a priori* probability of A—probability of A before the sampling—weighted by the probability to get result $\bar{A}$ given A:

$$|\langle A|\psi_{\bar{A}}(T)\rangle|^2\propto|\langle A|\psi(T)\rangle|^2|\Upsilon_{\bar{A}A}|^2\ . \tag{2.14}$$

This same conclusion would come from a classical Bayesian analysis, with one crucial difference: in a classical analysis one would not just infer information about the observable A after the sampling, but would also infer what was "really happening" before the sampling; in quantum mechanics no inference about the "real situation" before the sampling is possible.

These considerations acquire real power when generalized from probabilities to amplitudes. For that purpose, notice that $\Upsilon_{\bar{A}A}$ can be interpreted as the amplitude to obtain $\bar{A}$ as the result of the sampling, given that the system has the eigenvalue A (see Ref. 23 and Sec. II B 1). Then Eq. (2.9b) can be written as the amplitude analog of Eq. (2.12):

$$\langle A|\psi_{\bar{A}}(T)\rangle[P(\bar{A})]^{1/2}=\Upsilon_{\bar{A}A}\langle A|\psi(T)\rangle\ . \tag{2.15}$$

Both sides are the joint amplitude of A and $\bar{A}$, but the two sides refer to the situations before and after the sampling. From Eq. (2.15) comes an amplitude version of Bayes's theorem,

$$\frac{\langle A|\psi_{\bar{A}}(T)\rangle}{\langle A'|\psi_{\bar{A}}(T)\rangle}=\frac{\langle A|\psi(T)\rangle\Upsilon_{\bar{A}A}}{\langle A'|\psi(T)\rangle\Upsilon_{\bar{A}A'}}\ , \tag{2.16}$$

which is the key to understanding collapse. It says that the amplitude of A, given the result $\bar{A}$, is proportional to the *a priori* amplitude of A—amplitude of A before the sampling—weighted by the amplitude to get result $\bar{A}$ given A:

$$\langle A|\psi_{\bar{A}}(T)\rangle\propto\langle A|\psi(T)\rangle\Upsilon_{\bar{A}A}\ . \tag{2.17}$$

This amplitude relation describes how collapse takes into account the information acquired in a sampling of A. It comes from applying a Bayesian analysis to amplitudes in the same way one usually applies a Bayesian analysis to probabilities.

2. *Sequence of measurements*

To describe a sequence of samplings of A, add unitary evolution between samplings to the description (2.9) of a single sampling. The samplings are made at times t_q ($q=1,2,\ldots$) where $t_q\geq t_{q-1}$. Label the possible results of the qth sampling by $\bar{A}_q$. At some initial time $t_0<t_1$, let the state of the system be $|\psi_0(t_0)\rangle$. Consider now the first $q-1$ samplings with results $\bar{A}_1,\ldots,\bar{A}_{q-1}$. The state of the system just after the $(q-1)$th sampling depends, in general, on the results of all previous samplings, so denote it by $|\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_{q-1})\rangle$. Then the state just before the qth sampling is given by

$$|\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_q)\rangle=\hat{U}(t_q,t_{q-1})|\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_{q-1})\rangle\ , \tag{2.18}$$

where $\hat{U}(t,t')$ is the system's unitary evolution operator. Equation (2.9a) provides the conditional probability to obtain $\bar{A}_q$ as the result of the qth sampling, given the results of previous samplings:

$$P(\bar{A}_q|\bar{A}_1,\ldots,\bar{A}_{q-1})=\langle\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_q)|\hat{F}_{\bar{A}_q}|\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_q)\rangle\ . \tag{2.19}$$

Equation (2.9b) provides the state just after the qth sampling:

$$|\psi_{\bar{A}_1\cdots\bar{A}_q}(t_q)\rangle=\frac{\hat{\Upsilon}_{\bar{A}_q}|\psi_{\bar{A}_1\cdots\bar{A}_{q-1}}(t_q)\rangle}{[P(\bar{A}_q|\bar{A}_1,\ldots,\bar{A}_{q-1})]^{1/2}}\ . \tag{2.20}$$

Iterating Eqs. (2.18) and (2.20) yields

$$|\psi_{\bar{A}_1\cdots\bar{A}_q}(t_q)\rangle=\frac{\left[\prod_{r=1}^{q}\hat{\Upsilon}_{\bar{A}_r}\hat{U}(t_r,t_{r-1})\right]|\psi_0(t_0)\rangle}{[P(\bar{A}_1,\ldots,\bar{A}_q)]^{1/2}}\ , \tag{2.21}$$

where the product in the numerator is ordered with increasing times in the evolution operators on the left, and

$$P(\bar{A}_1,\ldots,\bar{A}_q)=\prod_{r=1}^{q}P(\bar{A}_r|\bar{A}_1,\ldots,\bar{A}_{r-1}) \tag{2.22}$$

is the joint probability for the sequence of results $\bar{A}_1,\ldots,\bar{A}_q$. That the state (2.21) is normalized implies[24]

$$P(\bar{A}_1,\ldots,\bar{A}_q)=\left\langle\psi_0(t_0)\right|\left[\prod_{r=1}^{q}\hat{\Upsilon}_{\bar{A}_r}\hat{U}(t_r,t_{r-1})\right]^{\dagger}\left[\prod_{r=1}^{q}\hat{\Upsilon}_{\bar{A}_r}\hat{U}(t_r,t_{r-1})\right]\left|\psi_0(t_0)\right\rangle\ . \tag{2.23}$$

This form for the joint probability is trivially normalized by the condition (2.8a).

Equation (2.23) is an example of a famous formula, first given by Wigner[3,5,6] in the case of measurements that can be described by projection operators (see Ref. 4 for the case of general operations). Explicit reference to wave-function collapse disappears from Eq. (2.23). What the standard description ultimately produces is a method for computing statistical correlations among the samplings in a sequence, with no explicit reference to collapse.[3,5,6]

Within the standard description, wave-function collapse is the formal device which takes into account the information acquired in a sampling and, hence, which describes correlations among successive samplings. Everett[25] realized that collapse disappears from questions about joint statistics, and he dispensed with it in the many-worlds interpretation of quantum mechanics.[13,25]

Perhaps the most important property of the joint probability (2.23) follows from a reduction in the number of samplings: the joint probability for the first $q-1$ samplings, defined as always by

$$P(\bar{A}_1,\ldots,\bar{A}_{q-1})\equiv\sum_{\bar{A}_q}P(\bar{A}_1,\ldots,\bar{A}_q)\;, \tag{2.24a}$$

has the form (2.23) with q replaced by $q-1$. This reduction property is obvious from the procedure leading to Eq. (2.23); it follows formally from Eq. (2.8a). The property conforms to the usual notion of time ordering; it means that the joint probability for a sequence of samplings is independent of the existence of subsequent samplings. Moreover, it allows the joint probability (2.23) to be constructed as a product of conditional probabilities [Eq. (2.22)], one for each sampling, *each independent of the existence of subsequent samplings.* The conditional probability (2.19), given as always by

$$P(\bar{A}_q\mid\bar{A}_1,\ldots,\bar{A}_{q-1})=\frac{P(\bar{A}_1,\ldots,\bar{A}_q)}{P(\bar{A}_1,\ldots,\bar{A}_{q-1})}\;, \tag{2.24b}$$

is clearly independent of the existence of subsequent samplings. This important reduction property does not hold, in general, for samplings of $y(t)$ (see Sec. III B).

B. Model for samplings of position

In this subsection I review a model for position measurements which goes back to von Neumann.[1] The system under consideration is a one-dimensional quantum-mechanical system with position x, momentum p, and Hamiltonian $\hat{H}$. The operators $\hat{x}$ and $\hat{p}$ satisfy the canonical commutation relation $[\hat{x},\hat{p}]=i\hbar$, and the δ-function normalized eigenstates of $\hat{x}$ I denote by $|x\rangle$. To model an instantaneous measurement of x, use the canonical model of the quantum theory of measurement: first couple the system to a "measuring apparatus" at some instant of time, the coupling producing a correlation such that some property of the measuring apparatus acquires information about x at that instant; then apply the projection-operator description (2.5) to a "readout" (arbitrarily precise measurement) on the measuring apparatus. The model incorporates automatically imprecision due to the measuring apparatus, and it leads[22,23] naturally to the standard description (2.9) involving effects and operations. Reviewing this model builds confidence in it, so that a generalization can be used without qualms in Sec. III A to model samplings of $y(t)$.

1. Single sampling

The objective here is to model a single sampling of position at time T. For that purpose one needs a single measuring apparatus, here modeled as a one-dimensional quantum-mechanical system with canonical coordinate $\bar{x}$ and canonical momentum $\bar{p}$. I refer to this measuring system as a "meter"; it can be regarded as the first stage of a genuine macroscopic measuring apparatus. The operators $\hat{\bar{x}}$ and $\hat{\bar{p}}$ satisfy $[\hat{\bar{x}},\hat{\bar{p}}]=i\hbar$, the δ-function normalized eigenstates of $\hat{\bar{x}}$ I denote by $|\bar{x}\rangle$, and for convenience the self-Hamiltonian of the meter is assumed to be unity, so that $\bar{x}$ and $\bar{p}$ are conserved in the absence of interactions. The interaction between system and meter is described by an interaction Hamiltonian $\delta(t-T)\hat{x}\hat{\bar{p}}$; this coupling of the meter to $\hat{x}$, localized in time by the δ function, ensures that the meter coordinate acquires information about the position of the system at time T. The total Hamiltonian is given by

$$\hat{H}_{\rm tot}=\hat{H}+\delta(t-T)\hat{x}\hat{\bar{p}}\;. \tag{2.25}$$

To avoid confusion, let me emphasize that in this subsection a superposed bar designates meter quantities; since the value of the meter coordinate will be the result of the sampling, this notation dovetails nicely with that of Sec. II A.

The first step in the measurement process specifies uncorrelated states for system and meter just before the interaction at time T. Denote the system state by $|\psi(T)\rangle$, with corresponding wave function $\psi(x,T)=\langle x\mid\psi(T)\rangle$. Prepare the meter in a state $|\Upsilon\rangle$, with wave function $\Upsilon(\bar{x})=\langle\bar{x}\mid\Upsilon\rangle$. Hence, the total state of the system plus meter just before time T is $|\Psi_i\rangle=|\Upsilon\rangle\otimes|\psi(T)\rangle$, with wave function $\langle\bar{x},x\mid\Psi_i\rangle=\Upsilon(\bar{x})\psi(x,T)$.

The second step takes into account the interaction $\delta(t-T)\hat{x}\hat{\bar{p}}$. The effect of the interaction is to produce a sudden change at time T, so that the total quantum state just after time T is given by[26]

$$\begin{aligned}|\Psi_f\rangle&=e^{-(i/\hbar)\hat{x}\hat{\bar{p}}}|\Psi_i\rangle\\&=\int dx\, e^{-(i/\hbar)x\hat{\bar{p}}}|\Upsilon\rangle\otimes|x\rangle\psi(x,t)\;,\end{aligned} \tag{2.26}$$

where the latter equality follows from

$$e^{-(i/\hbar)\hat{x}\hat{\bar{p}}}=\int dx\;|x\rangle e^{-(i/\hbar)x\hat{\bar{p}}}\langle x|\;. \tag{2.27}$$

By using the displacement property

$$\langle\bar{x}\,|\,e^{-(i/\hbar)x\hat{\bar{p}}}|\Upsilon\rangle=\langle\bar{x}-x\mid\Upsilon\rangle=\Upsilon(\bar{x}-x)\;, \tag{2.28}$$

one finds the wave function corresponding to $|\Psi_f\rangle$:

$$\begin{aligned}\langle\bar{x},x\mid\Psi_f\rangle&=\Upsilon(\bar{x}-x)\psi(x,T)\\&=\langle x\mid\Upsilon(\bar{x}-\hat{x})\,|\,\psi(T)\rangle\;.\end{aligned} \tag{2.29}$$

The interaction does indeed produce a displacement of the meter coordinate by x. This correlation of the meter coordinate with the position of the system does not, however, constitute a measurement. For that, proceed to the final step of the measurement process.

