# Three-plus-one formulation of Regge calculus

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Following the work of Lund and Regge for homogeneous spaces, we construct the action for Regge calculus in its three-plus-one form for general space-times. This is achieved in two ways: a first-order formalism and a second-order formalism. We describe the Regge-calculus analogue of solving the initial-value equations using conformal transformations. The second-order formalism is used to study the time development of two simple model universes.

## I. INTRODUCTION

Regge calculus[1–4] is an approach to general relativity in which simplicial decompositions of space-time are studied. Instead of considering $n$-dimensional smooth Riemannian manifolds, one investigates spaces which are flat everywhere except on $(n-2)$-dimensional subspaces. These may be regarded as approximations to smooth manifolds, which become more and more accurate as the simplicial decomposition becomes finer.

The study of four-dimensional space-times in Regge calculus involves dealing with four-dimensional blocks. In this three-plus-one formalism three-dimensional spacelike hypersurfaces are divided into simplices and the evolution of the hypersurfaces is described by the time development of the edge lengths in the simplicial decomposition. (Note that curvature is distributed discontinuously in the spacelike directions, but continuously in the time direction.) Just as in the three-plus-one continuum case[5] and unlike in the four-dimensional Regge calculus where all equations have the same structure and status, singling out the timelike direction here makes apparent the distinction between evolution equations and constraint equations. The three-plus-one Regge calculus provides an alternative approach for numerical relativity and it might also provide a potential, gauge-independent, basis for canonical quantization of gravity.

In an unpublished paper, Lund and Regge[6] set up a formalism for three-plus-one Regge calculus for homogeneous spaces. This forms the basis for the second-order formalism which we have set up for general space-times (see Sec. III). We have also constructed a first-order three-plus-one version of the theory involving the Hamiltonian and the conjugate variables (see Sec. IV). Which formulation will be more useful in practice will depend on the particular problem being studied.

As in the continuum three-plus-one formalism we define the lapse function,[5] which determines the distance between successive spacelike hypersurfaces. As a scalar field, the lapse is defined naturally on the vertices of the simplicial decomposition. The transcription is not so simple for the shift vector which describes how the coordinate system changes between different spacelike hypersurfaces. One would like to argue that the absence of spatial coordinates means that the shift vector (and therefore the momentum constraint) is redundant. However one can also see that the shift vector is related to the association of arbitrary velocities with the vertices of the simplicial decomposition ( and the momentum constraints somehow ensure the compatibility of these velocities). Therefore, one would like to include the shift vector in the formalism. Our difficulties in doing so seem to stem from the fact that although we know how to represent scalars and symmetric tensors on the simplicial decomposition we have not found a consistent way of representing vectors. Because of this difficulty the shift vector does not appear in our formalism which therefore contains a Hamiltonian constraint but not momentum constraints. As a result of that the Hamiltonian constraint will not be strictly conserved during the evolution. This may not be a serious problem if we consider this formalism as a method for approximating general relativity. It is a severe obstacle, though, if it is considered as a fundamental theory. In any case, a possible remedy for this problem was suggested by Friedman and Jack[7] and will be discussed elsewhere.

In a three-plus-one formulation of general relativity, it is necessary to solve the constraint equations on the initial hypersurface in order to specify proper initial data. In the continuum theory one can solve the Hamiltonian constraint using conformal transformations.[8,9] We describe in Sec. V the Regge-calculus analogue of this procedure.

A simple testing ground for a three-plus-one formulation is the time development of model universes. We have used our second-order formalism to study both homogeneous and inhomogeneous, anisotropic universes containing massive scalar fields. The equations and the numerical results are presented in Sec. VI. The first-order formalism could also be tested on such models.

 

## II. SIMPLICIAL CALCULUS

Consider a simplex of dimension $n$ and let $l_a^i$ denote the $i$th component of edge $a$, which has length $l_a$. Choose a coordinate system (see Fig. 1) in which one vertex 0 is represented by the origin and the other vertices $1,2,\ldots,n$ by the points $(1,0,0,\ldots,0)$, $(0,1,0,\ldots,0)$, $(0,0,1,\ldots,0),\ldots,(0,0,0,\ldots,1)$. Since

$$l_a \cdot l_b = g_{ij} l_a^i l_b^j ,$$

the metric tensor $g_{ij}$ for that simplex is given by

$$g_{ij} = \tfrac{1}{2}({l_{0i}}^2 + {l_{0j}}^2 - {l_{ij}}^2) , \tag{1}$$

where $l_{ij}$ is the length of the edge from vertex $i$ to vertex $j$.

Given any symmetric tensor $T_{ij}$, define the $\frac{1}{2}n(n+1)$ components of $T_{ij}$ along the edges of a simplex by

$$T_a = T_{ij} l_a^i l_a^j . \tag{2}$$

In particular, the $g_a$'s associated with the metric tensor are given by

$$g_a = g_{ij} l_a^i l_a^j = {l_a}^2 . \tag{3}$$

The number of independent components of a symmetric tensor equals the number of edges of a simplex of the corresponding dimension. Therefore the quantities $\{T_a, a \in \alpha\}$ describe uniquely a constant tensor inside the simplex $\alpha$. We will use, in what follows, the index $a$, $1 \le a \le \frac{1}{2}n(n+1)$, to replace the component indices of a symmetric tensor and in this sense $g_a$ is the metric tensor.

The identity

$$\sum_{a \in \alpha} g_a \frac{\partial \ln {V_\alpha}^2}{\partial g_a} = n ,$$

which follows from Euler's theorem, suggests the definition of $g^{\alpha a}$ as the inverse metric in the simplex $\alpha$:

$$g^{\alpha a} \equiv \frac{\partial \ln {V_\alpha}^2}{\partial g_a} . \tag{4}$$

[In what follows, we shall use (unless stated otherwise) the convention of summation over repeated lower and upper edge indices $a,b,\ldots$ within a given simplex.] Similarly the identity

$$g^{\alpha a} = -\frac{\partial^2 \ln {V_\alpha}^2}{\partial g_a \partial g_b} g_b$$

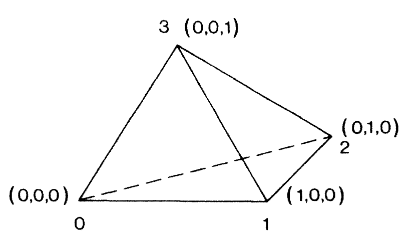


FIG. 1. A typical tetrahedron, with a particular choice of coordinate system.

suggests the definition of a double-component index-raising operator

$$G^{\alpha ab} \equiv -\frac{\partial^2 \ln {V_\alpha}^2}{\partial g_a \partial g_b} = -\frac{\partial}{\partial g_b} g^{\alpha a} . \tag{5}$$

The trace of a tensor $T$ in a simplex $\alpha$ is then given by

$$(\mathrm{Tr}T)_\alpha = g^{\alpha a} T_a \tag{6}$$

and

$$(\mathrm{Tr}T^2)_\alpha = G^{\alpha ab} T_a T_b . \tag{7}$$

A proof of these formulas, which were first derived by Lund and Regge,[6] is given in the Appendix.

