Vacuum energy density near static distorted black holes

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We investigate the contribution of massless fields of spins 0, $\frac{1}{2}$, and 1 to the vacuum polarization near the event horizon of static Ricci-flat space-times. We do not assume any particular spatial symmetry. Within the Page-Brown "ansatz" we calculate $\langle\phi^2\rangle^{\rm ren}$ and $\langle T_{\mu\nu}\rangle^{\rm ren}$ near static distorted black holes, for both the Hartle-Hawking ($|\ \rangle_H$) and Boulware ($|\ \rangle_B$) vacua. Using Israel's description of static space-times, we express these quantities in an invariant geometric way. We obtain that $\langle\phi^2\rangle_H^{\rm ren}$ and $\langle T_{\mu\nu}\rangle_H^{\rm ren}$ near the horizon depend only on the two-dimensional geometry of the horizon surface. We find $\langle\phi^2\rangle_H^{\rm ren}=(1/48\pi^2)K_0$, $\langle T_0^0\rangle_H^{\rm ren}=(7\alpha+12\beta)K_0{}^2-\alpha{}^{(2)}\Delta K_0$. K_0 is the Gaussian curvature of the horizon, and α and β are numerical coefficients depending on the spin of a field. The term in ${}^{(2)}\Delta K_0$ is characteristic of the distortion of the black hole. When the event horizon is not distorted, K_0 is a constant and this term disappears.

I. INTRODUCTION

The problem of vacuum polarization near black holes is of particular interest for several reasons. Knowledge of the renormalized vacuum expectation value of the stress-energy tensor $[\langle T_\mu{}^\nu(x)\rangle^{\rm ren}]$, which can be considered as a measure of the vacuum polarization, is crucial in order to determine the space-time evolution of an evaporating black hole. As a first step, one usually considers the situation when the space-time geometry is given; that is, one deals with the quantum field theory on a given space-time background. Such an approximation is expected to be rather good when the mass M of the black hole is much larger than the Planckian mass $m_{\rm Pl}=(\hbar c/G)^{1/2}$. In this case, one can use the one-loop approximation in which the contributions of different physical fields to $\langle T_\mu{}^\nu\rangle^{\rm ren}$ are summed additively and may be considered separately. The contributions of massive fields (with mass m) contain the additional factor $\epsilon=m_{\rm Pl}{}^4/m^2M^2$. The presence of a small parameter ϵ (for $\lambda=h/mc\ll 2GM/c^2$) and the fact that the contributions of massive fields are essentially local allow one to study them in detail.[1] The contributions of massless fields which are essentially nonlocal are much more complicated.[2] The aim of this paper is to investigate the contribution of massless fields of spin 0, $\frac{1}{2}$, and 1 to the vacuum polarization near the event horizon of static Ricci-flat space-times. We do not assume any particular spatial symmetry for the geometry. A rather simple approach for calculating $\langle T_\mu{}^\nu(x)\rangle^{\rm ren}$ in static space-times has been proposed by Page.[3] His approximation has been shown to be extremely good in the external space-time of a Schwarzschild black hole.[4–6] Another approach which gives, for a conformal scalar field in the Schwarzschild metric, the same approximative expressions as Page has been proposed by Brown.[7,8] These approaches are based on the possibility of obtaining $\langle T_\mu{}^\nu\rangle^{\rm ren}$ in the space-time of interest from that calculated in an appropriate conformally related space-time where trace anomalies vanish. A brief description of these approaches and their comparison is given in Sec. II. We analyze within Page's approximation the influence of an external gravitational field on the vacuum polarization near black holes. Such a field arises when there are massive bodies outside of the black hole. Their gravitational field changes the metric near the event horizon and distorts the black hole.[9,10] In the case of the scalar field ϕ, there is also some interest in the investigation of $\langle\phi^2\rangle^{\rm ren}$, which describes the quantum fluctuations of this field. We will denote by $\langle T_\mu{}^\nu\rangle^P$ and $\langle\phi^2\rangle^P$ the corresponding quantities obtained in Page and Brown's approximation. As is known, these expectation values depend on the choice of the vacuum state. We will deal here with the Hartle-Hawking ($|\ \rangle_H$) and the Boulware ($|\ \rangle_B$) vacua corresponding to a thermal and to an empty state at large radii, respectively. ($|\ \rangle_B$ is pathological at the horizon in the sense that $\langle T_\mu{}^\nu\rangle_B^{\rm ren}$ and $\langle\phi^2\rangle_B^{\rm ren}$ diverge there.) From a mathematical point of view the study of the behavior of $\langle\phi^2\rangle^{\rm ren}$ and $\langle T_\mu{}^\nu\rangle^{\rm ren}$ near the horizon in the framework of Page and Brown's approximation means the investigation of properties of scalars and tensor invariants constructed from the Weyl tensor, the Killing vector, and their derivatives near the fixed point of the Killing vector. The necessary information concerning the geometrical properties of static space-times is collected in Sec. III. In Sec. IV we describe the convenient choice of coordinates proposed by Israel[10] for studying the static metrics and we obtain the expansion of geometrical invariants near the event horizon. In Sec. V we obtain the explicit expressions of $\langle\phi^2\rangle_H^P$ and $\langle\phi^2\rangle_B^P$ near the event horizon of the distorted black hole. In Sec. VI the components of $\langle T_\mu{}^\nu\rangle_H^P$ and $\langle T_\mu{}^\nu\rangle_B^P$ near the horizon are calculated and their properties are discussed. We find that $\langle\phi^2\rangle_H^P$ and $\langle T_0^0\rangle_H^P$ near the horizon depend only on the two-dimensional geometry of the horizon surface. $\langle\phi^2\rangle_H^P$ is proportional to the Gaussian curvature K_0 of this surface [see Eq. (5.4)]. $\langle T_0^0\rangle_H^P$ is proportional to $K_0{}^2$ and ${}^{(2)}\Delta K_0$ [see Eq. (6.7)]. The term in ${}^{(2)}\Delta K_0$ is characteristic of the distortion of the black hole. When the event horizon is not distorted, K_0 is a constant and this term disappears.