The final step is the readout of the meter coordinate, which is an arbitrarily precise, instantaneous measurement of the meter coordinate, made by subsequent stages of the macroscopic measuring apparatus. The readout is described by the projection-operator formalism (2.5) applied to the total state $|\Psi_f\rangle$. The probability distribution to obtain the value $\bar{x}$ as the result of the readout is

$$P(\bar{x})=\int dx\ |\langle\bar{x},x\,|\,\Psi_f\rangle|^2$$
$$=\int dx\ |\Upsilon(\bar{x}-x)|^2|\psi(x,T)|^2$$
$$=\langle\psi(T)\,|\,[\Upsilon(\bar{x}-\hat{x})]^\dagger\Upsilon(\bar{x}-\hat{x})\,|\,\psi(T)\rangle \qquad (2.30a)$$

[cf. Eq. (2.9a)]; this probability distribution describes an imprecise measurement, with $|\Upsilon(\bar{x}-x)|^2$ interpreted as the conditional probability distribution to obtain the result $\bar{x}$, given that the system is at position x. The state of the system just after time T, if the result of the readout is the value $\bar{x}$, is obtained by projecting $|\Psi_f\rangle$ onto the eigenstate $|\bar{x}\rangle$, tracing out the meter, and normalizing [equivalently, by evaluating the total wave function (2.29) at $\bar{x}$ and normalizing]; the result is the state

$$|\psi_{\bar{x}}(T)\rangle=\Upsilon(\bar{x}-\hat{x})\,|\,\psi(T)\rangle/[P(\bar{x})]^{1/2} \qquad (2.30b)$$

[cf. Eq. (2.9b)], with wave function

$$\psi_{\bar{x}}(x,T)=\langle x\,|\,\psi_{\bar{x}}(T)\rangle$$
$$=\Upsilon(\bar{x}-x)\psi(x,T)/[P(\bar{x})]^{1/2}\,. \qquad (2.31)$$

As far as the system is concerned, this entire process, summarized by Eqs. (2.30), is equivalent to the standard description (2.9) of a single sampling of $x(T)$. Equations (2.30) are identical to Eqs. (2.9), with $\Upsilon(\bar{x}-\hat{x})$ playing the role of $\hat{\Upsilon}_{\bar{A}}$. The Hermitian operators $\hat{F}_{\bar{x}}\equiv[\Upsilon(\bar{x}-\hat{x})]^\dagger\Upsilon(\bar{x}-\hat{x})$ are effects (strictly speaking, an "effect-valued measure"[4,22]), satisfying $\hat{1}=\int d\bar{x}\,\hat{F}_{\bar{x}}$; the mappings

$$\mathscr{F}_{\bar{x}}(\hat{\rho})\equiv\Upsilon(\bar{x}-\hat{x})\hat{\rho}[\Upsilon(\bar{x}-\hat{x})]^\dagger$$

are corresponding operations (strictly speaking, an "operation-valued measure"[4,22]). In terms of initial and final system density operators, $\hat{\rho}(T)\equiv|\psi(T)\rangle\langle\psi(T)|$ and $\hat{\rho}_{\bar{x}}(T)\equiv|\psi_{\bar{x}}(T)\rangle\langle\psi_{\bar{x}}(T)|$, one can write

$$P(\bar{x})=\mathrm{tr}[\hat{\rho}(T)\hat{F}_{\bar{x}}]=\mathrm{tr}[\mathscr{F}_{\bar{x}}(\hat{\rho}(T))]$$

and

$$\hat{\rho}_{\bar{x}}(T)=\mathscr{F}_{\bar{x}}(\hat{\rho}(T))/\mathrm{tr}[\mathscr{F}_{\bar{x}}(\hat{\rho}(T))]$$

[cf. Eqs. (2.11)]. Everett[25] pointed out that this model of a position measurement includes imprecision, and he showed that it cannot be described by projection operators applied to the system alone.

The most natural interpretation of this sampling of position comes from noting that the total wave function $\langle\bar{x},x\,|\,\Psi_f\rangle=\Upsilon(\bar{x}-x)\psi(x,T)$ is the joint amplitude for the readout to yield the value $\bar{x}$ and for the system to be at position x. Thus one may interpret $\Upsilon(\bar{x}-x)$ as a conditional probability amplitude—the amplitude for a sampling to yield the result $\bar{x}$, given that the system is at x. Within the standard description, the conditional amplitude $\Upsilon(\bar{x}-x)$, which I call the resolution amplitude, characterizes the irresolution or imprecision of the measuring apparatus. It is useful here to review its roles. First, $P(\bar{x})$ is obtained by integrating the corresponding conditional probability distribution $|\Upsilon(\bar{x}-x)|^2$ over $|\psi(x,T)|^2$ [Eq. (2.30a)]. The upshot is imprecision; the result of the sampling cannot be predicted with certainty even if $|\psi(x,T)|^2$ is a δ function. Why square and then integrate instead of integrating and then squaring? Because the position of the system is potentially observable by an independent measurement. Second, the system is not left in an eigenstate of $\hat{x}$ after a sampling [Eq. (2.30b)], but has its post-sampling wave function $\psi_{\bar{x}}(x,T)$ spread by $\Upsilon(\bar{x}-x)$ [Eq. (2.31)]. Note again that the post-sampling wave function depends on the resolution amplitude $\Upsilon(\bar{x}-x)$, whereas the probability distribution $P(\bar{x})$ depends only on the absolute square $|\Upsilon(\bar{x}-x)|^2$. Finally, recall the Bayesian analysis of Sec. II A 1, which describes how wave-function collapse takes into account the information acquired in a sampling. The amplitude for the system to be at x, given a sampling with result $\bar{x}$, is proportional to the *a priori* amplitude of x—amplitude of x in the absence of a sampling—weighted by the resolution amplitude:

$$\psi_{\bar{x}}(x,T)\propto\psi(x,T)\Upsilon(\bar{x}-x) \qquad (2.32)$$

[cf. Eq. (2.17)].

2. *Sequence of samplings*

To analyze a sequence of samplings of $x(t)$, extend the above model by adding unitary evolution between samplings. The ultimate results of the model must duplicate the results of Sec. II A 2, obtained using the standard description, but here these results emerge without use of wave-function collapse. Consider then a sequence of samplings of position at times t_q ($q=1,2,\ldots$) where $t_q>t_{q-1}$. Each sampling requires a separate meter with canonical coordinate $\bar{x}_q$ and canonical momentum $\bar{p}_q$ ($[\hat{\bar{x}}_q,\hat{\bar{p}}_r]=i\hbar\delta_{qr}$). Each meter is coupled to the system at the appropriate time, so the total Hamiltonian for the first q samplings is

$$\hat{H}_{\mathrm{tot}}=\hat{H}+\sum_{r=1}^{q}\delta(t-t_r)\hat{x}\,\hat{\bar{p}}_r \qquad (2.33)$$

(unity self-Hamiltonian for the meters).

At some initial time $t_0<t_1$, let the state of the system be $|\psi_0(t_0)\rangle$, with wave function $\psi_0(x,t_0)=\langle x\,|\,\psi_0(t_0)\rangle$, and prepare the rth meter in the state $|\Upsilon_r\rangle$, with wave function $\Upsilon(\bar{x}_r)=\langle\bar{x}_r\,|\,\Upsilon_r\rangle$ (all meter wave functions the same). The initial state of the system plus q meters is

$$|\Psi(t_0)\rangle=|\Upsilon_1\rangle\otimes\cdots\otimes|\Upsilon_q\rangle\otimes|\psi_0(t_0)\rangle\,, \qquad (2.34a)$$

with wave function

$$\langle\bar{x}_1,\ldots,\bar{x}_q,x\,|\,\Psi(t_0)\rangle=\left[\prod_{r=1}^{q}\Upsilon(\bar{x}_r)\right]\psi_0(x,t_0)\,. \qquad (2.34b)$$

The unitary evolution operator corresponding to the total Hamiltonian (2.33) can be written in the forms

$$\hat{U}_{\rm tot}(t_q+,t_0)=\prod_{r=1}^{q} e^{-(i/\hbar)\hat{x}\hat{p}_r}\hat{U}(t_r,t_{r-1})$$

$$=\hat{U}(t_q,t_0)\left[\mathbf{T}\prod_{r=1}^{q} e^{-(i/\hbar)\hat{x}(t_r)\hat{p}_r}\right], \quad (2.35)$$

where t_q+ denotes a limit to t_q from above, and $\hat{U}(t,t')$ is the system's evolution operator (Hamiltonian $\hat{H}$). In the first form for $\hat{U}_{\rm tot}(t_q+,t_0)$, the product is ordered with increasing times in $\hat{U}(t,t')$ on the left. In the second form, $\mathbf{T}$ denotes time ordering (increasing times on the left) of the interaction-picture operators

$$\hat{x}(t)\equiv\hat{U}^{\dagger}(t,t_0)\hat{x}\hat{U}(t,t_0) . \quad (2.36)$$

The state of the system plus q meters just after time t_q is

$$|\Psi(t_q+)\rangle=\hat{U}_{\rm tot}(t_q+,t_0)|\Psi(t_0)\rangle , \quad (2.37a)$$

with wave function

$$\langle\bar{x}_1,\ldots,\bar{x}_q,x\,|\,\Psi(t_q+)\rangle\equiv\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q) , \quad (2.37b)$$

which is the joint amplitude just after time t_q for the meter coordinates to have values $\bar{x}_1,\ldots,\bar{x}_q$ and for the system to be at x.

More broadly, $\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q)$ can be interpreted as the joint amplitude for the first q samplings to yield the sequence of results $\bar{x}_1,\ldots,\bar{x}_q$ and for the system to be at position x at time t_q. Using Eqs. (2.27) and (2.28), one can write Φ as

$$\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q)=\left\langle x\left|\left[\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x})\hat{U}(t_r,t_{r-1})\right]\right|\psi_0(t_0)\right\rangle$$

$$=\left\langle x\left|\hat{U}(t_q,t_0)\left[\mathbf{T}\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x}(t_r))\right]\right|\psi_0(t_0)\right\rangle , \quad (2.38a)$$

where the comments about time ordering following Eq. (2.35) apply here as well. The joint probability distribution to obtain the results $\bar{x}_1,\ldots,\bar{x}_q$ for the first q samplings can now be derived as

$$P(\bar{x}_1,\ldots,\bar{x}_q)=\int dx\,|\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q)|^2$$

$$=\left\langle\psi_0(t_0)\left|\left[\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x})\hat{U}(t_r,t_{r-1})\right]^{\dagger}\left[\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x})\hat{U}(t_r,t_{r-1})\right]\right|\psi_0(t_0)\right\rangle$$

$$=\left\langle\psi_0(t_0)\left|\left[\mathbf{T}\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x}(t_r))\right]^{\dagger}\left[\mathbf{T}\prod_{r=1}^{q}\Upsilon(\bar{x}_r-\hat{x}(t_r))\right]\right|\psi_0(t_0)\right\rangle . \quad (2.38b)$$

This model makes no reference to wave-function collapse, but it yields the same joint probability distribution as does the standard description [cf. Eq. (2.23)], with $\hat{\Upsilon}_{\bar{A}_r}$ replaced by $\Upsilon(\bar{x}_r-\hat{x})$.

C. Path-integral formulation

Consider now a sequence of Q samplings. The fundamental quantity is the amplitude $\Phi(\bar{x}_1,\ldots,\bar{x}_Q;x,t_Q)$ of Eq. (2.38a); it was introduced in Sec. I [Eq. (1.1a)] as the joint amplitude for the sequence of results $\bar{x}_1,\ldots,\bar{x}_Q$ and for the system to be at x at time t_Q. The joint probability distribution $P(\bar{x}_1,\ldots,\bar{x}_Q)$ is derived from Φ according to Eq. (1.1b) [cf. Eq. (2.38b)]. By defining the (nonrelativistic) propagator for the system,

$$K(x,t\,|\,x',t')\equiv\langle x\,|\,\hat{U}(t,t')\,|\,x'\rangle , \quad (2.39)$$

the fundamental amplitude can be written as

$$\Phi(\bar{x}_1,\ldots,\bar{x}_Q;x_Q,t_Q)=\left\langle x_Q\left|\left[\prod_{q=1}^{Q}\Upsilon(\bar{x}_q-\hat{x})\hat{U}(t_q,t_{q-1})\right]\right|\psi_0(t_0)\right\rangle$$

$$=\int\left[\prod_{q=1}^{Q}\Upsilon(\bar{x}_q-x_q)K(x_q,t_q\,|\,x_{q-1},t_{q-1})dx_{q-1}\right]\psi_0(x_0,t_0) \quad (2.40)$$

(changing x to x_Q is just a label change, intended to make the right-hand side neater). This form for Φ has a simple interpretation. Begin with the product of propagators,

$$\prod_{q=1}^{Q} K(x_q,t_q \mid x_{q-1},t_{q-1}) ,$$

the amplitude for the system to assume the positions $x_1,\ldots,x_Q$ at the sampling times, given that it was at x_0 at time t_0. Multiply by the initial wave function to obtain the joint amplitude

$$\left[\prod_{q=1}^{Q} K(x_q,t_q \mid x_{q-1},t_{q-1})\right]\psi_0(x_0,t_0)$$

for the sequence of positions $x_0,x_1,\ldots,x_Q$. For each $q=1,\ldots,Q$, multiply by a resolution amplitude to find the joint amplitude

$$\left[\prod_{q=1}^{Q} \Upsilon(\bar{x}_q-x_q)K(x_q,t_q \mid x_{q-1},t_{q-1})\right]\psi_0(x_0,t_0)$$

for the sequence of results $\bar{x}_1,\ldots,\bar{x}_Q$ and for the sequence of positions $x_0,x_1,\ldots,x_Q$. Finally, integrate over unobservable quantities—i.e., integrate over the positions $x_0,x_1,\ldots,x_{Q-1}$—to obtain the probability amplitude (2.40).