While the matrix $G^{\alpha ab}$ raises indices $a,b,\ldots$ within simplex $\alpha$, indices will be lowered by the inverse matrix $G_{\alpha ab}$ which satisfies

$$G_{\alpha ab} G^{\alpha bc} = \delta_a^c . \tag{8}$$

Using $G^{\alpha ab}$ and $g^{\alpha a}$ we define the DeWitt matrix (for symmetric tensors only)

$$\mathcal{G}_\lambda^{\alpha ab} \equiv G^{\alpha ab} - \lambda g^{\alpha a} g^{\alpha b} . \tag{9}$$

For $\lambda = 1$, which we need for $\mathrm{Tr}K^2 - (\mathrm{Tr}K)^2$ terms, this becomes

$$\mathcal{G}^{\alpha ab} = -\frac{1}{{V_\alpha}^2} \frac{\partial^2 {V_\alpha}^2}{\partial g_a \partial g_b} . \tag{10}$$

The density equivalent to $\sqrt{g}\, G^{ijkl}$ is defined by

$$\mathfrak{G}^{\alpha ab} \equiv V_\alpha \mathcal{G}^{\alpha ab} . \tag{11}$$

Note that the inverse of $\mathcal{G}^{\alpha ab}$ is given by

$$\mathcal{G}_{\alpha ab} = G_{\alpha ab} - \tfrac{1}{2} g_a g_b \tag{12}$$

and that

$$\mathfrak{G}_{\alpha ab} = \frac{1}{V_\alpha} \mathcal{G}_{\alpha ab} . \tag{13}$$

Sometimes it is natural to define $T^a$, and not $T_a$, as the basic quantity (in which case $T_{\alpha a} \equiv G_{\alpha ab} T^b$ depends on $\alpha$). We then have

$$(\mathrm{Tr}T)_\alpha = g_a T^a \tag{14}$$

and

$$(\mathrm{Tr}T^2)_\alpha = G_{\alpha ab} T^a T^b . \tag{15}$$

## III. SECOND-ORDER FORMALISM FOR 3 + 1 REGGE CALCULUS

The starting point for most work on Regge calculus so far has been the expression in terms of a simplicial decomposition of the continuum Einstein action

$$S = \int d^4x \sqrt{(-g)} R , \tag{16}$$

where $R$ is the four-dimensional curvature scalar and $g$ the determinant of the four-dimensional metric tensor. An alternative is to take the action in its 3 + 1 form, in which time has been singled out as a preferred coordinate (see, for example, Chap. 21 of Ref. 3):

$$S=\int dt\,d^3x\{N(^3g)^{1/2}[^3R+\mathrm{Tr}(K^2)-(\mathrm{Tr}K)^2]+2N_i[(^3g)^{1/2}(K^{ij}-g^{ij}K)]_{|j}\}\;, \tag{17}$$

where

$$K_{ij}=-\frac{1}{2N}(^3\dot{g}_{ij}-N_{i\,|\,j}-N_{j\,|\,i}) \tag{18}$$

is the extrinsic curvature of a hypersurface of constant time, which has metric $^3g_{ij}$ and scalar curvature $^3R$. The overdot denotes differentiation with respect to time and the vertical bar covariant differentiation within the hypersurface; $N$ and $N_i$ are the lapse function and the shift vector, respectively. Time will be singled out further here as it will be kept continuous, while the spacelike hypersurface will be divided into flat simplices.

One may then proceed in two possible ways. First one may vary the action, written as in (17) as the integral of the Lagrangian, with respect to the basic variables ($g_{ij}$ in the continuum theory, $g_a$ here) to obtain second-order differential equations for the time development of these variables. Alternatively, one may express the action in its Hamiltonian form, written in terms of the basic variables and a set of conjugate variables. Variation of this action with respect to both sets of variables produces two sets of coupled first-order differential equations for the time development of the variables. (One set may be regarded as the definition of the conjugate variables.) This section will be concerned with the first of these alternatives.

The spacelike hypersurface is taken to be a surface composed of tetrahedra and the squared edge lengths, $g_a$, are the dynamical variables of the theory. The lapse function is defined on the vertices (labeled by $\mu,\nu,\ldots$) and its value on edges or inside tetrahedra will usually be obtained by linear interpolation:

$$N_a\equiv\tfrac{1}{2}\sum_{\mu\in a}N_\mu\;, \tag{19}$$

$$N_\alpha\equiv\tfrac{1}{4}\sum_{\mu\in\alpha}N_\mu\;. \tag{20}$$

The shift, which is a vector, might also be naturally defined on the vertices, but it is not clear how to relate a vector on a vertex in one simplex to its counterpart in another simplex. Furthermore, as we saw in Sec. II, symmetric tensors and scalars appear naturally in Regge calculus but this is not the case for vectors. Therefore we have chosen to omit the shift vector from our formalism.

Curvature resides at the edges, and the deficit angle $\epsilon_a$ at an edge $a$, which gives a measure of the curvature there, is defined by

$$\epsilon_a=2\pi-\sum_{\alpha\in a}(\text{dihedral angle in }\alpha\text{ at }a)\;. \tag{21}$$

Regge's formula for the integral of the scalar curvature,

$$\int d^3x(^3g)^{1/2}\,{}^3R=2\sum_a g_a{}^{1/2}\epsilon_a\;,$$

suggests the identification of the curvature on edge $a$ as

$$R_a=\frac{g_a{}^{1/2}\epsilon_a}{V_a^*}\;, \tag{22}$$

$V_a^*$ being the volume dual to edge $a$, corresponding to $\sqrt{g}\,d^3x$ (Ref. 10). Hence

$$\int d^3x(^3g)^{1/2}N\,^3R=2\sum_a N_a g_a{}^{1/2}\epsilon_a=\sum_\mu N_\mu\sum_{\mu\in a}g_a{}^{1/2}\epsilon_a\;. \tag{23}$$

To evaluate the part of the action involving the extrinsic curvature, we use the formalism set up in Sec. II. From Eqs. (6), (7), and (10), we see that within a tetrahedron $\alpha$,

$$[\mathrm{Tr}(K)^2-(\mathrm{Tr}K)^2]\,|_\alpha=\mathscr{G}^{\alpha ab}K_aK_b\;, \tag{24}$$

where, in the absence of the shift vector,

$$K_a=-\frac{1}{2N}\dot{g}_a\;. \tag{25}$$

Integrating (24) over the volume we obtain

$$\int d^3x\,N(^3g)^{1/2}[\mathrm{Tr}(K^2)-(\mathrm{Tr}K)^2]=\tfrac{1}{4}\sum_\alpha\frac{1}{N}\mathscr{G}^{\alpha ab}\dot{g}_a\dot{g}_b\;. \tag{26}$$

Since the factor $\mathscr{G}^{\alpha ab}\dot{g}_a\dot{g}_b$ is constant for a given simplex, we interpret the $1/N$ factor as

$$\left[\frac{1}{N}\right]_\alpha=\tfrac{1}{4}\sum_{\mu\in\alpha}\frac{1}{N_\mu}\;. \tag{27}$$

Our final form for the action is then

$$S=\int dt\left[\tfrac{1}{4}\sum_\alpha\left[\frac{1}{N}\right]_\alpha\mathscr{G}^{\alpha ab}\dot{g}_a\dot{g}_b+\sum_\mu N_\mu\sum_{a\in\mu}g_a{}^{1/2}\epsilon_a-16\pi\sum_\alpha N_\alpha V_\alpha(\mathscr{L}_{\mathrm{matter}})_\alpha\right]\;, \tag{28}$$

where $(\mathscr{L}_{\mathrm{matter}})_\alpha$ is the Regge-calculus analogue of the continuum matter Lagrangian. We do not include this term in what follows in this section. [For a homogeneous space, the action (28) reduces to the Lund-Regge expression.[6] Also Friedman and Jack[7] have obtained independently an action which is identical to (28) when $N=$const, by evaluating the continuum action for a piecewise linear metric.]

Variation of the action $S$ with respect to $g_a$ determines the time development of $g_a$, i.e., the time evolution of the edge lengths:

$$
\begin{aligned}
0 &= -\frac{\delta S}{\delta g_a} \\
&\equiv \frac{\partial S}{\partial g_a} - \frac{d}{dt}\left[\frac{\partial S}{\partial \dot{g}_a}\right] \\
&= \tfrac{1}{4}\sum_{\alpha\in a}\left[\left[\frac{1}{N}\right]_\alpha [(\mathscr{G}^{\alpha bc,a} - 2\mathscr{G}^{\alpha ab,c})\dot{g}_b\dot{g}_c - 2\mathscr{G}^{\alpha ab}\ddot{g}_b] - 2\left[\frac{1}{N}\right]_\alpha \mathscr{G}^{\alpha ab}\dot{g}_b\right] + \tfrac{1}{2}\sum_{\mu\in a} N_\mu g_a{}^{-1/2}\epsilon_a + \sum_\mu N_\mu \sum_{b\in\mu} g_b{}^{1/2}\epsilon_b{}^{,a} ,
\end{aligned}
\tag{29}
$$

where the superscript $,a$ denotes differentiation with respect to $g_a$. Note that although Regge showed[1] that

$$\sum_a g_a{}^{1/2}\epsilon_a{}^{,b} = 0 ,$$

this simplification does not hold when $N$ is included, i.e.,

$$\sum_\mu N_\mu \sum_{b\in\mu} g_b{}^{1/2}\epsilon_b{}^{,a} \neq 0 \quad \text{in general} .$$