In a number of cases, the formulas obtained here coincide identically with the exact values of $\langle\phi^2\rangle_H$ and $\langle T_\mu{}^\nu\rangle_H$. In particular, it happens for $\langle\phi^2\rangle_H$ (Ref. 1) and $\langle T_\mu{}^\nu\rangle_H$ of the electromagnetic field[11,12] at the horizon of the Schwarzschild black hole and for $\langle\phi^2\rangle_H$ at the pole of the event horizon of the axially symmetric distorted black hole.[13]

We use the sign conventions of Misner, Thorne, and Wheeler[14] and Planck's units $\hbar=c=G=1$.

II. PAGE AND BROWN APPROXIMATION FOR $\langle\phi^2\rangle$ AND $\langle T_\mu{}^\nu\rangle$

In 1982 Page[3] proposed a rather simple approach for calculating $\langle T_{\mu\nu}\rangle^{\text{ren}}$ in static spaces with the geometry obeying the equation

$$R_{\mu\nu}=\Lambda g_{\mu\nu} \quad (\Lambda=\text{const}) . \tag{2.1}$$

Page's approach is based on the following two facts. (i) Under conformal transformations

$$d\bar{S}^2=\Omega^2(x)dS^2 , \tag{2.2}$$

$\langle T_\mu{}^\nu\rangle^{\text{ren}}$ (for conformally related states) transforms in such a way that the following expression remains invariant:

$$g^{1/2}\{\langle T_\mu{}^\nu\rangle^{\text{ren}}+\alpha[(C^{\alpha\nu}{}_{\beta\mu}\ln g)^{;\beta}{}_{;\alpha}+\tfrac{1}{2}R_\alpha{}^\beta C^{\alpha\nu}{}_{\beta\mu}\ln g] +\beta(2H_\mu{}^\nu-4R_\alpha{}^\beta C^{\alpha\nu}{}_{\beta\mu})+(1/G)\gamma I_\mu{}^\nu\} ; \tag{2.3}$$

here,

$$H_\mu{}^\nu=-R_\mu{}^\alpha R_\alpha{}^\nu+\tfrac{2}{3}RR_\mu{}^\nu+(\tfrac{1}{2}R^\alpha{}_\beta R_\alpha{}^\beta-\tfrac{1}{4}R^2)\delta_\mu{}^\nu ,$$
$$I_\mu{}^\nu=2R_{;\mu}{}^{;\nu}-2RR_\mu{}^\nu+(\tfrac{1}{2}R^2-2R^{;\alpha}{}_{;\alpha})\delta_\mu{}^\nu , \tag{2.4}$$