Equation (2.40) can be translated immediately into the language of paths: use Eq. (1.3) to write the propagators as sums over paths, and then incorporate the integrals over $x_0,x_1,\ldots,x_{Q-1}$ into the sum over paths. The result is the path-integral formulation (1.1a) for Φ, whose interpretation follows Eqs. (1.1).

The path-integral formulation (1.1) nowhere invokes wave-function collapse, yet hidden within it must be a way of thinking in terms of collapse. To discover that way of thinking, a key idea is to ask conditional questions: pick a sampling—call it the qth—from among the total of Q samplings, and ask for its statistics, conditioned on the results of other samplings. For samplings of position, the joint statistics of the first q samplings are independent of the existence of later samplings; thus a natural question asks for the statistics of the qth sampling, given the results of previous samplings. Though natural, this is not the only conditional question one might ask; one could condition the statistics of the qth sampling on the results of subsequent samplings or on the results of all other samplings in the sequence of Q samplings. These more general questions, conditioned on the results of future as well as past measurements, have been investigated by Aharonov, Bergmann, and Lebowitz[27] (see also Ref. 17); I do not pursue them here.

Consider then the first q samplings, and think in terms of a conditional probability amplitude

$$\Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1}) ,$$

the amplitude for the qth sampling to yield the result $\bar{x}_q$ and for the system to be at x at time t_q, given the sequence of results $\bar{x}_1,\ldots,\bar{x}_{q-1}$ for the first $q-1$ samplings. In the language of the standard description (Sec. II A 2), this conditional amplitude can be written in terms of the system state $|\psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q)\rangle$ just before the qth sampling:

$$\Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1})=\Upsilon(\bar{x}_q-x)\langle x \mid \psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q)\rangle=\langle x \mid \Upsilon(\bar{x}_q-\hat{x}) \mid \psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q)\rangle . \tag{2.41}$$

Equation (2.19) supplies the conditional probability distribution,

$$\begin{aligned} P(\bar{x}_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1}) &= \langle \psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q) \mid [\Upsilon(\bar{x}_q-\hat{x})]^\dagger \Upsilon(\bar{x}_q-\hat{x}) \mid \psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q)\rangle \\ &= \int dx \, |\Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1})|^2 , \end{aligned} \tag{2.42}$$

the probability distribution to obtain $\bar{x}_q$ as the result of the qth sampling, given the previous results $\bar{x}_1,\ldots,\bar{x}_{q-1}$. By using Eqs. (2.18) and (2.21), one can write the conditional amplitude (2.41) in terms of the joint amplitude (2.38a):

$$\begin{aligned} \Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1}) &= \frac{\left\langle x \left| \left[\prod_{r=1}^{q} \Upsilon(\bar{x}_r-\hat{x})\hat{U}(t_r,t_{r-1})\right] \right| \psi_0(t_0)\right\rangle}{[P(\bar{x}_1,\ldots,\bar{x}_{q-1})]^{1/2}} \\ &= \frac{\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q)}{[P(\bar{x}_1,\ldots,\bar{x}_{q-1})]^{1/2}} . \end{aligned} \tag{2.43}$$

Equation (2.43) is actually a direct consequence of the definition of the conditional amplitude; it says that the conditional amplitude $\Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1})$ is proportional to the joint amplitude $\Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q)$, the normalization factor being the square root of the joint probability distribution for the first $q-1$ samplings [cf. Eqs. (2.24)]. Since one can always supply normalization at the end, it is convenient to write

$$\begin{aligned} \Phi(\bar{x}_q;x,t_q \mid \bar{x}_1,\ldots,\bar{x}_{q-1}) &\propto \Phi(\bar{x}_1,\ldots,\bar{x}_q;x,t_q) \\ &= \Upsilon(\bar{x}_q-x)\int_{t_0}^{(x,t_q)} \mathscr{D}x(t)\, e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\prod_{r=1}^{q-1}\Upsilon(\bar{x}_r-x(t_r)) , \end{aligned} \tag{2.44}$$

where Eq. (1.1a) provides the path integral.

Though the path-integral forms of the conditional and joint amplitudes are the same, their interpretations are different. Interpretation of the conditional amplitude (2.44) relies on the Bayesian analysis developed in Sec. II A 1. Begin with the amplitude $e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$, the *a priori* amplitude (amplitude in the absence of observations) for the path $x(t)$. For each $r=1,\ldots,q-1$, weight the *a priori* amplitude by a resolution amplitude $\Upsilon(\bar{x}_r-x(t_r))$ to take into account the information acquired in the rth sampling; thereby find (within normalization) the conditional amplitude

$$e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\prod_{r=1}^{q-1}\Upsilon(\bar{x}_r-x(t_r))$$

for the path $x(t)$, given the results $\bar{x}_1,\ldots,\bar{x}_{q-1}$ for the first $q-1$ samplings [cf. Eq. (2.32)]. Sum over all paths such that $x(t_q)=x$; thereby obtain (within normalization) the amplitude for the system to be at x at time t_q, given the previous results $\bar{x}_1,\ldots,\bar{x}_{q-1}$. In the language of the standard description (Sec. II A 2), this amplitude, proportional to the sum over paths in Eq. (2.44), is the system wave function $\langle x\mid\psi_{\bar{x}_1,\ldots,\bar{x}_{q-1}}(t_q)\rangle$ just before the qth sampling [cf. Eq. (2.41)]. Finally, multiply by the resolution amplitude $\Upsilon(\bar{x}_q-x)$ to obtain

$$\Phi(\bar{x}_q;x,t_q\mid\bar{x}_1,\ldots,\bar{x}_{q-1})$$

within normalization. Lack of normalization throughout this procedure is an inherent feature of a Bayesian analysis, which furnishes only relative amplitudes.

In this Bayesian analysis of Eq. (2.44), the resolution amplitude $\Upsilon(\bar{x}_r-x(t_r))$, for $r=1,\ldots,q-1$, restricts the sum over paths to be consistent with the result of the rth sampling. Because $\Upsilon(\bar{x}_r-x(t_r))$ depends only on a path's value at time t_r, the sum over paths is restricted only at the sampling times $t_1,\ldots,t_{q-1}$. Thus, from Eq. (2.44) emerges the picture painted by the standard description: between samplings an unrestricted sum over paths, corresponding to unitary evolution of a wave function; at each sampling a restriction of the sum, corresponding to an instantaneous disturbance of unitary evolution, the collapse of the wave function.

Restriction of the sum over paths, disturbance of unitary evolution, collapse of the wave function—all of these describe a single characteristic quantum-mechanical phenomenon, which is often described in still another way—as an inevitable "back-action disturbance" due to a system's interaction with a measuring apparatus. Returning to the model of Sec. II B 2, one notices that the resolution amplitudes in Eq. (2.44) do indeed arise from the system's interaction with the meters; for the first $q-1$ samplings they can be thought of as describing a back-action disturbance due to the interaction. In the language of the standard description, this same back-action disturbance appears in the guise of the collapse of the wave function. Instantaneous measurements have an important property: a given measurement feels a back-action disturbance only from previous measurements. In Eq. (2.44), which describes the qth sampling, this property shows up in that the sum over paths is restricted only by resolution amplitudes for the first $q-1$ samplings.

III. SAMPLINGS OF $y(t)$

A. Model for samplings of $y(t)$

In the case of samplings of $x(t)$, one can base an analysis on the standard description or on the model of Sec. II B. The two analyses give the same results, and both lead to the path-integral formulation (1.1). For samplings of $y(t)$, no standard description exists. One looks to a model to generate the path-integral formulation (1.9). In this subsection I develop a model for samplings of $y(t)$; it is a straightforward generalization of the model used for samplings of $x(t)$ in Sec. II B. The system considered here is the same as in Sec. II B: a one-dimensional system with position x, momentum p, and Hamiltonian $\hat{H}$. The approach here is also the same: couple the system to a "meter" in such a way that some property of the meter acquires information about $y(t)$ evaluated at a particular time. From the resulting model one abstracts the path-integral formulation for samplings of $y(t)$, in the same way one might abstract the formulation (1.1) from the model of Sec. II B.

Ultimately one wants to model a sequence of Q samplings of $y(t)$ at times $t_1,\ldots,t_Q$ ($t_1<t_2<\cdots<t_Q$). In this subsection I write the sampled quantities $y(t_q)$ as discrete-time sums; i.e., the integral in Eq. (1.8) is approximated as a sum of contributions from a set of discrete times:

$$y(t_q)=\sum_{j=1}^{N_q}Y_{qj}(x(t_{qj}))\ . \tag{3.1}$$

Here Y_{qj} denotes a real function of position, and the times t_{qj} ($j=1,\ldots,N_q$) that contribute to $y(t_q)$ are temporally ordered within the interval $[t_q-\Delta_{t_q},t_q]$—i.e., $t_q-\Delta_{t_q}\le t_{q1}<\cdots<t_{qN_q}\le t_q$. I regard the discrete-time sum (3.1) as equivalent to the integral (1.8) in the sense that the integral can be approximated arbitrarily well by a discrete sum with times spaced sufficiently closely.

Before dealing with a sequence of samplings, consider first a single sampling.

1. Single sampling

The objective here is to model a single sampling of $y(t)$ at time T. One can drop the reference to the qth sampling in Eq. (3.1) and write the sampled quantity as

$$y(T)=\sum_{j=1}^{N}Y_j(x(T_j))\ , \tag{3.2}$$

where Y_j denotes a real function of position, and the times T_j satisfy $T-\Delta_T\le T_1<\cdots<T_N\le T_{N+1}\equiv T$. Required for the single sampling is a single meter, a one-dimensional system with canonical coordinate $\bar{y}$, canonical momentum $\bar{p}$, and unity self-Hamiltonian. The operators $\hat{\bar{y}}$ and $\hat{\bar{p}}$ obey $[\hat{\bar{y}},\hat{\bar{p}}]=i\hbar$, and the δ-function normalized eigenstates of $\hat{\bar{y}}$ I denote by $|\bar{y}\rangle$. In this subsection a

superposed bar designates meter quantities; the value of the meter coordinate will be the result of the sampling. The symbols $y(T)$ and y are reserved for the sampled quantity—i.e., the quantity (3.2) constructed from positions at the times T_j.

The interaction between system and meter is described by an interaction Hamiltonian

$$\hat{\bar{p}}\sum_{j=1}^{N}\delta(t-T_j)Y_j(\hat{x}) ,$$

which leads to a total Hamiltonian

$$\hat{H}_{\text{tot}}=\hat{H}+\hat{\bar{p}}\sum_{j=1}^{N}\delta(t-T_j)Y_j(\hat{x}) . \tag{3.3}$$

Why choose this form? Each term in the interaction acts on the meter like the single interaction term in the Hamiltonian (2.25). The jth term displaces the meter coordinate by $Y_j(x(T_j))$; the net effect of all the terms is to displace the meter coordinate by $y(T)$. The Hamiltonian (3.3) is chosen because it ensures that the meter coordinate acquires the desired information.

At some initial time $T_0<T_1$, specify uncorrelated states for system and meter. Let the system state be $|\psi_0(T_0)\rangle$, with wave function $\psi_0(x,T_0)=\langle x\,|\,\psi_0(T_0)\rangle$. Prepare the meter in a quantum state $|\Upsilon\rangle$, with wave function $\Upsilon(\bar{y})=\langle\bar{y}\,|\,\Upsilon\rangle$. The total initial quantum state is $|\Psi(T_0)\rangle=|\Upsilon\rangle\otimes|\psi_0(T_0)\rangle$, with wave function $\langle\bar{y},x\,|\,\Psi(T_0)\rangle=\Upsilon(\bar{y})\psi_0(x,T_0)$.