Variation of $S$ with respect to $N_\mu$ gives the Hamiltonian constraint at each vertex

$$
\begin{aligned}
0 = \frac{\delta S}{\delta N_\mu} &\equiv H_\mu \\
&= +\frac{1}{16 N_\mu{}^2}\sum_{\alpha\in\mu}\mathscr{G}^{\alpha ab}\dot{g}_a\dot{g}_b - \sum_{\alpha\in\mu} g_a{}^{1/2}\epsilon_a .
\end{aligned}
\tag{30}
$$

This constraint is not conserved in time:

$$
\begin{aligned}
\frac{dH_\mu}{dt} = &-\frac{N_\mu}{8N_\mu{}^3}\sum_{\alpha\in\mu}\mathscr{G}^{\alpha ab}\dot{g}_a\dot{g}_b \\
&+ \frac{1}{16N_\mu{}^2}\sum_{\alpha\in\mu}(\mathscr{G}^{\alpha ab,c}\dot{g}_a\dot{g}_b\dot{g}_c + 2\mathscr{G}^{\alpha ab}\dot{g}_a\ddot{g}_b) \\
&- \tfrac{1}{2}\sum_{a\in\mu} g_a^{-1/2}\dot{g}_a\epsilon_a - \sum_{a\in\mu} g_a{}^{1/2}\sum_b \epsilon_a{}^{,b}\dot{g}_b .
\end{aligned}
\tag{31}
$$

However we may show that, under a global summation,

$$\sum_\mu N_\mu \frac{dH_\mu}{dt} = \sum_a \dot{g}_a \frac{\delta S}{\delta g_a} = 0 . \tag{32}$$

This means that although the individual constraints $H_\mu = 0$ are not automatically conserved in time, a weighted average of their time derivatives is zero. Moreover if we consider the sum of $N_\mu dH_\mu/dt$ over a finite region, only terms arising from the surface of this region will not cancel. When we approach the continuum limit and the number of points inside this fixed volume increases, the ratio of the number of noncanceling surface terms to the number of canceling volume terms vanishes. In this sense, the continuum limit is recovered.

The fact that $H$ is not conserved is not new in discrete versions of Einstein's equations[11] and so it is not necessarily an obstacle to the use of this theory as another discrete approximation to general relativity. However it would still be much preferable to obtain a formalism in which the constraints are satisfied identically. The nonvanishing of $dH/dt$ is related to the fact that $[H,H] \neq 0$ (see discussion in Sec. IV). In the continuum theory closure of the Hamiltonian constraints is obtained via the momentum constraints.[12] This suggests that the introduction of momentum constraints may solve the problem here. A procedure for writing the momentum constraints using the integral of $\pi^{ij}dS_j$ over surfaces of simplices was suggested by Friedmann and Jack[7] and will be discussed elsewhere, but as yet it is not clear whether this will indeed lead to a closure. Alternatively one may try another definition of a Regge calculus 3 + 1 Lagrangian.

We may use the methods described here to study the time development of general space-times. Two examples are described in Sec. VI. Lund and Regge[6] applied their formalism for homogeneous spaces to a number of cosmological models, including the Friedmann universe, the Kantowski-Sachs universe, and Bianchi Type-1 cosmologies.

## IV. FIRST-ORDER FORMALISM FOR 3 + 1 REGGE CALCULUS

In the previous section, we wrote down a lattice approximation to the Lagrangian and in this section we write down a lattice approximation to the Hamiltonian. However the two approximations are not related by a Legendre transformation. For example, variation of the action (28) (without the matter term) with respect to $\dot{g}_a$ gives

$$\frac{\partial S}{\partial \dot{g}_a} = \tfrac{1}{2}\sum_{\alpha\in a}\left[\frac{1}{N}\right]_\alpha \mathscr{G}^{\alpha ab}\dot{g}_b . \tag{33}$$

However, defining the Regge-calculus conjugate variable by such an expression would produce an impossibly cumbersome formalism, although (as we shall see later) there is indeed a relationship between our conjugate variable and the expression above.

The continuum form for the action in its Hamiltonian form is

$$S = \int dt\, d^3x(\pi^{ij}\,{}^3\dot{g}_{ij} - NH - N_i H^i) , \tag{34}$$

where

$$H = \frac{1}{({}^3g)^{1/2}}[\mathrm{Tr}(\pi^2) - \tfrac{1}{2}(\mathrm{Tr}\pi)^2] - ({}^3g)^{1/2}R \tag{35}$$

and

$$H^i = -2\pi^{ij}{}_{|j} . \tag{36}$$

We again set the shift vector to zero to obtain

$$S=\int dt\, d^3x\left[\pi^{ij}\,{}^3\dot{g}_{ij}-\frac{N}{({}^3g)^{1/2}}[\mathrm{Tr}(\pi^2)-\tfrac{1}{2}(\mathrm{Tr}\pi)^2]+N({}^3g)^{1/2}R\right] . \tag{37}$$

The extrinsic curvature $K_{ij}$ is related to $\pi_{ij}$ by

$$K_{ij}=\frac{1}{({}^3g)^{1/2}}[\tfrac{1}{2}\,{}^3g_{ij}(\mathrm{Tr}\pi)-\pi_{ij}] . \tag{38}$$

For the Regge-calculus variable conjugate to $g_a$, we define a quantity $\pi^a$ analogous to $\pi^{ij}$. Note that this is *not* obtained from a quantity $\pi_a$ by raising the index by $G^{\alpha ab}$. On the other hand, we may define a quantity $\pi_{\alpha a}$ by

$$\pi_{\alpha a}=G_{\alpha ab}\pi^b . \tag{39}$$

The action then takes the form

$$S=\int dt\left[\sum_a \pi^a\dot{g}_a-\sum_\alpha N_\alpha \mathcal{G}_{\alpha ab}\pi^a\pi^b+2\sum_a N_a g_a{}^{1/2}\epsilon_a\right] . \tag{40}$$

Variation of the action with respect to $\pi^a$ then gives

$$\dot{g}_a=-2\sum_\alpha N_\alpha \mathcal{G}_{\alpha ab}\pi^b=-2\sum_{\alpha\in a} N_\alpha K_{\alpha a} , \tag{41}$$

where

$$K_{\alpha a}=\frac{1}{V_\alpha}[\tfrac{1}{2}g_a(\mathrm{Tr}\pi)_\alpha-\pi_{\alpha a}] \tag{42}$$

which is the analogue of the continuum equation

$${}^3\dot{g}_{ij}=-2NK_{ij} .$$

Note that in the Regge-calculus expression, we obtain a weighted average of $K_{\alpha a}$ for the simplices on the edge $a$.

After integration by parts of the first term in (40), variation with respect to $g_a$ gives

$$\dot{\pi}^a=-\sum_{\alpha\in a} N_\alpha \mathcal{G}_{\alpha bc}{}^{,a}\pi^b\pi^c+N_a g_a{}^{-1/2}\epsilon_a+2\sum_b N_b g_b{}^{1/2}\epsilon_b{}^{,a} . \tag{43}$$

To see the correspondence with the continuum equation

$$\dot{\pi}^{ij}=\frac{1}{2}\frac{N}{({}^3g)^{1/2}}\,{}^3g^{ij}[\mathrm{Tr}(\pi^2)-\tfrac{1}{2}(\mathrm{Tr}\pi)^2]-2\frac{N}{({}^3g)^{1/2}}[\pi^{im}\pi_m{}^j-\tfrac{1}{2}(\mathrm{Tr}\pi)\pi^{ij}]-N({}^3g)^{1/2}({}^3R^{ij}-\tfrac{1}{2}\,{}^3g^{ij}\,{}^3R)+({}^3g)^{1/2}(N^{|ij}-{}^3g^{ij}N^{|m}{}_{|m}) , \tag{44}$$

we express $\mathcal{G}_{\alpha ab}$ in terms of $g_a$, $G_{\alpha ab}$, and $V_\alpha$ to obtain

$$\dot{\pi}^a=\tfrac{1}{2}\sum_{\alpha\in a}\frac{N_\alpha}{V_\alpha}g^{aa}[(\mathrm{Tr}\pi^2)_\alpha-\tfrac{1}{2}(\mathrm{Tr}\pi)_\alpha{}^2]-2\sum_{\alpha\in a}\frac{N_\alpha}{V_\alpha}[\tfrac{1}{2}G_{\alpha bc}{}^{,a}\pi^b\pi^c-\tfrac{1}{2}(\mathrm{Tr}\pi)_\alpha\pi^a]+N_a g_a{}^{-1/2}\epsilon_a+2\sum_b N_b g_b{}^{1/2}\epsilon_b{}^{,a} . \tag{45}$$

Notice again that the Regge-calculus equation contains weighted averages from the relevant tetrahedra. The term $\frac{1}{2}G_{\alpha ab}{}^{,a}\pi^b\pi^c$ must play the role of $\pi^{im}\pi_m{}^j$ and $2\sum_b N_b g_b{}^{1/2}\epsilon_b{}^{,a}$ must be related to $({}^3g)^{1/2}(N^{|ij}-{}^3g^{ij}N^{|m}{}_{|m})$.

