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Functional formulation of Regge gravity

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We consider the problem of quantizing gravity in a simplicial, Regge-calculus approach. Based on the idea of invariants, we derive the metric and the measure for integrating over Regge manifolds. These results are general; in two dimensions they exhibit the necessary agreement with the continuum (two-dimensional gravity and the Polyakov string model). This provides a basis for numerical Monte Carlo studies.

I. INTRODUCTION

In his original paper Regge[1] developed a simplicial discretization of gravity, formulated the Einstein-Hilbert action for discrete simplicial manifolds, and constructed the classical equations of motion. In this approach, the basic variables are the lengths of links (one-simplices), general discrete manifolds are built out of basic, flat simplices, and the curvature is represented by the appearance of deficit angles. Considerable work was done on approximating known solutions of general relativity in terms of Regge simplices (detailed references are given in earlier papers[2–9]).

At present, there is much interest in the quantum application of Regge calculus. There are a large number of important theoretical problems that can be approached through the Regge approximation, the most notable being, for example, the quantization of the lattice string,[2,3] study of higher-derivative gravitational theories,[4] simplicial minisuperspace,[5] general Monte Carlo investigations,[6] dynamical generation of symmetries,[9] and many others. In general, one would like to sum over discrete manifolds using a functional integral quantization.[10] This implies summing over topologies and distinct construction matrices, and most importantly one wants to integrate over the Regge lengths $\{L_i\}$ as dynamical variables. However, as yet no systematic quantum formulation has been given. Compared with the very successful Wilson lattice theory of non-Abelian Yang-Mills fields, there are some very basic issues in lattice gravity. They pertain to the origin of general coordinate symmetry and the form of the measure. Namely, in Yang-Mills theories, one has group elements on links $\{U(i,i+\hat{\mu})\}$ and the Wilson lattice construction possesses explicitly the gauge symmetry leading then to the functional integral

$$\int \prod dU\, e^{-S_W} \tag{1}$$

with the group-invariant Haar measure.

In Regge calculus the basic variables are the lengths of the bones $\{L_i\}$, their physical interpretation being the geodesic distance between vertices. The first basic question is then on the analogue of general coordinate transformations in Regge calculus. There are quite a few interpretations of this issue in the literature. For example, based on the phenomenon in flat space where the triangulations are manifestly redundant, it was suggested[7] that certain changes of edge lengths $\{L_i\}\to\{L_i'\}$ (which leave the action unchanged) correspond to general coordinate transformations. This means that one proposes the analogy $\{L_i\}\leftrightarrow\{U\}$. It is clear, though, that changes of lengths actually change the discrete manifold (with the exception of the already mentioned flat space). Approximate transformations were also suggested.[5,7] To continue along the line of exact symmetry one would need to argue that other invariant quantities besides the action possess the same symmetry and that there exists an invariant measure for integrations. At present, the simple form $\int \prod dL_i/L_i$ was used. Other, different interpretations on the form of general coordinate transformations were also discussed,[8,9] the last reference being most interesting with the original proposal that randomness is the origin of symmetry.

We shall in what follows present an interpretation of Regge calculus based on the idea of invariants and follow through with the corresponding approach to a functional integral and the measure. Basically, one considers summing over all different simplicial manifolds. In the present thinking, one would like to concentrate on the nonperturbative region, that means genuinely curved discrete manifolds. The magnitudes of lengths, $\{L_i\}$, together with the construction matrix then uniquely give the manifold in question. In general, here (except for some exceptions and obvious discrete symmetries), there are no symmetry transformations and the interpretation is that different lengths represent different manifolds. The lengths are therefore thought of as *invariants.* General coordinate symmetry *would* enter the picture only if one would attempt to construct a metric for the manifold $g_{\mu\nu}(x;L)$; it would come from arbitrariness associated with a construction.

Our analogy is therefore with the invariant variables in Yang-Mills theory. Those are the closed Wilson loops

$$\{L_i\}\leftrightarrow\{W_c\}\ . \tag{2}$$

In Yang-Mills theories, the set of loops is, in general, an overcomplete set, with the fact that for $N=\infty$ we have no redundancy (only the inequalities, analogous inequalities are present also in Regge calculus). Now in terms of

Wilson loops, the Euclidean functional integral Eq. (1) can be shown to become[11]

$$\int \prod_c dW_c J(W) e^{-S} \tag{3}$$

with a calculable, nontrivial measure J. It explicitly contains the gauge group volume and the Jacobian of transformations.