The evolution operator corresponding to the total Hamiltonian (3.3) has the forms

$$\begin{aligned}\hat{U}_{\text{tot}}(T+,T_0)&=\hat{U}(T,T_N)\prod_{j=1}^{N}e^{-(i/\hbar)Y_j(\hat{x})\hat{\bar{p}}}\hat{U}(T_j,T_{j-1})\\&=\hat{U}(T,T_0)\left[\mathbf{T}\prod_{j=1}^{N}e^{-(i/\hbar)Y_j(\hat{x}(T_j))\hat{\bar{p}}}\right] ,\end{aligned} \tag{3.4}$$

where $T+$ denotes a limit to T from above, and $\hat{U}(t,t')$ is the system's evolution operator. The first product in Eq. (3.4) is ordered with increasing times in $\hat{U}(t,t')$ on the left; in the second product, $\mathbf{T}$ time orders the interaction-picture operators

$$\hat{x}(T_j)\equiv\hat{U}^{\dagger}(T_j,T_0)\hat{x}\hat{U}(T_j,T_0) . \tag{3.5}$$

The total quantum state just after time T is

$$|\Psi(T+)\rangle=\hat{U}_{\text{tot}}(T+,T_0)\,|\,\Psi(T_0)\rangle . \tag{3.6}$$

The corresponding wave function,

$$\begin{aligned}\langle\bar{y},x\,|\,\Psi(T+)\rangle&=\langle\bar{y},x\,|\,\hat{U}_{\text{tot}}(T+,T_0)\,|\,\Psi(T_0)\rangle\\&\equiv\Phi(\bar{y};x,T) ,\end{aligned} \tag{3.7a}$$

is the fundamental amplitude introduced in Sec. I [Eq. (1.9a) with $Q=1$]—the joint amplitude for the sampling to yield the result $\bar{y}$ and for the system to be at x at time T. From this fundamental amplitude comes the probability distribution to obtain the value $\bar{y}$ as the result of the sampling:

$$P(\bar{y})=\int dx\,|\,\Phi(\bar{y};x,T)\,|^2 \tag{3.7b}$$

[cf. Eq. (1.9b) with $Q=1$].

Now construct for $\Phi(\bar{y};x,T)$ an explicit form by using in Eqs. (3.4) and (3.7a) the relation

$$e^{-(i/\hbar)Y_j(\hat{x})\hat{\bar{p}}}=\int dx_j\,|x_j\rangle e^{-(i/\hbar)Y_j(x_j)\hat{\bar{p}}}\langle x_j\,| , \tag{3.8}$$

the displacement property (2.28), and the definition (2.39) of the propagator. The desired form is

$$\Phi(\bar{y};x_{N+1},T)=\int\Upsilon\left[\bar{y}-\sum_{j=1}^{N}Y_j(x_j)\right]\left[\prod_{j=1}^{N+1}K(x_j,T_j\,|\,x_{j-1},T_{j-1})dx_{j-1}\right]\psi_0(x_0,T_0) \tag{3.9}$$

(recall that $T_{N+1}\equiv T$; the change of x to x_{N+1} is a label change which makes the right-hand side neater). The quantity

$$y=\sum_{j=1}^{N}Y_j(x_j)$$

in Eq. (3.9) one recognizes as the sampled quantity corresponding to the sequence of positions $x_1,\dots,x_N$ at times $t_1,\dots,t_N$ [cf. Eq. (3.2)]. The displaced meter wave function, $\Upsilon(\bar{y}-y)$, one calls on again to be a resolution amplitude—the conditional amplitude to get $\bar{y}$ as the result of the sampling of $y(T)$, given that $y(T)=y$. With these identifications Eq. (3.9) acquires a simple interpretation. Begin with the product

$$\left[\prod_{j=1}^{N+1}K(x_j,T_j\,|\,x_{j-1},T_{j-1})\right]\psi_0(x_0,T_0) ,$$

the joint amplitude for the sequence of positions $x_0,x_1,\dots,x_N,x_{N+1}$. Multiply by the resolution amplitude

$$\Upsilon\left[\bar{y}-\sum_{j=1}^{N}Y_j(x_j)\right] ,$$

the amplitude to obtain $\bar{y}$ as the result of the sampling, given the sequence of positions $x_1,\dots,x_N$; thereby find the joint amplitude for the sampling to yield the result $\bar{y}$ and for the sequence of positions $x_0,x_1,\dots,x_N,x_{N+1}$. Finally, integrate over the unobservable quantities $x_0,x_1,\dots,x_N$ to obtain the amplitude (3.9).

Translation of Eq. (3.9) into a path integral is immediate. Using Eq. (1.3) for the propagators, one finds that

$$\begin{aligned}\Phi(\bar{y};x,T)=\int_{T_0}^{(x,T)}&\mathscr{D}x(t)\,\Upsilon(\bar{y}-y(T))\\&\times e^{(i/\hbar)S[x(t)]}\psi_0(x(T_0),T_0) .\end{aligned} \tag{3.10}$$

This path-integral form for $\Phi(\bar{y};x,T)$ is identical to the formulation (1.9a) with $Q=1$. For each path $x(t)$ on the interval $[T_0,T]$, $y(T)$ in Eq. (3.10) is computed from Eq. (3.2). The total amplitude summed in Eq. (3.10) is the joint amplitude for the result $\bar{y}$ and for the path $x(t)$; summing over all paths $x(t)$ on the interval $[T_0,T]$ such that $x(T)=x$ yields the amplitude $\Phi(\bar{y};x,T)$.

2. *Sequence of samplings*

Generalize now to a model for a sequence of Q samplings of $y(t)$ [Eq. (3.1)]. For that purpose use Q meters, one for each sampling, with canonical coordinates $\bar{y}_q$ and canonical momenta $\bar{p}_q$ ($[\hat{\bar{y}}_q,\hat{\bar{p}}_r]=i\hbar\delta_{qr}$). Couple the meters to the system so that the total Hamiltonian is

$$\hat{H}_{\text{tot}}=\hat{H}+\sum_{q=1}^{Q}\hat{\bar{p}}_q\sum_{j=1}^{N_q}\delta(t-t_{qj})Y_{qj}(\hat{x})\ ; \tag{3.11}$$

for each meter the interaction with the system has the form used in Eq. (3.3) to model a single sampling.

At some initial time $t_0<t_{q1}$, for $q=1,\ldots,Q$, which precedes all times that contribute to the sampled quantities, let the state of the system be $|\psi_0(t_0)\rangle$, with wave function $\psi_0(x,t_0)=\langle x\,|\,\psi_0(t_0)\rangle$, and prepare the qth meter in a state $|\Upsilon_q\rangle$, with wave function $\Upsilon(\bar{y}_q)=\langle\bar{y}_q\,|\,\Upsilon_q\rangle$.[16] The total initial quantum state is

$$|\Psi(t_0)\rangle=|\Upsilon_1\rangle\otimes\cdots\otimes|\Upsilon_Q\rangle\otimes|\psi_0(t_0)\rangle\ , \tag{3.12a}$$

with wave function

$$\langle\bar{y}_1,\ldots,\bar{y}_Q,x\,|\,\Psi(t_0)\rangle=\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q)\right]\psi_0(x,t_0)\ . \tag{3.12b}$$

The evolution operator for the total Hamiltonian (3.11) has the form

$$\hat{U}_{\text{tot}}(t_Q+,t_0)=\hat{U}(t_Q,t_0)\left[\mathbf{T}\prod_{q=1}^{Q}\prod_{j=1}^{N_q}e^{-(i/\hbar)Y_{qj}(\hat{x}(t_{qj}))\hat{\bar{p}}_q}\right]\ , \tag{3.13}$$

where $\mathbf{T}$ time orders the interaction-picture operators (2.36). The total quantum state just after time t_Q is

$$|\Psi(t_Q+)\rangle=\hat{U}_{\text{tot}}(t_Q+,t_0)\,|\Psi(t_0)\rangle\ . \tag{3.14}$$

The corresponding wave function,

$$\langle\bar{y}_1,\ldots,\bar{y}_Q,x\,|\,\Psi(t_Q+)\rangle=\langle\bar{y}_1,\ldots,\bar{y}_Q,x\,|\,\hat{U}_{\text{tot}}(t_Q+,t_0)\,|\Psi(t_0)\rangle\equiv\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)\ , \tag{3.15}$$

is the fundamental amplitude introduced in Sec. I [Eq. (1.9a)]—the joint amplitude for the samplings to yield the sequence of results $\bar{y}_1,\ldots,\bar{y}_Q$ and for the system to be at x at time t_Q. The probability distribution $P(\bar{y}_1,\ldots,\bar{y}_Q)$ follows from Eq. (1.9b).

When one considers more than one sampling of $y(t)$, an important new possibility emerges—overlapping of the intervals $[t_q-\Delta_{t_q},t_q]$ for different q and consequent interleaving of times t_{qj} that contribute to different sampled quantities $y(t_q)$. Interleaving makes it inconvenient to write explicitly an expression for $\hat{U}_{\text{tot}}(t_Q+,t_0)$ [Eq. (3.13)] analogous to the first form in Eq. (3.4) or an expression for $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ [Eq. (3.15)] analogous to Eq. (3.9). To write such expressions requires a tedious specification of the temporal order of all the times t_{qj}. On the other hand, one can construct such expressions in imagination, and from the latter imagined expression one can proceed directly to the path-integral formulation (1.9a). The path-integral expression for $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ can be written conveniently because the path integral handles automatically questions of time ordering.

B. Path-integral formulation

Having developed a model that leads to the path-integral formulation in a discrete-time approximation, return now to the general formulation (1.9) for samplings of $y(t)$:[28]

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int_{t_0}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-y(t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\ , \tag{3.16a}$$

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=\int dx\ |\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)|^2\ . \tag{3.16b}$$

Recall that for each path $x(t)$, $y(t_q)$ in Eq. (3.16a) is evaluated using

$$y(t_q)=\mathscr{Y}_{t_q}[x(t)]=\int_{-\infty}^{\infty}dt\ Y_{t_qt}(x(t))=\int_{t_q-\Delta_{t_q}}^{t_q}dt\ Y_{t_qt}(x(t)) \tag{3.17}$$

[Eq. (1.8)]. In this subsection focus on the properties and interpretation of the formulation (3.16). At the same time, keep in mind the model, because some properties are transparent when viewed in terms of its intermediate results (3.13) and (3.15). A good example is the normalization of the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$. Not clear from the path-integral formulation (3.16), the normalization is obvious from Eq. (3.15), because $\hat{U}_{\text{tot}}(t_Q+,t_0)$ is unitary. Thus one can write immediately

$$\begin{aligned}\int d\bar{y}_1 \cdots d\bar{y}_Q P(\bar{y}_1,\ldots,\bar{y}_Q) &= \int d\bar{y}_1 \cdots d\bar{y}_Q dx\, |\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)|^2 \\ &= \int d\bar{y}_1 \cdots d\bar{y}_Q dx\, |\langle \bar{y}_1,\ldots,\bar{y}_Q,x\,|\Psi(t_0)\rangle|^2 \\ &= \left[\int d\bar{y}\, |\Upsilon(\bar{y})|^2\right]^Q \int dx\, |\psi_0(x,t_0)|^2 . \end{aligned} \tag{3.18}$$

Provided the resolution amplitude is normalized to unity (assumed throughout this paper), the normalization of the fundamental amplitude reflects the normalization of the initial wave function $\psi_0(x,t_0)$.

A minor point involves the upper time limit t_Q for the path integral (3.16a). It should be obvious, either from Eq. (3.16a) or from the discrete-time results (3.13) and (3.15), that the upper time limit can be pushed forward to a time $t>t_Q$, thereby yielding an amplitude

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t) = \int dx_Q\, K(x,t\,|\,x_Q,t_Q)\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x_Q,t_Q) , \tag{3.19}$$

from which the joint probability distribution follows just as in Eq. (3.16b).

A more substantial issue concerns the reduction property discussed at the end of Sec. II A 2. The joint probability distribution for the first $Q-1$ samplings, defined as always by

$$P(\bar{y}_1,\ldots,\bar{y}_{Q-1}) \equiv \int d\bar{y}_Q\, P(\bar{y}_1,\ldots,\bar{y}_Q) \tag{3.20}$$

[cf. Eq. (2.24a)], cannot in general be derived from Eqs. (3.16) with Q replaced by $Q-1$. The joint probability distribution for a sequence of samplings of $y(t)$ does not in general have the reduction property that applies to a sequence of instantaneous measurements. That it does not means that the joint statistics of a sequence of samplings of $y(t)$ can depend on the existence of subsequent samplings. Is there a violation of causality here? No, because the measurements are not instantaneous. The influence of a sampling of $y(t)$ extends over the interval $[t-\Delta_t,t]$ from which $y(t)$ acquires information about the position. Suppose the interval for the Qth sampling overlaps the interval for the $(Q-1)$th sampling—i.e., $t_Q-\Delta_{t_Q}<t_{Q-1}$; then one should not be surprised that the joint statistics for the first $Q-1$ samplings depend on the existence of the Qth sampling. In terms of the path integral (3.16a), the resolution amplitude $\Upsilon(\bar{y}_Q-y(t_Q))$ for the Qth sampling influences the sum over paths at times preceding t_{Q-1}. There is no violation of causality, because a decision to make the Qth sampling must be made before the $(Q-1)$th sampling is completed. On the other hand, if the Qth sampling does not overlap previous samplings, a reduction property should hold. In this case, i.e., $t_Q-\Delta_{t_Q}>t_{Q-1}$, split the sum over paths in Eq. (3.16a) into two parts with respect to time t_{Q-1}—a sum over paths on the interval $[t_0,t_{Q-1}]$ and a sum over paths on the interval $[t_{Q-1},t_Q]$:

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q) = \int_{t_{Q-1}}^{(x,t_Q)} \mathscr{D}x(t)\,\Upsilon(\bar{y}_Q-y(t_Q))e^{(i/\hbar)S[x(t)]}\Phi(\bar{y}_1,\ldots,\bar{y}_{Q-1};x(t_{Q-1}),t_{Q-1}) . \tag{3.21}$$

Here the integral denotes a sum over all paths $x(t)$ on the interval $[t_{Q-1},t_Q]$ such that $x(t_Q)=x$; the quantity $\Phi(\bar{y}_1,\ldots,\bar{y}_{Q-1};x,t_{Q-1})$ is defined by Eq. (3.16a) with Q replaced by $Q-1$ and contains the sum over paths on the interval $[t_0,t_{Q-1}]$. Notice that Eq. (3.21) looks like the path-integral expression for a single sampling [cf. Eq. (3.10)]; the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_{Q-1};x,t_{Q-1})$ for the first $Q-1$ samplings, regarded as a function of x alone, serves as initial wave function at time t_{Q-1}. Thus the normalization property (3.18) implies that the joint probability distribution (3.20) is given by

$$P(\bar{y}_1,\ldots,\bar{y}_{Q-1}) = \int dx\, |\Phi(\bar{y}_1,\ldots,\bar{y}_{Q-1};x,t_{Q-1})|^2 \tag{3.22}$$

[cf. Eq. (3.16b)]. Equation (3.22) is the desired reduction property for the case that the Qth sampling does not overlap the $(Q-1)$th sampling.