The constraint equations follow from variation of the action with respect to $N_\mu$:

$$0=\frac{\delta S}{\delta N_\mu}\equiv H_\mu=\tfrac{1}{4}\sum_{\alpha\in\mu}\mathcal{G}_{\alpha ab}\pi^a\pi^b-\sum_{a\in\mu}g_a{}^{1/2}\epsilon_a=+\tfrac{1}{4}\sum_{\alpha\in\mu}\frac{1}{V_\alpha}[(\mathrm{Tr}\pi^2)_\alpha-\tfrac{1}{2}(\mathrm{Tr}\pi)_\alpha{}^2]-\sum_{a\in\mu}g_a{}^{1/2}\epsilon_a . \tag{46}$$

This is a semilocal constraint as each vertex, corresponding to the continuum constraint

$$0=+\frac{1}{({}^3g)^{1/2}}[\mathrm{Tr}\pi^2-\tfrac{1}{2}(\mathrm{Tr}\pi)^2]-({}^3g)^{1/2}\,{}^3R . \tag{47}$$

The sense in which the constraints are preserved in time is the same as in the second-order formalism. We have

$$\begin{aligned}\frac{dH_\mu}{dt}&=+\tfrac{1}{4}\sum_{\alpha\in\mu}(\mathcal{G}_{\alpha ab,c}\dot{g}_c\pi^a\pi^b+2\mathcal{G}_{\alpha ab}\dot{\pi}^a\pi^b)-\tfrac{1}{2}\sum_{a\in\mu}g_a{}^{-1/2}\dot{g}_a\epsilon_a-\sum_{a\in\mu}g_a{}^{1/2}\sum_b\epsilon_a{}^{,b}\dot{g}_b\\
&=+\tfrac{1}{2}\sum_{\alpha\in\mu}\sum_{c\in\alpha}\sum_{\beta\in c}\mathcal{G}_{\alpha ab}{}^{,c}\pi^a\pi^b N_\beta\mathcal{G}_{\beta cd}\pi^d-\tfrac{1}{2}\sum_{\alpha\in\mu}\sum_{a\in\alpha}\sum_{\beta\in a}\mathcal{G}_{\alpha ab}\pi^b N_\beta\mathcal{G}_{\beta cd}{}^{,a}\pi^c\pi^d\\
&\quad+\tfrac{1}{2}\sum_{\alpha\in\mu}\sum_{a\in\alpha}\mathcal{G}_{\alpha ab}\pi^b N_a g_a^{-1/2}\epsilon_a-\sum_{a\in\mu}\sum_{\alpha\in a}g_a{}^{-1/2}\epsilon_a N_\alpha\mathcal{G}_{\alpha ab}\pi^b\\
&\quad+\sum_{\alpha\in\mu}\sum_{a\in\alpha}\sum_c\mathcal{G}_{\alpha ab}\pi^b N_c g_c{}^{1/2}\epsilon_c{}^{,a}+2\sum_{a\in\mu}\sum_c\sum_{\beta\in c}g_a{}^{1/2}\epsilon_a{}^{,c}N_\beta\mathcal{G}_{\beta bc}\pi^b .\end{aligned} \tag{48}$$

Although this expression does not vanish, we may multiply it by $N_\mu$ and sum over all vertices, and after an elaborate series of cancellations between terms, we obtain

$$\sum_{\mu} N_\mu \frac{dH_\mu}{dt} = 0 \; . \tag{49}$$

Thus we see again that a weighted average of the time derivates of the constraints is zero.

A further examination of (48) reveals that the six terms appear in pairs which look identical but with opposite signs. However these pairs do not cancel since the summation is over different elements of the lattice. Part of the summation is over the same elements, and this part, of course, cancels; the rest, which is like a surface term, remains. If we consider $\sum_{\mu \in U} N_\mu dH_\mu / dt$ with the summation over a small region, $U$, we notice that terms associated with internal elements of $U$ cancel. We can write

$$\sum_{\mu \in U} N_\mu \frac{dH_\mu}{dt} = \text{surface terms} \; . \tag{50}$$

In the limit when there are many vertices within $U$, the ratio of these surface terms to the other terms, that cancel, becomes small. In this sense the constraints are conserved in the continuum limit of our formalism.

Additional insight on the structure of this formalism is obtained from consideration of the Poisson brackets, in particular the Poisson brackets with $H_\mu$. Simple calculation reveals that

$$\dot{g}_a = \sum_{\mu} N_\mu [g_a, H_\mu] \tag{51}$$

and

$$\dot{\pi}_a = \sum_{\mu} N_\mu [\pi_a, H_\mu] \tag{52}$$

as required. The Poisson bracket between two constraints is

$$\begin{aligned} [H_\mu, H_\nu] &\equiv \sum_d \left[ \frac{\delta H_\mu}{\delta g_d} \frac{\delta H_\nu}{\delta \pi^d} - \frac{\delta H_\mu}{\delta \pi^d} \frac{\delta H_\nu}{\delta g_d} \right] \\ &= \tfrac{1}{8} \sum_d \sum_{\alpha \in \mu} \sum_{\substack{\beta \in \nu \\ \alpha, \beta \in d}} (\hat{\mathcal{G}}_{\alpha a b}{}^{,d} \hat{\mathcal{G}}_{\beta c d} - \hat{\mathcal{G}}_{\alpha c d} \hat{\mathcal{G}}_{\beta a b}{}^{,d}) \pi^a \pi^b \pi^c \\ &\quad + \tfrac{1}{4} \sum_d \left[ \sum_{\alpha \in \mu} \sum_{c \in \nu} - \sum_{\alpha \in \nu} \sum_{c \in \mu} \right] \hat{\mathcal{G}}_{\alpha a d} \pi^a (g_c{}^{-1/2} \epsilon_c \delta^{dc} + 2 g_c{}^{1/2} \epsilon_c{}^{,d}) \; . \end{aligned} \tag{53}$$

This expression vanishes if the simplices containing $\mu$ and the simplices containing $\nu$ have no edge in common ($\mu^* \cap \nu^* = \phi$). However $[H_\mu, H_\nu] \neq 0$ if $\mu^* \cap \nu^* \neq \phi$, i.e., if $\mu$ and $\nu$ are nearest neighbors or next-nearest neighbors. Again we find that

$$\dot{H}_\mu = \sum_{\nu} N_\nu [H_\mu, H_\nu] \tag{54}$$

and the nonconservation of the constraints follows, in this context, from the nonvanishing of their commutators. In the continuum theory, the momentum constraints ensure the closure of the commutation relation between the Hamiltonian constraints. This can potentially suggest that examination of (53) will provide a clue to the formulation of the momentum constraints. However one has to remember that whatever constraint is chosen, it should be conserved by itself. It is not clear that a choice that will lead to a closed system is possible here.

## V. THE INITIAL-VALUE EQUATION AND CONFORMAL TRANSFORMATIONS

Before studying the time development of a system, one must solve the constraint equations on the initial slice. This is likely to be a nontrivial problem for simplicial decompositions involving more than two edge lengths (see Ref. 13 for examples where only two edge lengths are involved).