and α,β,γ are numerical coefficients depending on the spin s and on the number of polarizations $h(s)$ of the fields. Their values (calculated by dimensional regularization) are

$$\alpha=\frac{1}{5760\pi^2}[3h(0)+\tfrac{9}{2}h(\tfrac{1}{2})+18h(1)] ,$$
$$\beta=\frac{1}{5760\pi^2}[-h(0)-\tfrac{11}{4}h(\tfrac{1}{2})-31h(1)] , \tag{2.5}$$
$$\gamma=\frac{1}{5760\pi^2}[2h(0)+3h(\tfrac{1}{2})+12h(1)] .$$

(ii) If the static space-time dS^2 (with Killing vector ξ^μ) obeys the relation (2.1) and the conformal factor in Eq. (2.2) is

$$\Omega^2=-\xi_\mu\xi^\mu , \tag{2.6}$$

then conformal anomalies in the space $d\bar{S}^2$ are absent. Page restricted himself to the conformal massless scalar field and used the Gaussian approximation for the propagator in the conformal space $d\bar{S}^2$. For the Schwarzschild space-time and the Hartle-Hawking vacuum state $|\ \rangle_H$, this approximation gives

$$\langle T_\mu{}^\nu(r)\rangle_H^P =\frac{1}{(90)8^4\pi^2M^4}\left[\frac{1-\eta^6(4-3\eta)^2}{(1-\eta)^2}(\delta_\mu{}^\nu-4\delta_0{}^\nu\delta_\mu{}^0) +24\eta^6(3\delta_0{}^\nu\delta_\mu{}^0+\delta_1{}^\nu\delta_\mu{}^1)\right] , \tag{2.7}$$

where

$$\eta=2M/r . \tag{2.8}$$

The analogous approximation can also be used for the expectation value of $T_\mu{}^\nu$ in the Boulware vacuum state $|\ \rangle_B$ with the result[15]

$$\langle T_\mu{}^\nu(r)\rangle_B^P=\frac{\eta^6}{(90)8^4\pi^2M^4}\left[-\left[\frac{4-3\eta}{1-\eta}\right]^2(\delta_\mu{}^\nu-4\delta_0{}^\nu\delta_\mu{}^0) +24(3\delta_0{}^\nu\delta_\mu{}^0+\delta_1{}^\nu\delta_\mu{}^1)\right] . \tag{2.9}$$

For a scalar field ϕ, Page's approximation in the static Ricci-flat space-time gives for $\langle\phi^2\rangle$ in the Boulware vacuum and in the Hartle-Hawking vacuum the following results:

$$\langle\phi^2\rangle_B^P=-\frac{1}{192\pi^2}\frac{X^{;\alpha}X_{;\alpha}}{X^2} , \tag{2.10}$$

$$\langle\phi^2\rangle_H^P=\frac{1}{48\pi^2X}\left[\kappa_0{}^2-\frac{X^{;\alpha}X_{;\alpha}}{4X}\right] . \tag{2.11}$$

Here

$$X\equiv-\xi_\mu\xi^\mu \tag{2.12}$$

and κ_0 is the surface gravity of the black hole $(1/4M)$.

Brown[7,8] proposed another approach which gives, for a conformal scalar field in the Schwarzschild space-time, the same approximative expressions (2.7) and (2.9). His approach is based on the possibility to choose, in the space where conformal anomalies are absent, the state for which the average value of $\langle T_\mu{}^\nu\rangle^{\text{ren}}$ vanishes. In the static Ricci-flat $(R_{\alpha\beta}=0)$ space-time for the choice of the conformal factor (2.6) this ansatz gives

$$\langle T_\mu{}^\nu\rangle_B^P=\alpha A^{(0)}{}_\mu{}^\nu+\beta B^{(0)}{}_\mu{}^\nu , \tag{2.13}$$