These ideas can be extended in the following way. Pick a particular sampling—call it the qth—from among the total of Q samplings. For samplings after the qth sampling ($r=q+1,\ldots,Q$), decompose the sampled quantity $y(t_r)$ [Eq. (3.17)] with respect to time t_q:

$$y(t_r)=y_<(t_r,t_q)+y_>(t_r,t_q) , \quad r=q+1,\ldots,Q , \tag{3.23}$$

$$y_<(t_r,t_q) \equiv \int_{-\infty}^{t_q} dt\, Y_{t_r t}(x(t)) , \tag{3.24a}$$

$$y_>(t_r,t_q) \equiv \int_{t_q}^{t_r} dt\, Y_{t_r t}(x(t)) . \tag{3.24b}$$

Here $y_<(t_r,t_q)$ is the contribution to $y(t_r)$ from times before t_q, and $y_>(t_r,t_q)$ is the contribution to $y(t_r)$ from times after t_q. One can characterize overlapping of samplings precisely by saying that for $r>q$ there is no overlap between the rth sampling and the qth sampling if and only if $y_<(t_r,t_q)=0$ for all paths $x(t)$. Now split the sum over paths in Eq. (3.16a) into two parts with respect to time t_q, thereby obtaining

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int d\bar{y}'_{q+1}\cdots d\bar{y}'_Q\int_{t_q}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{r=q+1}^{Q}\delta(\bar{y}'_r-\bar{y}_r+y_>(t_r,t_q))\right]e^{(i/\hbar)S[x(t)]}$$
$$\times\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x(t_q),t_q)\,, \tag{3.25}$$

where

$$\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q)\equiv\int_{t_0}^{(x,t_q)}\mathscr{D}x(t)\left[\prod_{r=1}^{q}\Upsilon(\bar{y}_r-y(t_r))\right]$$
$$\times\left[\prod_{r=q+1}^{Q}\Upsilon(\bar{y}'_r-y_<(t_r,t_q))\right]e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)\,. \tag{3.26a}$$

In Eq. (3.25) the sum over paths includes all paths $x(t)$ on the interval $[t_q,t_Q]$ such that $x(t_Q)=x$; similarly, in Eq. (3.26a) the sum over paths includes all paths $x(t)$ on the interval $[t_0,t_q]$ such that $x(t_q)=x$. A generalization of the normalization property (3.18) yields the joint probability distribution for the first q samplings:

$$P(\bar{y}_1,\ldots,\bar{y}_q)=\int d\bar{y}_{q+1}\cdots d\bar{y}_Q\,P(\bar{y}_1,\ldots,\bar{y}_Q)$$
$$=\int d\bar{y}'_{q+1}\cdots d\bar{y}'_Q dx\;|\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q)|^2\,. \tag{3.26b}$$

Equations (3.26) generalize the basic equations (3.16) to the case where one is interested in the joint statistics of the first q samplings in a sequence of Q samplings. One sees clearly that a sampling after the qth sampling influences the joint statistics of the first q samplings if and only if it overlaps the qth sampling.

Equations (3.26) can be derived directly using the model of Sec. III A 2. The amplitude (3.26a) is the total wave function for the system and the Q meters, evolved to time t_q+; it leads to the joint probability distribution (3.26b) in the obvious way. One can obtain the path-integral form (3.26a) for the total wave function at time t_q+ in the same way one obtains the path-integral form (3.16a) for the total wave function (3.15) at time t_Q+.

Interpretation of Eqs. (3.26) generalizes the interpretation given Eqs. (3.16). Select a path $x(t)$ on the interval $[t_0,t_q]$, and begin with the amplitude $e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0)$, the unconditioned amplitude for the path. For each $r=1,\ldots,q$, multiply by the resolution amplitude $\Upsilon(\bar{y}_r-y(t_r))$, the amplitude for the rth sampling to yield the value $\bar{y}_r$, given the path's value $y(t_r)$ for the sampled quantity; further, for each $r=q+1,\ldots,Q$, multiply by $\Upsilon(\bar{y}'_r-y_<(t_r,t_q))$, the amplitude for the rth sampling to yield the value $\bar{y}'_r$, given the partial information $y_<(t_r,t_q)$ available from the path up to time t_q; thereby find the joint amplitude for the sequence of results $\bar{y}_1,\ldots,\bar{y}_q$, for the sequence of "partial results" $\bar{y}'_{q+1},\ldots,\bar{y}'_Q$, and for the path $x(t)$. Sum over all paths such that $x(t_q)=x$ to obtain the amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q)$ [Eq. (3.26a)], the joint amplitude for the results $\bar{y}_1,\ldots,\bar{y}_q$, for the partial results $\bar{y}'_{q+1},\ldots,\bar{y}'_Q$, and for the system to be at x at time t_q. Finally, square and integrate over the partial results and the position of the system to obtain the joint probability distribution $P(\bar{y}_1,\ldots,\bar{y}_q)$ [Eq. (3.26b)].

Suppose now that the qth sampling overlaps no later samplings [$y_<(t_r,t_q)=0$, $y(t_r)=y_>(t_r,t_q)$, for $r=q+1,\ldots,Q$]. Then Eq. (3.26a) simplifies to

$$\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q)=\left[\prod_{r=q+1}^{Q}\Upsilon(\bar{y}'_r)\right]\Phi(\bar{y}_1,\ldots,\bar{y}_q;x,t_q)\,, \tag{3.27}$$

where $\Phi(\bar{y}_1,\ldots,\bar{y}_q;x,t_q)$ is the fundamental amplitude for the first q samplings, defined by Eq. (3.16a) with Q replaced by q. In this case the joint probability distribution (3.26b) for the first q samplings becomes

$$P(\bar{y}_1,\ldots,\bar{y}_q)=\int dx\;|\Phi(\bar{y}_1,\ldots,\bar{y}_q;x,t_q)|^2\,, \tag{3.28}$$

a result that generalizes the reduction property (3.22). Notice also that in this case Eq. (3.25) takes the form

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int_{t_q}^{(x,t_Q)}\mathscr{D}x(t)\left[\prod_{r=q+1}^{Q}\Upsilon(\bar{y}_r-y(t_r))\right]e^{(i/\hbar)S[x(t)]}\Phi(\bar{y}_1,\ldots,\bar{y}_q;x(t_q),t_q) \tag{3.29}$$

[cf. Eq. (3.21)], a form that looks like the path-integral expression for a sequence of $Q-q$ samplings, with the role of initial wave function at time t_q played by $\Phi(\bar{y}_1,\ldots,\bar{y}_q;x,t_q)$.

For samplings of position the path-integral formulation conceals a way of thinking in terms of wave-function collapse. Is there some similar way of thinking—some analog of wave-function collapse—buried within the path-integral formulation for samplings of $y(t)$? To investigate this question, adopt the approach used in Sec. III C: ask conditional questions, and apply to the conditional questions a Bayesian analysis. Consider then the qth sampling, and ask for its statistics, conditioned on the results of previous samplings. The required conditional probability distribution

$$P(\bar{y}_q \mid \bar{y}_1,\ldots,\bar{y}_{q-1}) = \frac{P(\bar{y}_1,\ldots,\bar{y}_q)}{P(\bar{y}_1,\ldots,\bar{y}_{q-1})} \tag{3.30}$$

[cf. Eqs. (2.24)] is the probability distribution to obtain $\bar{y}_q$ as the result of the qth sampling, given the previous results $\bar{y}_1,\ldots,\bar{y}_{q-1}$. Think in terms of an associated conditional probability amplitude

$$\Phi(\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q \mid \bar{y}_1,\ldots,\bar{y}_{q-1}) \equiv \frac{\Phi(\bar{y}_1,\ldots,\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q)}{[P(\bar{y}_1,\ldots,\bar{y}_{q-1})]^{1/2}} \tag{3.31}$$

[cf. Eq. (2.43)], from which the conditional probability distribution (3.30) can be derived as

$$P(\bar{y}_q \mid \bar{y}_1,\ldots,\bar{y}_{q-1}) = \int d\bar{y}'_{q+1}\cdots d\bar{y}'_Q dx \; |\Phi(\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q \mid \bar{y}_1,\ldots,\bar{y}_{q-1})|^2 \tag{3.32}$$

[Eq. (3.26b); cf. Eq. (2.42)]. It is convenient to use relative amplitudes (normalization supplied later) and to write the amplitude (3.31) as

$$\Phi(\bar{y}_q;\bar{y}'_{q+1},\ldots,\bar{y}'_Q,x,t_q \mid \bar{y}_1,\ldots,\bar{y}_{q-1}) \propto \int_{t_0}^{(x,t_q)} \mathscr{D}x(t)\,\Upsilon(\bar{y}_q - y(t_q)) \left[\prod_{r=q+1}^{Q} \Upsilon(\bar{y}'_r - y_<(t_r,t_q))\right] \times e^{(i/\hbar)S[x(t)]}\psi_0(x(t_0),t_0) \left[\prod_{r=1}^{q-1} \Upsilon(\bar{y}_r - y(t_r))\right] \tag{3.33}$$

[Eq. (3.26a); cf. Eq. (2.44)].

Interpretation of the conditional amplitude (3.33) is the same as that given the joint amplitude (3.26a), except that the resolution amplitudes $\Upsilon(\bar{y}_r - y(t_r))$ for samplings before the qth sampling ($r=1,\ldots,q-1$) require a Bayesian interpretation appropriate for conditional questions. As in the analogous expression (2.44) for samplings of $x(t)$, these resolution amplitudes weight the *a priori* amplitude for a path and thereby restrict the sum over paths to be consistent with the results of the first $q-1$ samplings. Each of these amplitudes produces a disturbance of the system—a back-action disturbance—which, in general, affects the qth sampling. Analogy with samplings of $x(t)$ might suggest viewing these resolution amplitudes as generalizing the notion of wave-function collapse to samplings of $y(t)$. There is, however, a crucial difference: the resolution amplitude $\Upsilon(\bar{y}_r - y(t_r))$, for $r=1,\ldots,q-1$, depends on a path's history; it restricts the sum over paths—it disturbs the system—not just at a single time, but at all times from which $y(t_r)$ acquires information about $x(t)$. In Eq. (3.33) one finds no sharp demarcation between unitary evolution and the disturbances produced by measurements.

Moreover, there is another difference between samplings of $y(t)$ and samplings of $x(t)$. Equation (3.33) presents a new phenomenon, absent from Eq. (2.44)—restriction of the sum over paths due to the qth and later samplings. In Eq. (3.33) the resolution amplitude $\Upsilon(\bar{y}_q - y(t_q))$ for the qth sampling restricts the sum over paths and produces a back-action disturbance by which the qth sampling disturbs itself. Nothing comparable occurs for samplings of $x(t)$, because the resolution amplitude $\Upsilon(\bar{x}_q - x)$ in Eq. (2.44) does not affect the sum over paths. If a sampling after the qth sampling ($r=q+1,\ldots,Q$) overlaps the qth sampling, then its resolution amplitude $\Upsilon(\bar{y}'_r - y_<(t_r,t_q))$ also restricts the sum over paths in Eq. (3.33); it produces a back-action disturbance of the qth sampling by a later sampling—a situation that cannot occur for samplings of $x(t)$. In Eq. (3.33) the restrictions on the sum over paths due to the qth and later samplings should not be thought of as some sort of generalized wave-function collapse, partly because they have no analog in samplings of $x(t)$, but more importantly because they do not have the Bayesian interpretation that goes with collapse.