In the continuum theory, there are four initial-value equations, the Hamiltonian constraint and the momentum constraints. A standard way of solving the Hamiltonian constraint in the continuum case[8,9] is by using a conformal transformation. The idea is to start with a simple trial metric $\gamma_{ij}$ and make a conformal transformation $\phi$ to

$$g_{ij} = \phi^4 \gamma_{ij} \; , \tag{55}$$

where $g_{ij}$ is the required solution. (Note that we have dropped superscripts 3 on three-dimensional quantities.) The scalar curvatures for $g_{ij}$ and $\gamma_{ij}$ are related by

$$R(g) = \frac{R(\gamma)}{\phi^4} - \frac{8}{\phi^5} \Box\phi \; , \tag{56}$$

where $\Box\phi = \gamma^{ij} \phi_{|ij}$ and $|$ denotes a covariant derivative with respect to $\gamma_{ij}$. For an initial slice with $K_{ij} = 0$, the Hamiltonian constraint reduces to

$$R(g) = 0 \tag{57}$$

and so we solve

$$\Box\phi = \tfrac{1}{8} R \phi \tag{58}$$

for $\phi$ and then use (55) to obtain $g_{ij}$ on the initial slice.

The momentum constraints do not exist in our formalism. To solve the Hamiltonian constraint, when $K=0$, we start with some trial set of edge lengths $l_a^{(0)}=(g_a^{(0)})^{1/2}$ with deficit angles $\epsilon_a^{(0)}$. We then make a "conformal transformation" (as described later in this section) to edge lengths $l_a={g_a}^{1/2}$, with deficit angles $\epsilon_a$, which satisfy the Hamiltonian constraint at each point. At vertex $\mu$, for a slice with $K=0$, this is given by Regge-calculus terms, by

$$0=R=\frac{\sum_{a\in\mu} {g_a}^{1/2}\epsilon_a}{V_\mu^*}\ , \tag{59}$$

where $V_\mu^*=\frac{1}{2}\sum_{a\in\mu}V_a^*$ is the "volume per vertex" at $\mu$. (See p. 472 of Ref. 2. The usual factor of 2 in the definition of $R$ is not needed here; it reappears when we sum over vertices.) Similarly the scalar curvature for the trial edge lengths is given by

$$R^{(0)}=\frac{\sum_{a\in\mu}(g_a^{(0)})^{1/2}\epsilon_a^{(0)}}{V_\mu^{*(0)}}\ . \tag{60}$$

The Regge-calculus analogues of conformal transformations[14] are defined as follows. We assign a scalar field $\phi_\mu$ to each vertex $\mu$. An edge variable $g_a={l_a}^2$ between vertices $\mu$ and $\nu$, then transforms according to

$$g_a\to g_a'={\phi_\mu}^2{\phi_\nu}^2 g_a \tag{61}$$

(we require $g_a$ to transform like the metric tensor, i.e., with $\phi^4$; taking the geometric mean of the $\phi$'s at either end of the edge gives the required group property for infinitesimal conformal transformations[14]). Note that the conformal factors $\phi_\mu$ must be restricted so that the new edge lengths do not violate the triangle or tetrahedral inequalities. (This is analogous in the continuum case to the condition that $\phi$ does not vanish since this leads to a singularity in $g_{ij}$.) To solve the Hamiltonian constraint we insert this conformal transformation into Eq. (55) to obtain a nonlinear equation for the $\phi_\mu$'s.

The Regge-calculus form for the derivative term in (52) at the vertex $\mu$ is[15]

$$-\Box\phi=\frac{1}{V_\mu^{*(0)}}\sum_{a\in\mu}\lambda_a^{(0)}(\phi_\mu-\phi_\nu)\ , \tag{62}$$

where $\nu\equiv\nu(\mu,a)$ is the vertex connected to $\mu$ by the edge $a$. The quantity $\lambda_a^{(0)}$ is the trial value of $\lambda_a$, which is given by

$$\lambda_a=\frac{A_a^*}{l_a}\ , \tag{63}$$

where $A_a^*$ is the area of the dual loop[10,15] associated with the edge $a$, which is related to the dual volume by

$$V_a^*=\tfrac{1}{3}A_a^* l_a\ . \tag{64}$$

Note that it is not strictly necessary to define the volume per vertex in terms of dual volumes; one could also use, for example, the volumes per hinge obtained from a barycentric subdivision. However Eq. (64) *must* involve the dual volume. We find then that

$$\lambda_a=\frac{3V_a^*}{g_a}\ . \tag{65}$$

Collecting the information in Eqs. (60)–(65) we see that the Regge-calculus analogue of Eq. (58) is

$$\frac{1}{{\phi_\mu}^5\sum_{a\in\mu}V_a^{*(0)}}\sum_{a\in\mu}\left\{\phi_\mu(g_a^{(0)})^{1/2}\epsilon_a+24\frac{V_a^{*(0)}}{g_a^{(0)}}(\phi_\mu-\phi_\nu)\right\}=0 \tag{66}$$

at each vertex $\mu$.

Equation (66) is not identical to Eq. (59). Equation (66) involves a linear function of $\phi_\mu$, divided by ${\phi_\mu}^5$, while Eq. (59) is a very complicated function of $\phi_\mu$ involving arccosines. However the equations become equivalent in the continuum limit, when the simplicial decomposition becomes finer and finer. In this limit all edges and all deficit angles become small. At any point, space is almost flat (assuming that we are approaching a smooth Riemannian manifold). Therefore we may take the trial edge length variables $g_a^{(0)}$ to correspond to flat space (which means that $\epsilon_a^{(0)}=0$) and the transformed edge length variables $g_a$ to differ only slightly from their flat-space values. This is equivalent [see Eq. (61)] to allowing each conformal transformation to differ from unity by only a small amount. Thus we set

$$\phi_\mu=1+\psi_\mu \ \text{ with } \ |\psi_\mu|\ll 1\ . \tag{67}$$

Under these circumstances, the Regge-calculus analogue of Eq. (56) becomes

$$\frac{\sum_{a\in\mu}{g_a}^{1/2}\epsilon_a}{\sum V_a^*}=\frac{24}{(1+\psi_\mu)^5\sum V_a^{*(0)}}\sum_{a\in\mu}\frac{V_a^{*(0)}}{g_a^{(0)}}(\psi_\mu-\psi_\nu) \tag{68}$$

and we must now show that this holds to lowest order in $\psi$, at each vertex $\mu$.

For simplicity, we take the triangulation in the neighborhood of a vertex, the origin say, to be a cubic lattice divided into tetrahedra.[14] Let the length scale be $l$ [the vertices of the flat-space lattice are then of the form $l(p,q,r)$ where $p$, $q$, and $r$ are integers]. We study Eq. (68) at the origin. Since the right-hand side of the equation is linear in $\psi_\nu$, $\nu\neq 0$, we take only three typical nonzero $\psi_\nu$'s: $\psi_1$ at $(l,0,0)$, $\psi_2$ at $(l,l,0)$, and $\psi_3$ at $(l,l,l)$ [and of course $\psi_0\neq 0$ at $(0,0,0)$]. After evaluation of $g_a$, $\epsilon_a$, and $V_a^*$ for all the edges based on the origin, we obtain

$$\frac{\sum_{a\subset 0}{g_a}^{1/2}\epsilon_a}{\sum_{a\subset 0}V_a^*}=\frac{4}{l^2}(6\psi_0-\psi_1)+O(\psi^2)$$
$$=\frac{24}{(1+\psi_0)^5\sum_{a\subset 0}V_a^{*(0)}}\sum_{a\subset 0}\frac{V_a^{*(0)}}{g_a^{(0)}}(\psi_0-\psi_\nu) \tag{69}$$

as required. Thus the Regge-calculus analogue of Eq. (56) becomes exact in the continuum limit.

To summarize, we can find suitable initial data by solving the nonlinear equation (59) or alternatively we can obtain an approximate solution using (66) to solve for $\phi_\mu$.

## VI. EXAMPLES OF THE USE OF THE SECOND-ORDER FORMALISM: THE TIME DEVELOPMENT OF TWO MODEL UNIVERSES

We now illustrate the use of the second-order formalism by studying the time development of two simple model universes,[16] both with the topology of a three-sphere and containing a massive scalar field.