Therefore we come to think that quantized Regge gravity is to be formulated and the lengths integrated with a nontrivial measure

$$\int \prod d\mu(L) e^{-S[L]} \,. \tag{4}$$

This measure in a sense explicitly incorporates the relevance of the general coordinate group. Once we consider the premise that L_i's are invariants, the presence of the measure is natural: indeed, one can present an argument for the measure in general. It appears always when reducing a given system to invariants. The idea is to basically evaluate the effect of eliminating the variables coupled with the additional effect of a change of variables.

II. REDUCTION OF SYMMETRIES

We summarize now the general procedure for elimination of variables and reduction of a dynamical system to invariants. Elements of what follows appeared in various previous applications.[12–16] The general result we reach will be suitable for application to gravity and Regge calculus.

Consider a dynamical system $\{q_i(t)\}$ with a symmetry group G parametrized by $\{\alpha_a\}$. One can think of t as artificial time; the group parameters and the symmetry are time independent. Let the symmetry transformations $q_i \to f_i(q,\alpha)$ generate the orbits of G. Consider now a set of invariant variables $\{L_m(t)\}$ and the problem of specifying the measure for this set. One starts from a given metric product in terms of original (noninvariant) variables

$$(\dot q, \dot q) = \dot q^{\,i} G_{ij}(q) \dot q^{\,j} \tag{5}$$

and performs a separation through a change

$$q_i(t) = f_i[\bar q(L(t)), \alpha(t)] \,, \tag{6}$$

where $\alpha = \alpha(t)$ are now promoted to be dynamical and $\bar q = \bar q(L)$ is constructed from the invariants $\{L_m\}$. Clearly, for this reconstruction to be unique, one ought to supply a set of subsidiary conditions. We shall comment on these shortly. This issue was discussed in detail in Ref. 13. In the present case, we have the additional situation that the new variables are taken to be the invariants: this fact will allow an invariant final formula, the explicit construction of the gauge will be avoided. Now a direct substitution leads to the metric

$$(\dot q, \dot q) = \dot L_m (f_{,m} f_{,n}) \dot L_n + \dot\alpha_a (f_{,a} f_{,b}) \dot\alpha_b \tag{7}$$

with a self-evident derivative notation. Most importantly, one drops the cross term which would be of the form

$$\dot\alpha_a (f_{,a} f_{,m}) \dot L_m = \dot\alpha_a Q_a[\bar q] \,. \tag{8}$$

It would equal the charge associated with $\bar q$, but the gauge can be taken[13] that this vanishes. In the present case, this is even more appropriate since one does not expect to carry charge based on time dependence in the invariants. We therefore have a decoupling with symmetry and invariant variable metrics

$$G_{ab} = (f_{,a} f_{,b}) \,, \tag{9a}$$

$$K_{mn} = (f_{,m} f_{,n}) \,. \tag{9b}$$

The first metric can be shown to be covariant; the second would seem to depend on the gauge; we can convert it, however, into a manifestly invariant form $K = \bar G^{-1}$ with

$$\bar G_{mn} = \frac{\partial L_m}{\partial q_i} G_{ij} \frac{\partial L_n}{\partial q_j} \;; \tag{10}$$

the derivatives here are meant as covariant derivatives and in the spirit of inversion one now considers the invariants $\{L\}$ as functions of the original variables $\{q\}$. This clearly is a natural formulation which also exhibits manifestly that $\bar G$ is a function of invariants only, $\bar G = \bar G(L)$. Now while the original integration measure was the usual form

$$\int (\det G)^{1/2} \prod_i dq_i \tag{11}$$

it follows from the quadratic form (7) that the measure in terms of invariants shall read

$$\int d\mu(L) = \int (\det G^{(\alpha)})^{1/2} (\det \bar G)^{-1/2} \prod dL_i \,. \tag{12}$$

Concerning this result, we mention first the following. Even though some nontrivial steps are required to reach (12), this form could have been arrived at through the following heuristic arguments. First, one has the contribution $\det G^{(\alpha)}$ representing the volume of the symmetry group. Furthermore, one has $\det \bar G$ corresponding to the Riemannian nature of the space of invariants. The final formula (10) contains the following important requirement which could have been used for its independent construction. Namely, concerning the choice of a set of invariants (it is not unique) one could consider either going with a different set $\{\tilde L_i\}$. Obviously, it must be related to the set $\{L_i\}$. One can always express one complete set of invariants in terms of another

$$L_i = F_i(\tilde L) \tag{13}$$

by a functional point transformation. The measure should be form invariant under such point transformations. Indeed the form (10) assures this invariance.