$$A^{(0)}{}_\mu{}^\nu=-8C_\mu{}^{\alpha\nu\beta}\omega_{\alpha\beta}-\tfrac{4}{3}(\omega_\alpha\omega^\alpha)_{;\mu}{}^{;\nu} -4\omega_\mu(\omega_\alpha\omega^\alpha)^{;\nu}-4\omega^\nu(\omega_\alpha\omega^\alpha)_{;\mu} -8\omega_\mu\omega^\nu(\omega_\alpha\omega^\alpha)+2\delta_\mu{}^\nu[2\omega^\beta(\omega_\alpha\omega^\alpha)_{;\beta}+(\omega_\alpha\omega^\alpha)^2 +\tfrac{2}{3}(\omega_\alpha\omega^\alpha)_{;\beta}{}^{;\beta}] , \tag{2.14}$$

$$B^{(0)}{}_\mu{}^\nu=-8C_\mu{}^{\alpha\nu\beta}\omega_{\alpha\beta}-8C_\mu{}^{\alpha\nu\beta}\omega_\alpha\omega_\beta -8\omega_{\mu\alpha}\omega^{\alpha\nu}-4\omega_\mu(\omega_\alpha\omega^\alpha)^{;\nu}-4\omega^\nu(\omega_\alpha\omega^\alpha)_{;\mu} -8\omega_\mu\omega^\nu(\omega_\alpha\omega^\alpha)+4\delta_\mu{}^\nu[\omega_{\alpha\beta}\omega^{\alpha\beta}+(\omega_\alpha\omega^\alpha)_{;\beta}\omega^\beta +\tfrac{1}{2}(\omega_\alpha\omega^\alpha)^2] . \tag{2.15}$$

Here

$$\omega=\ln\Omega=\tfrac{1}{2}\ln(-\xi_\mu\xi^\mu) \tag{2.16}$$

and coefficients α and β are given by Eq. (2.5). This expression for the stress-energy tensor is related with the Boulware vacuum choice and in the case of the scalar field near the Schwarzschild black hole it reproduces Eq. (2.9).

In the general case, the conformal factor ω relating the physical space-time and the space without conformal anomalies is defined up to some freedom associated with the choice of the vacuum state. In particular for the static Ricci-flat space-time, ω allows the following transformation:

$$\tilde{\omega}=\omega+at \tag{2.17}$$

(i.e., $e^{2\tilde{\omega}}\equiv\tilde{\Omega}^2=\Omega^2e^{2at}$), where t is the Killing time $[\xi^\mu=(\partial/\partial t)^\mu]$ and a is an arbitrary constant. Brown and Ottewill[8] have shown that the choice

$$a=2\pi\kappa_0\ , \tag{2.18}$$

where κ_0 is the surface gravity of a black hole, corresponds to the choice of the Hartle-Hawking vacuum state. The expectation value of the stress-energy tensor in this case is of the form

$$\langle T_\mu{}^\nu\rangle_H^P=\alpha A_\mu{}^\nu+\beta B_\mu{}^\nu\ , \tag{2.19}$$

where

$$A_\mu{}^\nu=A^{(0)}{}_\mu{}^\nu+A^{(1)}{}_\mu{}^\nu+A^{(2)}{}_\mu{}^\nu\ ,$$
$$B_\mu{}^\nu=B^{(0)}{}_\mu{}^\nu+B^{(1)}{}_\mu{}^\nu+B^{(2)}{}_\mu{}^\nu\ , \tag{2.20}$$

$A(0)_\mu{}^\nu$ and $B(0)_\mu{}^\nu$ are given by Eqs. (2.14) and (2.15), and

$$A^{(1)}{}_\mu{}^\nu=-\frac{\kappa_0{}^2}{3}\frac{1}{X^2}\left[X^{;\alpha}{}_{;\alpha}\left[\delta_\mu{}^\nu+\frac{4\xi_\mu\xi^\nu}{X}\right]+8C_\mu{}^{\alpha\nu\beta}\xi_\alpha\xi_\beta\right]\ ,$$
$$A^{(2)}{}_\mu{}^\nu=\frac{2\kappa_0{}^4}{X^2}\left[\delta_\mu{}^\nu+\frac{4\xi_\mu\xi^\nu}{X}\right]\ ,$$
$$B^{(1)}{}_\mu{}^\nu=3A^{(1)}{}_\mu{}^\nu\ ,$$
$$B^{(2)}{}_\mu{}^\nu=A^{(2)}{}_\mu{}^\nu\ . \tag{2.21}$$