The continuum action for a massive scalar field:

$$-16\pi\int dt\, d^3x\, N({}^3g)^{1/2}\left[\tfrac{1}{2}g^{ij}\partial_i\phi\partial_j\phi-\frac{1}{2}\frac{\dot\phi^2}{N^2}+\frac{1}{2}m^2\phi^2\right] \tag{70}$$

becomes

$$-16\pi\int dt\left[\sum_\mu N_\mu\sum_{a\in\mu}\lambda_a(\phi_\mu-\phi_\nu)^2 -\tfrac{1}{2}\sum_\mu\frac{V_\mu^*\dot\phi_\mu{}^2}{N_\mu{}^2}+\tfrac{1}{2}m^2\sum_\mu N_\mu V_\mu^*\phi_\mu{}^2\right], \tag{71}$$

where the scalar field $\phi$ (not to be confused with the conformal factor in the previous section) takes the value $\phi_\mu$ at vertex $\mu$.

First we consider an anisotropic but homogeneous universe, modeled by $\alpha_4$, the tessellation of $S^3$ which consists of five tetrahedra. The tetrahedra are identical,[13] each having three edges with length squared $g_a$ and three with length squared $g_b$. Neither set of equal edges forms a triangle [see Fig. 2(a)]. All five vertices are equivalent, so there is only one $N$ and one $\phi$. Also the vertex volume is equal to the volume $V$ of each tetrahedron. The action for this model is

$$S=\int dt\left[-\frac{5}{4NV}[\dot g_a{}^2(V^2)^{,aa}+2\dot g_a\dot g_b(V^2)^{,ab}+\dot g_b{}^2(V^2)^{,bb}] +10N(g_a{}^{1/2}\epsilon_a+g_b{}^{1/2}\epsilon_b) -40\pi NV\left[m^2\phi^2-\frac{\dot\phi^2}{N^2}\right]\right]. \tag{72}$$

The deficit angles are given by[13]

$$\epsilon_a=2\pi-2\theta_a-\theta_a',\quad \epsilon_b=2\pi-2\theta_b-\theta_b', \tag{72a}$$

with

$$\cos\theta_a=\frac{g_a{}^{1/2}(3g_b-2g_a)}{g_b{}^{1/2}[(4g_a-g_b)(4g_b-g_a)]^{1/2}},\qquad \cos\theta_{a'}=\frac{2g_a{}^2-2g_ag_b+g_b{}^2}{g_b(4g_a-g_b)}, \tag{72b}$$

and similar expressions for $\theta_b$ and $\theta_{b'}$, with $a$ and $b$ interchanged. The volume of each tetrahedron is

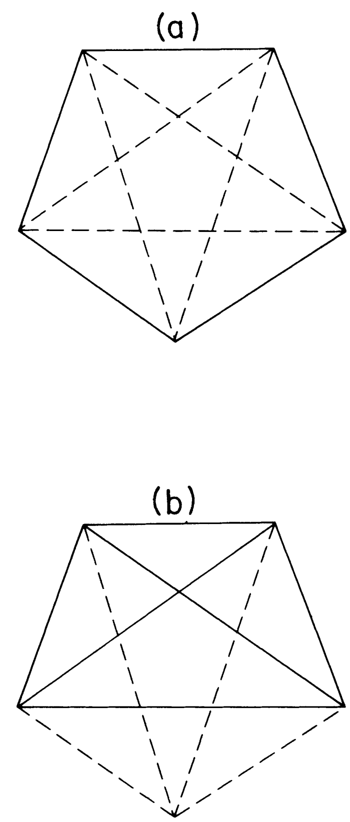


FIG. 2. Five tetrahedra, which form the surface of a four-simplex. (a) A homogeneous but anisotropic configuration with all vertices equal and two different edge lengths (solid lines and dashed lines). (b) A configuration with one peculiar vertex and two types of edge lengths (solid lines and dashed lines).

$$V=\tfrac{1}{12}(-g_a{}^3+2g_a{}^2g_b+2g_ag_b{}^2-g_b{}^3)^{1/2}. \tag{73}$$

Instead of working with the squared edge lengths, we find it more convenient to use the geometric mean of the lengths, the scale factor $r=(g_ag_b)^{1/4}$, and their ratio, $\omega=(g_a/g_b)^{1/2}$ which measures the anisotropy. We also define the Hubble constant $H=\dot r/r$. In terms of these variables, the action (69) becomes

$$S=\int dt\left[-\frac{5r^3\alpha}{2N}\left[H^2+\tfrac{2}{3}H\dot\omega\frac{\alpha'}{\alpha}-\frac{\dot\omega^2}{12\omega^2}\frac{\beta}{\alpha}\right] +10Nr\left[\sqrt{\omega}\,\epsilon_a+\frac{\epsilon_b}{\sqrt{\omega}}\right] -\tfrac{10}{3}\pi N\alpha r^3\left[m^2\phi^2-\frac{\dot\phi^2}{N^2}\right]\right] \tag{74}$$

with

$$\alpha(\omega)=(-\omega^3+2\omega+2\omega^{-1}-\omega^{-3})^{1/2},$$
$$\beta(\omega)=(3\omega^3+2\omega+2\omega^{-1}+3\omega^{-3})/\alpha, \tag{74a}$$

$$\epsilon_a=2\pi-\arccos\left[\frac{2\omega^4-2\omega^2+1}{4\omega^2-1}\right] -2\arccos\left[\frac{\omega(3-2\omega^2)}{[(4\omega^2-1)(4-\omega^2)]^{1/2}}\right],$$
$$\epsilon_b=2\pi-\arccos\left[\frac{2-2\omega^2+\omega^4}{\omega^2(4-\omega^2)}\right] -2\arccos\left[\frac{3\omega^2-2}{\omega[(4\omega^2-1)(4-\omega^2)]^{1/2}}\right]. \tag{74b}$$

Variation of (74) with respect to $N$ leads to the constraint equation

$$H^2+\frac{2}{3}H\dot{\omega}\frac{\alpha'}{\alpha}-\frac{\dot{\omega}^2}{12\omega^2}\frac{\beta}{\alpha}+\frac{4}{r^2\alpha}\left[\epsilon_a\sqrt{\omega}+\frac{\epsilon_b}{\sqrt{\omega}}\right]=\frac{4\pi}{3}(\dot{\phi}^2+m^2\phi^2), \tag{75}$$

where a prime denotes differentiation with respect to $\omega$. The equation of motion of the scalar field is

$$\ddot{\phi}+\left[3H+\frac{\dot{\omega}\alpha'}{\alpha}\right]\dot{\phi}+m^2\phi=0. \tag{76}$$

Variation of (74) with respect to $r$ and $\omega$ leads to evolution equations for $H$ and $\omega$. Since the space is homogeneous with all vertices identical, we see from (32) that the constraint (75) is a first integral of these evolution equations so it is sufficient to use the constraint and one of the evolution equations, that for the anisotropy factor say

$$\ddot{\omega}\left[\frac{\omega\alpha'}{3\alpha}+\frac{\beta}{4\omega\alpha'}\right]+\dot{\omega}H\left[\frac{\omega\alpha'}{\alpha}+\frac{3\beta}{4\omega\alpha'}\right]+\dot{\omega}^2\left[\frac{\alpha''\omega}{3\alpha}+\frac{\beta}{8\alpha\omega}+\frac{\beta'}{8\alpha'\omega}-\frac{\beta}{4\alpha'\omega^2}\right] =\frac{3}{r^2\alpha'}\left[\sqrt{\omega}\,\epsilon_a-\frac{\epsilon_b}{\sqrt{\omega}}\right]-\frac{2\omega}{r^2\alpha}\left[\sqrt{\omega}\,\epsilon_a+\frac{\epsilon_b}{\sqrt{\omega}}\right]. \tag{77}$$

Note that we have taken $N=1$ throughout.

In the isotropic case, Eqs. (75) and (76) reduce to the evolution equations for a Friedmann universe coupled to a massive homogeneous scalar field:

$$H^2=\frac{8\pi}{3}\left[\frac{m^2}{2}\phi^2+\tfrac{1}{2}\dot{\phi}^2\right]+\frac{k}{r^2} \tag{75a}$$

and

$$\ddot{\phi}+3H\dot{\phi}+m^2\phi=0. \tag{76a}$$

The only difference is the numerical factor: $4(\epsilon_a+\epsilon_b)/\alpha$ which should be compared with $k=-1$ in the continuum case. In any case this curvature term decreases rapidly when the universe expands and, as we show later, its inclusion with or without the numerical factor leads to the same evolution pattern, which was studied recently by Belinsky *et al.*[17] and independently by us.[16]

If the initial energy density $E_\phi$ of the scalar field is large enough, the configuration evolves into a de Sitter (inflationary) phase. Only in a very limited range of initial $(\phi,\dot{\phi})$ phase space, the size of which vanishes rapidly as the energy density of the scalar field increases, an inflationary phase does not exist. Later the configuration evolves toward the massive dustlike configuration and the universe goes out of the de Sitter phase. This happens without an apparent phase transition or a change in the scalar potential $v(\phi)=\frac{1}{2}m^2\phi^2$.