In summary, let us emphasize that the major contribution (generally of nonlocal nature) is usually given by the volume of the continuous group $\det G^{(\alpha)}$. It is clear that further contributions can arrive from the discrete subgroups which still could, in principle, remain in the problem. This last fact is relevant for proper understanding of the loop-space measure and the Gribov problem in Yang-Mills theories. In lattice gravity we shall not pursue it.

III. REGGE GRAVITY

Let us now go to the gravity problem. Actually, it was DeWitt who emphasized the fact that in gravity one has first the manifold M of matrix functions $g_{\mu\nu}(x)$ and then the orbits under the group of coordinate transformations lead to the physical orbit manifold (Refs. 16 and 17). On this manifold of orbits, one has an induced metric whose form is rather nontrivial. In what follows we shall apply the general procedure outlined above to the specific problem of discrete simplicial manifolds and exhibit the nontrivial measure of Regge calculus.

The starting point is the simple metric on the space M of metric tensors

$$(\dot{g},\dot{g})=\int dx\, \dot{g}_{\mu\nu}(x)G^{\mu\nu\alpha\beta}(g)\dot{g}_{\alpha\beta}(x) \tag{14}$$

which was given by DeWitt[17] and discussed further by Vilkovisky:[18]

$$G^{\mu\nu,\alpha\beta}=\tfrac{1}{2}\sqrt{g}\,(g^{\mu\alpha}g^{\nu\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\nu}g^{\alpha\beta}) . \tag{15}$$

Above, we again have an artificial fifth time parameter. Now given a simplicial manifold specified by the lengths $\{L_i\}$ one can in principle construct (in some gauge) a metric tensor $g_{\mu\nu}(x;\{L\})$; with subsequent gauge transformations one could transform this into a general configuration on the orbit:

$$g_{\mu\nu}=g_{\mu\nu}(x;L;\xi) \tag{16}$$

with $\xi_\mu(x)$ representing the parameters of the transformation. Again, let the variables $L_i(\tau)$, $\xi_\mu(\tau,x)$ depend on an artificial fifth time. The metric for these new variables follows after substitution in (14)

$$(\dot{g},\dot{g})=\dot{L}_i\left[\frac{\partial g}{\partial L_i},\frac{\partial g}{\partial L_j}\right]\dot{L}_j +\dot{\xi}^{\mu}(x)G^{(\xi)}(\mu x,\nu y)\dot{\xi}^{\nu}(y) . \tag{17}$$

Here integration over x,y is assumed. Following our earlier discussion, the invariant version for the first term reads

$$K^{-1}{}_{ij}=\int d^4x\frac{\delta L_i}{\delta g_{\mu\nu}(x)}G_{\mu\nu,\alpha\beta}\frac{\delta L_j}{\delta g_{\alpha\beta}(x)} , \tag{18}$$

where $G_{\mu\nu,\alpha\beta}$ is the inverse of G. K is invariant since L_i are and so is the derivative form in (18). This last step avoids the explicit discussion of gauge fixing which would be needed in a direct substitution.

Let us now concentrate on the metric of the group degrees. Using the infinitesimal transformation property $\delta g_{\mu\nu}=\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu$, we have from (14) the form

$$-4\dot{\xi}_\mu\nabla_\nu G^{\mu\nu,\alpha\beta}\nabla_\alpha\dot{\xi}_\beta \tag{19}$$

which gives

$$G^{(\xi)}=(-4)\sqrt{g}\,(g^{\mu\nu}\nabla^2+R^{\mu\nu})\delta^{(4)}(x-y) . \tag{20}$$

These arguments imply that, apart from local $\prod_x g(x)^a$ type of terms,[19] the measure in Regge calculus should include the following:

$$\int d\mu(L)=\int\prod_i dL_i(\mathrm{Det}K)^{1/2}(\mathrm{Det}G^{(\xi)})^{1/2} . \tag{21}$$

The physical meaning of nonlocal determinant forms is the following: K basically represents a Jacobian for a change of L's while $\mathrm{Det}G^{(\xi)}$ represents the volume of the gauge orbits.