Restriction of the sum over paths, disturbance of unitary evolution, back-action disturbance due to the act of measurement—these phrases apply to all the resolution amplitudes which appear in Eq. (3.33). Use these phrases, and discard the notion of wave-function collapse as not appropriate for measurements distributed in time.

All these considerations speak now to the same conclusion. For samplings of position, Eq. (2.44) breaks down into the conventional machinery of nonrelativistic quantum mechanics—a system wave function that undergoes unitary evolution, punctuated by wave-function collapses. For samplings of $y(t)$, Eq. (3.33) offers no way to introduce a system wave function evolving in time. The language of quantum states, confronted with measurements distributed in time, yields to the more powerful language of paths.

C. Special cases

In this subsection I examine several examples aimed at illustrating peculiar features of the path-integral formulation (3.16).

As an interesting first case, consider a sequence of Q samplings of velocity. For that purpose, choose as sampled quantity a two-point approximation to velocity:

$$y(t) = \frac{x(t) - x(t-\epsilon)}{\epsilon} . \tag{3.34}$$

Assume that $\epsilon > 0$ is small enough that during an interval of duration ϵ, the system acts essentially like a free particle; then a sampling of $y(t)$ can indeed be construed as a sampling of velocity. In addition, assume that the sampling times are sufficiently far apart that there is no overlap between samplings ($t_q-\epsilon>t_{q-1}$, for $q=1,\ldots,Q$).

An obvious question asks whether this sequence of velocity samplings is equivalent to a sequence of momentum samplings—or, rather, a sequence of samplings of p/μ, where μ is the mass of the system. The fundamental amplitude and joint probability distribution are, of course, given by the path-integral formulation (3.16), with $y(t_q)$ in Eq. (3.16a) evaluated using Eq. (3.34). Samplings of p/μ can be given an independent analysis, based on the standard description of instantaneous measurements (Sec. II A). Thus the question may be phrased more precisely: does the path-integral formulation (3.16) applied to the case (3.34) yield the same probability distribution as does the standard description applied to a sequence of samplings of p/μ? To investigate this question, use the model of Sec. III A 2 to write the fundamental amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)$ in a form that allows direct comparison with the standard description.

The case (3.34) fits the model nicely, because $y(t)$ has the discrete-time form (3.1), with $t_{qj}=t_q-(2-j)\epsilon$ ($j=1,2$; $N_q=2$) and $Y_{q1}(x)=-Y_{q2}(x)=-x/\epsilon$. The total evolution operator (3.13) can be written as

$$\hat{U}_{\rm tot}(t_Q+,t_0)=\prod_{q=1}^{Q}e^{-(i/\hbar)\hat{x}\hat{p}_q/\epsilon}\hat{U}(t_q,t_q-\epsilon)\times e^{(i/\hbar)\hat{x}\hat{p}_q/\epsilon}\hat{U}(t_q-\epsilon,t_{q-1})\ , \qquad (3.35)$$

where each factor in the product arises from a particular sampling, and the product is ordered with later samplings on the left. Now simplify each factor in the product. First, move the system evolution operator $\hat{U}(t_q,t_q-\epsilon)$ to the right by using the relation

$$\hat{U}(t_q,t_q-\epsilon)\hat{x}\hat{U}^\dagger(t_q,t_q-\epsilon)=\hat{x}-(\hat{p}/\mu)\epsilon\ , \qquad (3.36)$$

which restates and makes precise the assumption that the system acts like a free particle during the short time interval $[t_q-\epsilon,t_q]$. Second, combine the two exponentials in each factor by using the Baker-Campbell-Hausdorff relation: $e^{\hat{A}}e^{\hat{B}}=e^{\hat{A}+\hat{B}}e^{[\hat{A},\hat{B}]/2}$ if $\hat{A}$ and $\hat{B}$ commute with $[\hat{A},\hat{B}]$. The resulting form for the evolution operator is

$$\hat{U}_{\rm tot}(t_Q+,t_0)=\prod_{q=1}^{Q}e^{-(i/\hbar)(\hat{p}/\mu)\hat{p}_q}\hat{U}(t_q,t_{q-1})\times e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}\ , \qquad (3.37)$$

which looks much the same as the evolution operator (2.35) for a sequence of samplings of position, except for the presence of the meter operator $e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}$ in each factor. By plugging the evolution operator (3.37) into Eq. (3.15) for the total wave function and following the steps that lead from Eq. (2.35) to Eq. (2.38a), one finds for the fundamental amplitude

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\left\langle x\left|\left[\prod_{q=1}^{Q}\tilde{\Upsilon}(\bar{y}_q-\hat{p}/\mu)\hat{U}(t_q,t_{q-1})\right]\right|\psi_0(t_0)\right\rangle\ , \qquad (3.38a)$$

where the product is ordered with increasing times in $\hat{U}(t_q,t_{q-1})$ on the left, and

$$\tilde{\Upsilon}(\bar{y}_q)\equiv\langle\bar{y}_q|e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}|\Upsilon_q\rangle=\int d\bar{y}'_q\,\langle\bar{y}_q|e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}|\bar{y}'_q\rangle\Upsilon(\bar{y}'_q)\ . \qquad (3.38b)$$

The form (3.38a) for the fundamental amplitude is equivalent to the path-integral form (3.16a). From it comes a joint probability distribution [Eq. (3.16b)]

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=\left\langle\psi_0(t_0)\left|\left[\prod_{q=1}^{Q}\tilde{\Upsilon}(\bar{y}_q-\hat{p}/\mu)\hat{U}(t_q,t_{q-1})\right]^\dagger\left[\prod_{q=1}^{Q}\tilde{\Upsilon}(\bar{y}_q-\hat{p}/\mu)\hat{U}(t_q,t_{q-1})\right]\right|\psi_0(t_0)\right\rangle \qquad (3.38c)$$

[cf. Eq. (2.38b)]. Comparison with the standard description [Eq. (2.23)] shows that this is the probability distribution for a sequence of samplings of p/μ, with $\tilde{\Upsilon}(\bar{y}-y)$ playing the role of resolution amplitude, the amplitude for a sampling to yield the value $\bar{y}$, given that the system has momentum $p=\mu y$.

The conclusion of this exercise is somewhat surprising. A sequence of samplings of velocity with resolution amplitude $\Upsilon(\bar{y}-y)$ is equivalent to a sequence of samplings of momentum with a different resolution amplitude $\tilde{\Upsilon}(\bar{y}-y)$, related to $\Upsilon(\bar{y}-y)$ by Eq. (3.38b). The culprit is not hard to find; for each factor in $\hat{U}_{\rm tot}(t_Q+,t_0)$, it is the meter operator $e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}$, which in effect endows the qth meter with mass $\mu\epsilon^2$ during the interval $[t_q-\epsilon,t_q]$. Language used previously to describe this curious result holds that "a momentum coupling is equivalent to a velocity coupling plus a negative capacitance in the readout circuit."[29] Translation into present language goes as follows. To achieve a sequence of samplings of p/μ with resolution amplitude $\Upsilon(\bar{y}-y)$, give each meter negative mass $-\mu\epsilon^2$ for a time of duration ϵ; i.e., give the qth meter a self-Hamiltonian $-\hat{p}_q^2/2\mu\epsilon^2$ during the interval $[t_q,t_q-\epsilon]$. This self-Hamiltonian cancels the effect of the culprit operator $e^{-(i/\hbar)\hat{p}_q^2/2\mu\epsilon}$ in Eq. (3.37). Thus formulate the present language: a momentum sampling is equivalent to a velocity sampling plus a negative meter mass.

As a second example, consider the problem of detecting a force acting on a free particle. For that purpose, specialize the system to be a free particle with mass μ, exposed to a classical force $F(t)$; the corresponding system Hamiltonian is

$$\hat{H}=\hat{p}^2/2\mu-\hat{x}F(t)\ . \tag{3.39}$$

In addition, choose the sampled quantity to be

$$\begin{aligned} y(t)&=x(t)-2x(t-\tau)+x(t-2\tau)\\ &=[x(t)-x(t-\tau)]-[x(t-\tau)-x(t-2\tau)]\ , \end{aligned} \tag{3.40}$$

where $\tau>0$. In the language of Eq. (1.7), this choice for $y(t)$ corresponds to a time-stationary linear filter with filter function

$$g(t-t')=\delta(t-t')-2\delta(t-t'-\tau)+\delta(t-t'-2\tau)\ .$$

Why consider the form (3.39)? Because $y(t)/\tau^2$ is a three-point approximation to acceleration. Thus sampling of $y(t)$ gives direct information about $F(t)$ for frequencies $\nu\lesssim\tau^{-1}$. Indeed, Eq. (3.39) corresponds to the simplest experimental procedure for detecting a force. Unruh[30] has stressed that for analyses of force detection, one should consider experimental procedures which are directly sensitive to the force—not procedures sensitive to position or velocity.

For simplicity in this example, restrict consideration to a single sampling of $y(t)$ at time T. The path-integral formulation (3.10) provides the fundamental amplitude $\Phi(\bar{y};x,T)$ for a single sampling, but here focus instead on an equivalent form for $\Phi(\bar{y};x,T)$, which comes from the model of a single sampling in Sec. III A 1. The model applies directly to this example, because $y(T)$ has the discrete-time form (3.2), with $T_j=T-(3-j)\tau$ ($j=1,2,3$; $N=3$) and $Y_1(x)=-Y_2(x)/2=Y_3(x)=x$. For this case the total evolution operator (3.4) becomes

$$\begin{aligned} \hat{U}_{\rm tot}(T+,T_0)&=\hat{U}(T,T_0)e^{-(i/\hbar)\hat{x}(T_3)\hat{p}}e^{2(i/\hbar)\hat{x}(T_2)\hat{p}}e^{-(i/\hbar)\hat{x}(T_1)\hat{p}}\\ &=\hat{U}(T,T_0)\exp\{-(i/\hbar)[\hat{x}+(\hat{p}/\mu)(T_3-T_0)]\hat{\bar{p}}\}\exp\{2(i/\hbar)[\hat{x}+(\hat{p}/\mu)(T_2-T_0)]\hat{\bar{p}}\}\\ &\quad\times\exp\{-(i/\hbar)[\hat{x}+(\hat{p}/\mu)(T_1-T_0)]\hat{\bar{p}}\}e^{-(i/\hbar)\eta\hat{\bar{p}}}\ . \end{aligned} \tag{3.41}$$

For the system Hamiltonian (3.39), the interaction-picture operators (3.5) are given by

$$\begin{aligned} \hat{x}(T_j)&=\hat{U}^\dagger(T_j,T_0)\hat{x}\hat{U}(T_j,T_0)\\ &=\hat{x}+(\hat{p}/\mu)(T_j-T_0)+\xi(T_j,T_0)\ , \end{aligned} \tag{3.42}$$

where

$$\begin{aligned} \xi(t,t')&\equiv\int_{t'}^{t}du\,(t-u)F(u)/\mu\\ &=\int_{t'}^{t}du\int_{t'}^{u}dv\,F(v)/\mu \end{aligned} \tag{3.43}$$

is the classical motion induced by the force during the interval from time t' to time t [$\xi(t',t')=0=\dot{\xi}(t',t')$]. In the second part of Eq. (3.41), the quantity

$$\begin{aligned} \eta&\equiv\xi(T_3,T_0)-2\xi(T_2,T_0)+\xi(T_1,T_0)\\ &=\xi(T_3,T_1)-2\xi(T_2,T_1) \end{aligned} \tag{3.44}$$

is the classical value of the sampled quantity $y(T)$ [cf. Eq. (3.40)]; the second form for η shows explicitly that it depends only on what the force does during the interval $[T_1,T_3]$. Now simplify Eq. (3.41) by using the Baker-Campbell-Hausdorff relation to combine the exponentials; thereby find the surprising result

$$\hat{U}_{\rm tot}(T+,T_0)=\hat{U}(T,T_0)e^{-(i/\hbar)(\hat{\bar{p}}^2/2\mu)2\tau}e^{-(i/\hbar)\eta\hat{\bar{p}}}\ . \tag{3.45}$$

Surprising because $\hat{U}_{\rm tot}(T+,T_0)$ factors into a product of the system evolution operator $\hat{U}(T,T_0)$ and a meter operator $e^{-(i/\hbar)(\hat{\bar{p}}^2/2\mu)2\tau}e^{-(i/\hbar)\eta\hat{\bar{p}}}$. The total quantum state at time $T+$ [Eq. (3.6)] similarly factors:

$$|\Psi(T+)\rangle=|\Phi\rangle\otimes|\psi(T)\rangle\ , \tag{3.46a}$$

$$|\psi(T)\rangle\equiv\hat{U}(T,T_0)|\psi_0(T_0)\rangle\ , \tag{3.46b}$$

$$|\Phi\rangle\equiv e^{-(i/\hbar)(\hat{\bar{p}}^2/2\mu)2\tau}e^{-(i/\hbar)\eta\hat{\bar{p}}}|\Upsilon\rangle\ . \tag{3.46c}$$

After the interaction, particle and meter are left uncorrelated. The particle proceeds along its way in precisely the state $|\psi(T)\rangle$ it would have been in had there been no interaction, and the meter is in a state $|\Phi\rangle$ that is independent of the particle's initial state. All this does not mean that the interaction has no effect. During the interval $[T_1,T_3]$, particle and meter are correlated. Although the particle eventually escapes the interaction bearing no trace, the meter bears the interaction's imprint in two ways: (i) the operator $e^{-(i/\hbar)\eta\hat{\bar{p}}}$ displaces the meter coordinate by η, the quantity one wants to determine; (ii) the operator $e^{-(i/\hbar)(\hat{\bar{p}}^2/2\mu)2\tau}$ changes the meter state as though the meter had mass μ during the interval $[T_1,T_3]$.