An immediate feature of the anisotropic case is that asymptotically, i.e., for large $r$, $\ddot{\omega}=\dot{\omega}=0$ is an approximate solution of Eq. (77). Asymptotically the anisotropy freezes at a given value (depending on the initial conditions). A similar behavior occurs, incidentally, in a Kasner universe when a cosmological constant is included; eventually the anisotropy freezes and the universe expands exponentially. Here this feature remains even after the universe goes out of the inflationary phase provided that $r$ is large enough. When $\dot{\omega}$ vanishes, Eqs. (75) and (76) reduce to the isotropic equations (up to the curvature term which is now small). $H$ and the scalar field follow the same pattern described earlier.

A typical evolution of this anisotropic universe is shown in Fig. 3, in which the de Sitter phase, with an almost constant $H$, and the freezing of the anisotropy are apparent. We have not neglected the curvature term in

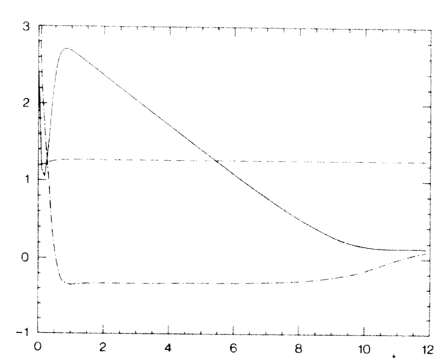


FIG. 3. Hubble's constant (solid line), the anisotropy parameter (dashed line), and normalized $\phi$ (dashed-dotted line) during a typical evolution of an anisotropic but homogeneous toy universe with a scalar field. The initial values were chosen so that the universe will get into (at $t\approx 1$) and out of (at $t\approx 9.5$) an inflationary phase.

solving these equations; however the solution is not affected by it all.

Incidentally, in the isotropic case, one can replace the massive scalar field by pressureless dust. The constraint and evolution equations then reduce to the Lund-Regge equations[6] for the Friedman cosmology

$$0=\frac{1}{4\sqrt{2}}\frac{\ddot{g}}{g^{1/2}}-\frac{1}{16\sqrt{2}}\frac{\dot{g}^2}{g^{3/2}}-g^{-1/2}\epsilon \tag{78a}$$

and

$$0=\frac{1}{4\sqrt{2}}\frac{\dot{g}^2}{g^{1/2}}+4g^{1/2}\epsilon-\frac{16}{5}\pi M\ , \tag{78b}$$

where $M$ is the total rest mass. The equations again have the same form as the continuum theory equations (unlike the four-dimensional Regge-calculus equations for the Friedmann universe[18]) but with different constants.

Our second model also uses $\alpha_4$. In this case, one vertex $\mu$ is different from the other four $\nu$ with all edges meeting at $\mu$ having length squared $g_a$ and all others $g_b$ [see Fig. 2(b)]. Four of the tetrahedra $\alpha$ have equal volumes

$$V_a=\tfrac{1}{12}g_b(3g_a-g_b)^{1/2} \tag{79}$$

and the fifth $\beta$ is equilateral with

$$V_\beta=\frac{1}{6\sqrt{2}}g_b^{3/2}\ . \tag{80}$$

The action for this model universe is given by

$$S=\int dt\Bigg[-\frac{1}{N_aV_a}[2\dot{g}_a\dot{g}_b(V_a^2)^{,ab}+\dot{g}_b^2(V_a^2)^{,bb}]-\frac{1}{4N_\nu V_\beta}\dot{g}_b^2(V_\beta^2)^{,bb}+4(N_\mu+N_\nu)g_a^{1/2}\epsilon_a+12N_\nu g_b^{1/2}\epsilon_b$$
$$-8\pi\left[2(N_\mu+N_\nu)\lambda_a(\phi_\mu-\phi_\nu)^2+N_\mu V_\mu^*\left[m^2\phi_\mu^2-\frac{\dot{\phi}_\mu^2}{N_\mu^2}\right]+4N_\nu V_\nu^*\left[m^2\phi_\nu^2-\frac{\dot{\phi}_\nu^2}{N_\nu^2}\right]\right]\Bigg] \tag{81}$$

with

$$\frac{1}{N_a}=\frac{1}{4}\left[\frac{1}{N_\mu}+\frac{3}{N_\nu}\right],\quad V_\mu^*=\frac{g_a^2g_b}{2(4g_a-g_b)(3g_a-g_b)^{1/2}},\quad V_\nu^*=\frac{1}{24\sqrt{2}}g_b^{3/2}+\frac{g_b(21g_a^2-14g_ag_b+2g_b^2)}{24(4g_a-g_b)(3g_a-g_b)^{1/2}},$$
$$\lambda_a=\frac{3g_ag_b}{4(4g_a-g_b)(3g_a-g_b)^{1/2}},\quad \epsilon_a=2\pi-3\arccos\left[\frac{2g_a-g_b}{4g_a-g_b}\right],\quad \epsilon_b=2\pi-\arccos(\tfrac{1}{3})-2\arccos\left[\frac{g_b}{3(4g_a-g_b)}\right]^{1/2}.$$

Again we define new variables

$$p\equiv g_b\ ,\quad r\equiv g_a/g_b\ ,\quad \phi\equiv\phi_\nu\ ,\quad \psi\equiv\phi_\mu/\phi_\nu\ , \tag{82}$$

and the action (81) becomes

$$S=\int dt\Bigg\{-\frac{1}{4\sqrt{2}N_\nu}\frac{\dot{p}^2}{p^{1/2}}-\frac{1}{2N_ap^{1/2}(3r-1)^{1/2}}[2p\dot{p}\dot{r}+\dot{p}^2(3r-1)]+4p^{1/2}[(N_\mu+N_\nu)r^{1/2}\epsilon_a+3N_\nu\epsilon_b]$$
$$-4\pi\left[(N_\mu+N_\nu)p^{1/2}A\phi^2(1-\psi)^2+N_\mu p^{3/2}B\left[m^2\phi^2\psi^2-\frac{(\dot{\phi}\psi+\phi\dot{\psi})^2}{N_\mu^2}\right]+N_\nu p^{3/2}C\left[m^2\phi^2-\frac{\dot{\phi}^2}{N_\nu^2}\right]\right]\Bigg\}, \tag{83}$$

with

$$A(r)=\frac{3r}{(4r-1)(3r-1)^{1/2}},\quad B(r)=\frac{r}{3}A(r),\quad C(r)=\frac{1}{3}\left[\frac{1}{\sqrt{2}}+\frac{(21r^2-14r+2)}{(4r-1)(3r-1)^{1/2}}\right]. \tag{83a}$$

The constraint equations are obtained by varying (83) with respect to $N_\mu$ and $N_\nu$:

$$\frac{1}{8p(3r-1)^{1/2}}[2p\dot{p}\dot{r}+\dot{p}^2(3r-1)]+4r^{1/2}\epsilon_a=4\pi\{A\phi^2(1-\psi)^2+pB[m^2\phi^2\psi^2+(\dot{\phi}\psi+\phi\dot{\psi})^2]\}\ , \tag{84}$$

$$\frac{3}{8p(3r-1)^{1/2}}[2p\dot{p}\dot{r}+\dot{p}^2(3r-1)]+\frac{1}{4\sqrt{2}}\frac{\dot{p}^2}{p}+4(r^{1/2}\epsilon_a+3\epsilon_b)=4\pi[A\phi^2(1-\psi)^2+pC(m^2\phi^2+\dot{\phi}^2)]\ . \tag{85}$$