Let us now say a few words on the explicit form of the measure. First of all, the metric K is in first approximation diagonal and the most important nonlocal effects are given by the group volume determinant. The differential operator $G^{(\alpha)}$ acts on vector fields. It can be represented in Regge calculus, it can be split into two parts: $R_{\mu\nu}$ (this contribution can be written directly in terms of the parallel transport matrix) and the derivative form:

$$\hat{L}_{\mu\nu}=-(g_{\mu\nu}\nabla^\lambda\nabla_\lambda-R_{\mu\nu}) \tag{22}$$

representing the Lichnerowicz operator on vector fields. This operator has an elegant representation on Regge simplices. In Refs. 20–22 the simplicial version of the exterior differential calculus was developed. Basically, one has the exterior derivative d defined on a simplicial n-form as

$$(dA)^i_{(n+1)}=\sum_i \eta^i_j(n+1)A^j_{(n)} , \tag{23}$$

where η are coefficients found with the help of the boundary map of oriented simplices

$$\partial[P_0P_1\cdots P_n]=\sum(-1)^i[P_0P_{i-1}P_{i+1}\cdots P_n]$$

so that for enumerated set of basic n-simplices $\partial\sigma^i_n=\eta^i_j(n)\sigma^j_{(n-1)}$. One can also define the dual operator δ and this can be used for constructing the Laplacian: $\hat{L}=d\delta+\delta d$. This gives the explicit construction. One could conclude with the following remarks. In the flat limit the group volume could actually be thought of as a Faddeev-Popov determinant, but this with some specific way of eliminating the redundancy. For general manifolds, there is no need for such subsidiary conditions. It might be useful to think in terms of a hybrid picture treating separately the weakly curved and the strongly curved contributions. Some numerical studies will enlighten these aspects.[23–25]

IV. TWO DIMENSIONS

Considerable explicit support and a nontrivial check for the arguments presented come if one considers the problem of two-dimensional gravity (and the string model). Here the fundamental work of Polyakov[26] outlined the nontrivial dynamics of a matter and gravity system:

$$L=\sqrt{g}R+\sqrt{g}+\sqrt{g}g^{\mu\nu}\sum_{a=1}^{D}\partial_\mu\phi^a\partial_\nu\phi^a . \tag{24}$$

This nontrivial dynamics comes from the measure involved and the conformal anomaly. Already pure gravity is nontrivial at the quantum level: even though $\sqrt{g}R$ is a topological invariant, dynamical terms come from the measure given by Polyakov:

$$\int dg_{\mu\nu}e^{-S}=\int d\phi\,\exp\left[-\left[\frac{26}{48\pi}(\partial\phi)^2+e^{\phi}\right]\right]e^{-S} \tag{25}$$

in the conformal gauge. So overall one gets a Liouville theory with nontrivial dynamics:

$$L=\frac{1}{2}(\partial\phi)^2+e^{\phi}\ . \tag{26}$$

With matter fields, the integration over them provides through the conformal anomaly an additional contribution to the kinetic term. This gives the coupling coefficient of $(26-D)/48\pi$; it is only for $D=26$ that a cancellation occurs resulting in a simple theory.

Now on the lattice, the situation becomes analogous and goes as follows: in our earlier work (Ref. 2) we have considered matter fields on two-dimensional Regge nets with emphasis on the string model and the conformal anomaly. We argued that the conformal anomaly is there and arrived to a discretization for the Liouville theory which was shown to read

$$-\frac{D}{8\hbar}\sum_n \epsilon_n(L)\Delta_{nm}(L)\epsilon_m(L)\ , \tag{27}$$

where $\epsilon_n(L)$'s are deficit angles and $\Delta_{nm}(\{L\})$ is the propagator on the Regge net. So in this matter contribution we have appearance of the dynamical conformal anomaly effect. Clearly, the gravitational lattice measure ought to provide a matching contribution.

The functional integral measure of Regge manifolds that we have established in the present paper indeed provides the necessary contribution in two dimensions. Namely, in two dimensions our group volume metric $G^{(\xi)}$ given in Eq. (20) is the Polyakov determinant. Its explicit form (in the continuum) is the Liouville Lagrangian which can be represented through our discretization Eq. (27). We have the two-dimensional Regge calculus integral in the form

$$\int \prod_i dL_i(\mathrm{Det}K)^{-1/2}e^{(-26/48\pi)\epsilon\cdot\Delta\cdot\epsilon}e^{-S} \tag{28}$$

exhibiting the required effects. It provides the matching of the conformal anomaly and the simplification of the string model for $D=26$. We emphasize that the measure (which is nonlocal in form) is crucial, however, for simulation of two-dimensional gravity and any-dimensional string model. It specifies the nontrivial universality class.

To summarize, we have in this work presented, in general, a discussion of the measure and formulation of the functional integral for Regge calculus. Apart from the relevance of understanding some fundamental questions, these considerations are of practical relevance. The measure being nonlocal is quite complex to handle. In turn, however, it seems necessary for approaching the continuum theory of quantum gravity.

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