The total wave function corresponding to the state (3.46)—i.e., the fundamental amplitude (3.7a)—has the form

$$\Phi(\bar{y};x,T)=\langle\bar{y},x\,|\,\Psi(T+)\rangle=\Phi(\bar{y})\psi(x,T)\ , \tag{3.47a}$$

$$\begin{aligned} \psi(x,T)&\equiv\langle x\,|\,\psi(T)\rangle\\ &=\int dx_0\,K(x,T\,|\,x_0,T_0)\psi_0(x_0,T_0)\ , \end{aligned} \tag{3.47b}$$

$$\begin{aligned} \Phi(\bar{y})&\equiv\langle\bar{y}\,|\,\Phi\rangle\\ &=\int d\bar{y}'\,\langle\bar{y}\,|\,e^{-(i/\hbar)(\hat{\bar{p}}^2/2\mu)2\tau}\,|\,\bar{y}'\rangle\Upsilon(\bar{y}'-\eta)\ . \end{aligned} \tag{3.47c}$$

The probability distribution (3.7b) to obtain $\bar{y}$ as the result of the sampling,

$$P(\bar{y})=\int dx\,|\Phi(\bar{y};x,T)|^2=|\Phi(\bar{y})|^2\ , \tag{3.47d}$$

is independent of the particle's initial state. This is undoubtedly the most felicitous feature of this example. No preparation of the particle's initial state is required; preparation is directed wholly at preparing the meter's initial state. This situation corresponds closely to what happens in real experiments to detect a force acting on some system: elaborate procedures are used to prepare the measuring apparatus; the system takes care of itself.

Contrast this example with the language used by the canonical model[1-4,22] of the quantum theory of measurement (see Sec. II B 1 and Ref. 23). There one imagines coupling the system to a measuring apparatus. The coupling produces a correlation between system and apparatus, so that some property of the apparatus is correlated with the desired information about the system. A "readout" of this property completes the measurement and disturbs the system—i.e., collapses the system's wave function. Essential to this language is correlation between system and measuring apparatus, as expressed in the total quantum state. No way can be found to fit the example considered here into the language of the canonical model; particle (system) and meter (measuring apparatus) are uncorrelated after their interaction. Within the language of the canonical model, no correlation means no possibility of a measurement. Yet the example has clear potential for a measurement, manifested in the displacement of the meter coordinate by η. Within the language of the canonical model, no correlation means no possibility to disturb the system; in the example, readout of the meter coordinate after time $T+$ can have no effect on the particle—cannot disturb it, cannot affect its wave function. Yet the example incorporates a back-action disturbance, manifested after the interaction in the operator $e^{-(i/\hbar)(\hat{p}^2/2\mu)2\tau}$, which spreads the meter wave function as though the meter had mass μ during the interval $[T_1,T_3]$. Recall now how these features are manifested in the path-integral formulation (3.10). There one looks to the resolution amplitude; its restriction of the sum over paths expresses both the potential for measurement and the back-action disturbance. Thus reinforce an argument made in Sec. III B. Drop the phrase "collapse of the wave function"; in this example it can have no meaning. Adopt the phrases "restriction of the sum over paths" and "back-action disturbance"; they apply to measurements distributed in time and characterize the role of the resolution amplitude in the path-integral formulation.

The preceding is not an isolated example, but rather exhibits properties that characterize a class of force-detection schemes for linear systems. To investigate these schemes, let the system be a simple harmonic oscillator with mass μ and frequency ω, exposed to a classical force $F(t)$. For a path $x(t)$ on the interval $[t_0,t_Q]$, the action is given by

$$S[x(t)]\equiv S_0[x(t)]+\int_{t_0}^{t_Q} dt\, x(t)F(t)\ , \tag{3.48a}$$

where

$$S_0[x(t)]\equiv\int_{t_0}^{t_Q} dt\, \tfrac{1}{2}\mu(\dot{x}^2-\omega^2x^2) \tag{3.48b}$$

is the action for a free (unforced) oscillator. Throughout this discussion let $x_c(t)$ denote a classical path for a *free* oscillator on the interval $[t_0,t_Q]$ $(\ddot{x}_c+\omega^2x_c=0)$. The space of all paths on the interval $[t_0,t_Q]$ can be divided into equivalence classes by defining two paths to be equivalent if they differ by a free classical path. Each equivalence class contains one path for every set of initial and final values. A class is specified uniquely by its element $\mathcal{X}(t)$ that satisfies $\mathcal{X}(t_0)=\mathcal{X}(t_Q)=0$; it consists of the paths $x(t)=x_c(t)+\mathcal{X}(t)$, where $x_c(t)$ runs over all free classical paths (arbitrary initial and final values). The specifying path $\mathcal{X}(t)$ can be regarded as a particular kind of deviation from free classical motion. A special feature of linear systems is that

$$S[x_c(t)+\mathcal{X}(t)]=S[x_c(t)]+S[\mathcal{X}(t)]\ . \tag{3.49}$$

Now choose the sampled quantity $y(t)=\mathcal{Y}_t[x(t')]$ [Eq. (1.6)] to have the property that it has the same value for all paths in the same equivalence class (one must assume $t>t_0+\Delta_t$):

$$\mathcal{Y}_t[x_c(t')+\mathcal{X}(t')]=\mathcal{Y}_t[\mathcal{X}(t')]\ . \tag{3.50a}$$

For convenience, also choose the sampled quantity to be zero for the equivalence class of free classical paths [specified by $\mathcal{X}(t)=0$]:

$$\mathcal{Y}_t[x_c(t')]=0\ . \tag{3.50b}$$

This sort of sampled quantity is clearly designed to reveal the presence of a force $F(t)$, since it is sensitive only to deviations from free classical motion. The simplest way to devise a sampled quantity of this sort is to use a time-stationary linear filter [Eq. (1.7)] whose filter function $g(t-t')$ has the property that its Fourier transform vanishes at frequency ω.

The fundamental amplitude (3.16a) for a sequence of Q samplings can be written in the form

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int dx_0\,\psi_0(x_0,t_0)\int_{(x_0,t_0)}^{(x,t_Q)}\mathcal{D}x(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-\mathcal{Y}_{t_q}[x(t)])\right]e^{(i/\hbar)S[x(t)]}\ , \tag{3.51}$$

where the sum over paths includes all paths $x(t)$ on the interval $[t_0,t_Q]$ such that $x(t_0)=x_0$ and $x(t_Q)=x$ (one from each equivalence class), and the dependence of the sampled quantity on the path $x(t)$ is indicated explicitly. Now use a trick[7] that works for linear systems: let $x_c(t)$ be the free classical path that satisfies $x_c(t_0)=x_0$ and $x_c(t_Q)=x$, and write each path $x(t)$ as $x(t)=x_c(t)+\mathcal{X}(t)$, where $\mathcal{X}(t)$ is the specifying path for the equivalence class of $x(t)$. Then, using Eqs. (3.49) and (3.50a), one can write the fundamental amplitude as

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\int dx_0\, e^{(i/\hbar)S[x_c(t)]}\psi_0(x_0,t_0)\int_{(0,t_0)}^{(0,t_Q)}\mathscr{D}\mathcal{X}(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-\mathscr{Y}_{t_q}[\mathcal{X}(t)])\right]e^{(i/\hbar)S[\mathcal{X}(t)]}, \tag{3.52}$$

where the sum over paths includes all specifying paths $\mathcal{X}(t)$—i.e., all paths on the interval $[t_0,t_Q]$ such that $\mathcal{X}(t_0)=\mathcal{X}(t_Q)=0$. For linear systems the propagator is given by[7]

$$K(x,t_Q\mid x_0,t_0)\propto e^{(i/\hbar)S[x_c(t)]} \tag{3.53}$$

[obvious from Eq. (3.52) in the absence of samplings]. Thus one can write Eq. (3.52) as

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q;x,t_Q)=\Phi(\bar{y}_1,\ldots,\bar{y}_Q)\psi(x,t_Q)\,, \tag{3.54a}$$

where

$$\psi(x,t_Q)=\int dx_0\, K(x,t_Q\mid x_0,t_0)\psi_0(x_0,t_0) \tag{3.54b}$$

is an oscillator wave function, and

$$\Phi(\bar{y}_1,\ldots,\bar{y}_Q)\propto\int_{(0,t_0)}^{(0,t_Q)}\mathscr{D}\mathcal{X}(t)\left[\prod_{q=1}^{Q}\Upsilon(\bar{y}_q-\mathscr{Y}_{t_q}[\mathcal{X}(t)])\right]e^{(i/\hbar)S[\mathcal{X}(t)]} \tag{3.54c}$$

(path integral not normalized) is a probability amplitude that is independent of the oscillator's initial state. The joint probability distribution (3.16b) for the Q samplings becomes

$$P(\bar{y}_1,\ldots,\bar{y}_Q)=|\Phi(\bar{y}_1,\ldots,\bar{y}_Q)|^2\,. \tag{3.54d}$$

This class of force-detection schemes for a harmonic oscillator shares with the preceding example the pleasant property that the sampling statistics are independent of the system's initial state. In these schemes the force reveals itself by the deviation it produces from free classical motion. How is the deviation to be judged? Which free classical path is to be held up as fiducial, relative to which the deviation is reckoned? Independence of initial state means that it matters not which free classical path is chosen as fiducial. In particular, one may always choose to judge the deviation relative to the path of an oscillator at rest at its equilibrium position. This possibility receives explicit expression in the amplitude (3.54c), where the fiducial free classical path is clearly that of an oscillator at rest.

Independence of initial state has a further important consequence for the case where the qth sampling overlaps no other samplings—earlier or later—in the sequence of Q samplings. In this case, repeated use of Eqs. (3.29) and (3.54) shows that the qth sampling is statistically independent of all the other samplings. Its statistics are determined by the absolute square of a probability amplitude

$$\Phi(\bar{y}_q)\propto\int_{(0,t_{q-1})}^{(0,t_q)}\mathscr{D}\mathcal{X}(t)\,\Upsilon(\bar{y}_q-\mathscr{Y}_{t_q}[\mathcal{X}(t)])\times e^{(i/\hbar)S[\mathcal{X}(t)]}\,, \tag{3.55}$$

where the integral denotes a sum over all paths $\mathcal{X}(t)$ on the interval $[t_{q-1},t_q]$ such that $\mathcal{X}(t_{q-1})=\mathcal{X}(t_q)=0$. One recognizes Eq. (3.55) as Eq. (3.54c) specialized to a single sampling. This result has an obvious generalization to a sequence of samplings, none of which overlaps samplings before or after the sequence.

These force-detection schemes share with the preceding example another important property. Factoring of the fundamental amplitude (3.54a) means no correlation after time t_Q between the oscillator state and the sampling statistics [cf. Eq. (3.47a)]. Just after time t_Q, one may regard the oscillator as having wave function $\psi(x,t_Q)$, the same wave function it would have had in the absence of samplings; the sampling statistics, contained in the amplitude $\Phi(\bar{y}_1,\ldots,\bar{y}_Q)$, are independent of the oscillator's state after time t_Q. Factoring means that none of these force-detection schemes can be fitted into the language used by the canonical model of the quantum theory of measurement. These schemes thus illustrate again the power of a path-integral formulation. The language of the canonical model fails not because the model is bad, but because the language is bad. The language is the language of quantum states, which severely restricts the notion of correlation. When the canonical model speaks of a correlation between system and measuring apparatus, it means a correlation at a particular time, embodied in the total quantum state at that time. In these force-detection schemes, one may say that the sampling statistics are correlated with the oscillator's behavior—correlated with what the oscillator was doing during the interval $[t_0,t_Q]$, but correlated not at all with the oscillator's behavior after time t_Q. This kind of correlation with past behavior is characteristic of measurements distributed in time. It finds its natural expression not in the language of quantum states, but in the more powerful language of paths.