The evolution equations for $\phi$ and $\psi$ follow from variation of (83) with respect to these quantities:

$$0=2p\ddot{\phi}(B\psi^2+C)+2Bp\phi\psi\ddot{\psi}+3\dot{p}\dot{\phi}(B\psi^2+C)+3B\dot{p}\phi\psi\dot{\psi}+2p\dot{r}\dot{\phi}(B'\psi^2+C')+2B'p\dot{r}\phi\psi\dot{\psi}+4Bp\psi\dot{\phi}\dot{\psi}$$
$$+4A\phi(1-\psi)^2+2pm^2\phi(B\psi^2+C)\ , \tag{86}$$

$$0=2Bp\phi(\psi\ddot{\phi}+2\dot{\phi}\dot{\psi}+\phi\ddot{\psi})+\phi(3B\dot{p}+2B'p\dot{r})(\dot{\phi}\psi+\dot{\phi}\psi)-4A\phi^2(1-\psi)+2Bpm^2\phi^2\psi\ . \tag{87}$$

Finally the action is varied with respect to $p$ and $r$ to obtain their evolution equations:

$$\left[\frac{1}{2\sqrt{2}}+(3r-1)^{1/2}\right]\left[\frac{\ddot{p}}{p}-\frac{\dot{p}^2}{4p^2}\right]+\frac{1}{(3r-1)^{1/2}}\left[\ddot{r}-\frac{3\dot{r}^2}{2(3r-1)}+\frac{3}{2}\frac{\dot{r}\dot{p}}{p}\right]+\frac{2}{p}(2r^{1/2}\epsilon_a+3\epsilon_b)$$
$$=4\pi\left[\frac{A\phi^2}{p}(1-\psi)^2+\tfrac{3}{2}B[m^2\phi^2\psi^2-(\phi\dot{\psi}+\dot{\phi}\psi)^2]+\tfrac{3}{2}C(m^2\phi^2-\dot{\phi}^2)\right], \quad (88)$$

$$\frac{1}{(3r-1)^{1/2}}\left[\frac{\ddot{p}}{p}-\frac{\dot{p}^2}{4p^2}\right]+\frac{4}{pr^{1/2}}\epsilon_a=4\pi\left[\frac{2A'}{p}\phi^2(1-\psi)^2+B'[m^2\phi^2\psi^2-(\dot{\phi}\psi+\phi\dot{\psi})^2]+C'(m^2\phi^2-\dot{\phi}^2)\right]. \quad (89)$$

Again we have set $N=1$ throughout.

As initial data we specify $p$, $r$, $\phi$, $\psi$, $\dot{\phi}$, and $\dot{\psi}$ and then use the constraint equations to solve for $\dot{p}$ and $\dot{r}$. (This is equivalent to solving for $\dot{g}_a$ and $\dot{g}_b$, having specified the other variables.)

The evolution of this model resembles the previous cases. Again a de Sitter phase appears naturally under most initial conditions. As the radius of the universe increases, the curvature terms (and the spatial gradient terms) become unimportant and the inhomogeneity freezes both in the different edge lengths and the values of the scalar field. A typical evolution of this sort is shown in Fig. 4.

The anisotropy and inhomogeneity that appear in these toy models is very coarse. In other words we consider here only a large-scale inhomogeneity or a large-scale anisotropy. This is due to the small number of edge lengths involved. The observation that these become frozen does not mean that the same will be true for small-scale deviations from homogeneity. These may be better studied using perturbation calculations or much more detailed Regge-calculus models.

## VII. CONCLUSIONS AND OPEN QUESTIONS

We have described a way of extending the three-plus-one formulation of Regge calculus set up by Lund and Regge, to general space-times, using a second-order Lagrangian method. We have also set up a Hamiltonian formulation for Regge calculus. Thus we have two methods of writing down the action and deriving the field equations, by constructing spacelike hypersurfaces composed of tetrahedra, with edge lengths which vary with time. We have also constructed a method of solving the constraint equations on the initial simplicial hypersurface.

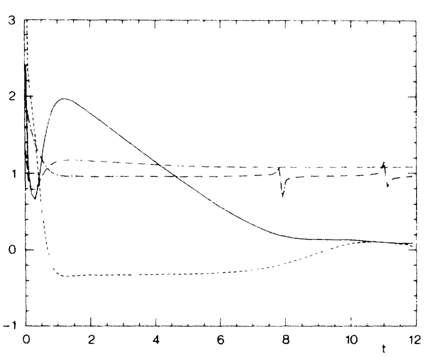


FIG. 4. Hubble's constant (solid line), the anisotropy parameter (dashed line), the scalar-field ratio (dotted-dashed line), and the normalized average $\dot{\phi}$ (short dashed line) during a typical evolution of the inhomogeneous toy universe with a scalar field. The initial values were chosen so that the universe will get into (at $t\approx 1$) and out of (at $t\approx 8$) an inflationary phase.

Possible applications of our formalism fall into two main categories. First, as we have seen from the examples described here, it may be used to study the time evolution of systems like model universes, in classical general relativity. Second, it might be possible to use the Hamiltonian formulation to investigate the canonical quantization of lattice gravity.

There are many open questions which need to be studied. In particular, this three-plus-one formulation of Regge calculus is not really complete without the inclusion of the shift vector and the momentum constraints. A prerequisite here is the representation of a vector field within a simplex. Another aspect of formalism which is lacking and would be specially useful for classical problems is a way of representing matter flow between Regge-calculus blocks, which is clearly necessary for inhomogeneous spaces containing perfect fluid or dust. These questions and others will be addressed in future work.

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## APPENDIX: PROOF OF TRACE FORMULAS IN SEC. II

Formulas (6) and (7) may be proved as follows.[19] For the symmetric tensor $T_{ij}$, define the quantity $T_{ab}$ by

$$T_{ab}=T_{ij}l_a^i l_b^j \quad (\text{A1})$$

in the simplex with edge vectors $l_a^i$ and the volume $V$. Then for all $T$,

$$\det(T_{ab})=(\det l_a^i)^2\det T_{ij}\ , \quad (\text{A2})$$

and, in particular,

$$\det[(1+zT)_{ab}]=(\det l_a^i)^2\det[(1+zT)_{ij}]\ , \tag{A3}$$

where $z$ is a variable. This gives

$$\det[(1+zT)_{ij}]=\frac{\det[(1+zT)_{ab}]}{\det(l_a^i)^2}=\frac{\det[(1+zT)_{ab}]}{\det(1_{ab})}\ , \tag{A4}$$

where

$$\det(1_{ab})=\begin{vmatrix} l_1{}^2 & l_1\cdot l_2 & \\ l_1\cdot l_2 & l_2{}^2 & \cdots \\ \cdots & \cdots & \end{vmatrix}=\frac{1}{(n!)^2}V^2(g_1,\ldots,g_{n(n+1)/2})\ . \tag{A5}$$

To obtain $\det[(1+zT)_{ab}]$ from this, we must replace $l_a\cdot l_b$ by $l_a\cdot l_b+zT_{ab}$. We then have

$$\det[(1+zT)_{ij}]=\frac{V^2(g_a+zT_a)}{V^2(g_a)}\ . \tag{A6}$$

We now expand both sides in $z$: the left-hand side gives

$$\det[(1+zT)_{ij}]=\sum z^iS_i=\prod_{i=1}^{n}(1+z\lambda_i)\ , \tag{A7}$$

where the $\lambda_i$ are the eigenvalues of $T_{ij}$. The coefficients $S_i$ are given by

$$S_0=1\ ,\quad S_1=\sum_{i=1}^{n}\lambda_i=\mathrm{Tr}T\ ,$$
$$S_2=\sum_{i>j}\lambda_i\lambda_j=\tfrac{1}{2}[(\mathrm{Tr}T)^2-\mathrm{Tr}(T^2)]\ , \tag{A8}$$

and so on. For the right-hand side, we have

$$\frac{V^2(l_a{}^2+zT_a)}{V^2(l_a{}^2)}=\frac{1}{V^2}\sum_i\frac{z^i}{i!}\left[\sum_a T_a\frac{\partial}{\partial g_a}\right]^iV^2\ . \tag{A9}$$

Comparing sides, we obtain

$$\mathrm{Tr}T=\frac{1}{V^2}\sum_a T_a\frac{\partial V^2}{\partial g_a}\ , \tag{A10}$$

$$(\mathrm{Tr}T)^2-\mathrm{Tr}(T^2)=\frac{1}{V^2}\sum_{a,b}T_aT_b\frac{\partial^2V^2}{\partial g_a\partial g_b}\ . \tag{A11}$$


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