IV. CONCLUSION

The only reality permitted by quantum mechanics is tied directly to the network of observations we make. No hidden reality lurks behind the observations. As Wheeler[31] puts it, "What we call 'reality' consists of an elaborate papier-mâché construction of imagination and theory fitted in between a few iron posts of observation." Quantum mechanics describes reality by giving statistical correlations among observations.[3,5,6] The path-integral formulation provides a route straight and true, leading directly to these correlations. An alternate, more circuitous route exists for instantaneous measurements; it constructs the same correlations by using the conventional machinery of quantum mechanics—a system quantum state evolving in time. How much simpler to go directly to the result. How much more elegant not to watch the conventional machinery clanking, its gears straining to crank out a simple result. For measurements distributed in time, not just simpler, not just more elegant—there is no machinery to clank, no gears to turn. The route to reality is the path-integral formulation.

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[1]J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), especially Chap. 6 [English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, New Jersey, 1955)].

[2]F. London and E. Bauer, *La théorie de l'observation en mécanique quantique* (Hermann, Paris, 1939) [English translation in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University, Princeton, New Jersey, 1983), p. 217].

[3]E. P. Wigner, Am. J. Phys. **31**, 6 (1963).

[4]References 1—3 deal with measurements described by projection operators. A more general description, better suited to measurements of position (or any observable with a continuous spectrum), uses "effects" and "operations" in place of projection operators (see Sec. II A and Ref. 22). This more general description is outlined succinctly by A. Barchielli, L. Lanz, and G. M. Prosperi, in *Foundations of Quantum Mechanics*, edited by S. Kamefuchi *et al.* (Physical Society of Japan, Tokyo, 1984), p. 165.

[5]E. P. Wigner, in *Contemporary Research in the Foundations and Philosophy of Quantum Mechanics*, edited by C. A. Hooker (Reidel, Dordrecht, 1973), p. 369.

[6]E. P. Wigner, in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University, Princeton, New Jersey, 1983), p. 260.

[7]R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965), especially Chaps. 1—3.

[8]More precisely, for a path $x(t)$ on the interval $[t_0,t_Q]$, $e^{(i/\hbar)S[x(t)]}$ is the joint amplitude for all the path's values $x(t)$ at times t such that $t_0<t\le t_Q$, given that the system is at $x(t_0)$ at time t_0.

[9]It should be clear that the wave function does not collapse to a δ function unless the resolution amplitude is a δ function.

[10]For simplicity, the discussion of the standard description in this paper is given in terms of pure states and wave functions; it could easily be generalized to mixed states and density matrices (see Ref. 24). The crucial feature of the standard description—that it relies on quantum states defined at each instant of time—has nothing to do with the distinction between pure states and mixed states.

[11]P. C. Marzio, *Rube Goldberg: His Life and Work* (Harper and Row, New York, 1973), especially Chap. 11.

[12]For a statement of one physicist's discomfort with wave-function collapse, see W. E. Lamb, in *The Centrality of Science and Absolute Values*, proceedings of the Fourth International Conference on the Unity of the Sciences (International Cultural Foundation, New York, 1975), p. 297, or W. E. Lamb, in *Science of Matter*, edited by S. Fujita (Gordon and Breach, New York, 1979).

[13]The disappearance of wave-function collapse from questions about joint statistics is the foundation for the many-worlds interpretation of quantum mechanics; see Ref. 25 and *The Many-Worlds Interpretation of Quantum Mechanics*, edited by B. S. DeWitt and N. Graham (Princeton University, Princeton, New Jersey, 1973).

[14]E. Schrödinger, Naturwissenschaften **23**, 807 (1935); **23**, 823 (1935); **23**, 844 (1935) [English translation by J. D. Trimmer, Proc. Am. Philos. Soc. **124**, 323 (1980)].

[15]The time t_q labeling the sampled quantity $y(t_q)$ [Eq. (1.8)] is only a label. There is no compelling choice for the time to label one of the sampled quantities; indeed, there is no compelling reason to use time as a label at all. The chosen time labeling, however, does satisfy a natural "causality" condition: $y(t_q)$ contains information about $x(t)$ only for times $t\le t_q$. This causal time labeling ensures—as must be true—that t_Q, the upper time limit for the path integral (1.9a), is greater than or equal to all times t that contribute to the sampled quantities $y(t_q)$.

[16]Implicit in this discussion is the assumption that the sampled quantities $y(t_q)$ are measured by apparatuses having identical properties. The path-integral formulation (1.9a) need not be restricted to this situation. The sampled quantities could be measured by apparatuses with different properties; moreover, the sampled quantities could be so different that their measurements would require apparatuses with different properties. One could handle this more general situation by introducing a different resolution amplitude $\Upsilon_q(\bar{y}_q-y_q)$ for each sampling.

[17](a) Y. Aharonov and D. Z. Albert, Phys. Rev. D **29**, 223 (1984); (b) Y. Aharonov, D. Z. Albert, and S. S. D'Amato, Phys. Rev. D **32**, 1975 (1985).

[18]A. Peres and W. K. Wootters, Phys. Rev. D **32**, 1968 (1985).

[19]M. B. Mensky, Phys. Rev. D **20**, 384 (1979).

[20]M. B. Mensky, Zh. Eksp. Teor. Fiz. **77**, 1326 (1979) [Sov. Phys. JETP **50**, 667 (1979)].

[21]The restriction to instantaneous measurements is a bit too severe. If the measured observable is conserved, then the standard description applies even to measurements of finite duration. This paper is primarily concerned with measurements that provide information about position, which is not conserved; then, as has been emphasized by Wigner (Refs. 3, 5, and 6), the standard description applies only to instantaneous measurements.

[22]Readers interested in effects and operations and their role in quantum mechanics are urged to consult the review by K.

Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).

[23]The description (2.9) arises from the canonical model of the quantum theory of measurement (Refs. 1–4 and 22): first the system is coupled to a "meter," the coupling producing a correlation, and then the projection-operator description (2.5) is applied to a "readout" (arbitrarily precise measurement) of some property of the meter (see Sec. II B 1 for the example of a position measurement). Let the meter have a complete, orthonormal set of states $|\bar{A}\rangle$. An initial, uncorrelated state $|\Lambda\rangle\otimes|\psi(T)\rangle=\sum_A|\Lambda\rangle\otimes|A\rangle\langle A|\psi(T)\rangle$, where $|\Lambda\rangle$ is a meter state, evolves during a short interaction time to

$$\sum_A|\Upsilon_A\rangle\otimes|A\rangle\langle A|\psi(T)\rangle=\sum_{\bar{A},A}|\bar{A}\rangle\otimes|A\rangle\Upsilon_{\bar{A}A}\langle A|\psi(T)\rangle ,$$

where $|\Upsilon_A\rangle=\sum_{\bar{A}}|\bar{A}\rangle\Upsilon_{\bar{A}A}$ is a meter state correlated with A (different $|\Upsilon_A\rangle$ are not necessarily orthogonal). Notice that $\Upsilon_{\bar{A}A}\langle A|\psi(T)\rangle$ is the joint amplitude of $\bar{A}$ and A; hence, $\Upsilon_{\bar{A}A}$ can be interpreted as the conditional amplitude of $\bar{A}$, given A. The probability to obtain $\bar{A}$ as the result of a readout on the meter is $P(\bar{A})=\sum_A|\Upsilon_{\bar{A}A}|^2|\langle A|\psi(T)\rangle|^2$ [cf. Eq. (2.9a)]. The state of the system after a readout yielding $\bar{A}$ is obtained by projecting onto $|\bar{A}\rangle$, tracing out the meter, and normalizing:

$$|\psi_{\bar{A}}(T)\rangle=\sum_A|A\rangle\Upsilon_{\bar{A}A}\langle A|\psi(T)\rangle/[P(\bar{A})]^{1/2}$$

[cf. Eq. (2.9b)].

[24]One way to generalize to mixed operations is to introduce into the description (2.11) a further "classical" irresolution. Let $\tilde{A}$ label the possible results of a sampling, and introduce a conditional probability $W(\tilde{A}\,|\,\bar{A})$—the probability to get $\tilde{A}$ as the result, given the value $\bar{A}$. Define a new set of operations $\mathscr{F}_{\tilde{A}}(\hat{\rho})\equiv\sum_{\bar{A}}W(\tilde{A}\,|\,\bar{A})\hat{\Upsilon}_{\bar{A}}\hat{\rho}\hat{\Upsilon}^{\dagger}_{\bar{A}}$ and associated effects $\hat{F}_{\tilde{A}}\equiv\sum_{\bar{A}}W(\tilde{A}\,|\,\bar{A})\hat{\Upsilon}^{\dagger}_{\bar{A}}\hat{\Upsilon}_{\bar{A}}$. Equations (2.11) are replaced by the following: (i) the probability to get $\tilde{A}$ as the result of the sampling is

$$P(\tilde{A})=\mathrm{tr}[\hat{\rho}(T)\hat{F}_{\tilde{A}}]=\mathrm{tr}[\mathscr{F}_{\tilde{A}}(\hat{\rho}(T))]$$
$$=\sum_{\bar{A}}W(\tilde{A}\,|\,\bar{A})P(\bar{A}) ;$$

(ii) the state of the system just after a sampling with result $\tilde{A}$ is

$$\hat{\rho}_{\tilde{A}}(T)=\mathscr{F}_{\tilde{A}}(\hat{\rho}(T))/P(\tilde{A})$$
$$=\sum_{\bar{A}}[W(\tilde{A}\,|\,\bar{A})P(\bar{A})/P(\tilde{A})]\hat{\rho}_{\bar{A}}(T) .$$

The operations $\mathscr{F}_{\tilde{A}}$ are mixed operations. Though not the most general operations, they are the most general that can be easily interpreted as describing a measurement of a single, conventional quantum-mechanical observable A. This description arises from the model in Ref. 23 by adding a second "meter" that "measures" $\bar{A}$ just as the first meter "measures" A. When this description is applied to a sequence of samplings, the resulting joint probability $P(\tilde{A}_1,\ldots,\tilde{A}_q)$ is related to the joint probability (2.23) in the obvious way.

[25]H. Everett, Rev. Mod. Phys. **29**, 454 (1957).

[26]Equation (2.26) can be derived as follows: (i) consider an interaction Hamiltonian $\hat{x}\hat{p}/\epsilon$, turned on from $t=T-\epsilon/2\equiv T_-$ to $t=T+\epsilon/2\equiv T_+$; (ii) obtain the total evolution operator $\hat{U}_{\mathrm{tot}}(t,T_-)$ for $T_-\le t\le T_+$ by solving

$$i\hbar d\hat{U}_{\mathrm{tot}}(t,T_-)/dt=(\hat{H}+\hat{x}\hat{p}/\epsilon)\hat{U}_{\mathrm{tot}}(t,T_-) ,$$

with initial condition $\hat{U}_{\mathrm{tot}}(T_-,T_-)=\hat{1}$; (iii) let ϵ go to zero. Physically, as ϵ goes to zero, the interaction becomes so strong that it dominates the system Hamiltonian $\hat{H}$, which can then be neglected. Formally, one can introduce a new time variable $\tau=(t-T)/\epsilon$, which varies from $\tau=-1/2$ to $\tau=+1/2$; letting ϵ go to zero, one finds the equation

$$i\hbar d\hat{U}_{\mathrm{tot}}(T+\epsilon\tau,T_-)/d\tau=\hat{x}\hat{p}\,\hat{U}_{\mathrm{tot}}(T+\epsilon\tau,T_-) ,$$

with solution $\hat{U}_{\mathrm{tot}}(T_+,T_-)=e^{-(i/\hbar)\hat{x}\hat{p}}$.

[27]Y. Aharonov, P. G. Bergmann, and J. L. Lebowitz, Phys. Rev. **134**, B1410 (1964).

[28]In addition to irresolution characterized by the resolution amplitude, one may introduce a further "classical" irresolution into the path-integral formulation (3.16) for samplings of $y(t)$. Let $\tilde{y}$ label the possible results of a sampling. Characterize the classical irresolution by a conditional probability distribution $W(\tilde{y}\,|\,\bar{y})$—the probability distribution to obtain the value $\tilde{y}$ as the result of a sampling, given the value $\bar{y}$. A joint probability distribution $P(\tilde{y}_1,\ldots,\tilde{y}_Q)$ can then be derived from Eq. (3.16b) in the obvious way. Including this classical irresolution is analogous to a procedure, sketched in Ref. 24, for going from pure operations (pure states) to mixed operations (mixed states) within the standard description.

[29]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).

[30]W. G. Unruh, in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1983), p. 637.

[31]J. A. Wheeler, in *Quantum Theory and Measurement*, edited by J. A. Wheeler and W. H. Zurek (Princeton University, Princeton, New Jersey, 1983), p. 182.