For the scalar massless field in the static Ricci-flat space-time Eqs. (2.19)–(2.21) give the same result as that obtained from Page's approximation. It should be stressed that the Brown-Ottewill formula (2.19) at far distances from the black hole gives

$$\langle T_\mu{}^\nu\rangle_H^P\approx 2\kappa_0{}^4(\alpha+\beta)(\delta_\mu{}^\nu-4\delta_\mu{}^0\delta_0{}^\nu)\ . \tag{2.22}$$

For the values of coefficients α and β given by Eq. (2.5) this expression reproduces correctly the behavior of the thermal (with temperature $\theta=\kappa_0/2\pi$) stress-energy density for the scalar and spinor fields. However, it gives the wrong sign and value for the electromagnetic field. It should be noted that the Brown-Ottewill formula was obtained using the values of α, β, and γ as given by Eq. (2.5). Different renormalization procedures give the same values for all the coefficients in Eq. (2.5) except the coefficient before $h(1)$ in γ. The possibility exists that using this freedom one may "improve" the approximation in the case of the electromagnetic field. Nevertheless, it appears that at the horizon of a Schwarzschild black hole the Brown-Ottewill formula gives the result which coincides with the exact value of $\langle T_\mu{}^\nu\rangle_H^{\text{ren}}$.

III. GEOMETRICAL PROPERTIES OF STATIC DISTORTED BLACK HOLES

Here we collect results connected with the geometrical properties of static black holes. We begin by discussing the properties of the scalar and tensor invariants in the static space-time which are constructed with the help of the Killing vector (for a general discussion see the paper by Boyer[16]).

Let ξ^μ be a Killing vector field in the static space-time with metric $g_{\mu\nu}$; $\xi_\mu\xi^\mu<0$. It means that the following relations are satisfied:

$$\xi_{(\mu;\nu)}\equiv\tfrac{1}{2}(\xi_{\mu;\nu}+\xi_{\nu;\mu})=0\ , \tag{3.1}$$

$$\xi_{[\mu}\xi_{\nu;\lambda]}=0\ , \tag{3.2}$$

$$\xi_{\alpha;\beta;\gamma}=R_{\alpha\beta\gamma\delta}\xi^\delta\ . \tag{3.3}$$

Denote $X=-\xi_\mu\xi^\mu$, then using the Killing equation (3.1) one can obtain the following relations:

$$X_{,\alpha}\xi^\alpha=0\ , \tag{3.4}$$

$$\tfrac{1}{2}X_{,\alpha}X^{,\alpha}=-X\xi_{\alpha;\beta}\xi^{\alpha;\beta}-3\xi_{[\alpha}\xi_{\beta;\gamma]}\xi^{[\alpha}\xi^{\beta;\gamma]}\ . \tag{3.5}$$

Equations (3.2) and (3.3) allow one to rewrite Eq. (3.5) in the following two equivalent forms:

$$\frac{X_{,\alpha}X^{,\alpha}}{X}=-2D\ , \tag{3.6}$$

$$X_{,\alpha}X^{,\alpha}=X(X_{;\alpha}{}^{;\alpha}-2R_{\alpha\beta}\xi^\alpha\xi^\beta)\ , \tag{3.7}$$

where

$$D\equiv\xi_{\alpha;\beta}\xi^{\alpha;\beta}\ . \tag{3.8}$$

We restrict ourselves by considering the Ricci-flat static metrics. In this case Eqs. (3.3) and (3.7) read

$$\xi_{\alpha;\beta;\gamma}=C_{\alpha\beta\gamma\delta}\xi^\delta\ , \tag{3.9}$$

$$X_{;\alpha}{}^{;\alpha}=-2D\ . \tag{3.10}$$

Equation (3.2) implies that

$$\xi_{\alpha;\beta}=X^{-1}\xi_{[\alpha}X_{,\beta]}\equiv\tfrac{1}{2}X^{-1}(\xi_\alpha X_{,\beta}-\xi_\beta X_{,\alpha})\ . \tag{3.11}$$

Using this relation one can obtain

$$\xi_{\alpha;\beta}\xi^{\beta;\gamma}=\frac{1}{4X}(2D\xi_\alpha\xi^\gamma+X_{,\alpha}X^{,\gamma})\ , \tag{3.12}$$

$$\xi_{\alpha;\beta}\xi^{\beta;\gamma}\xi_{\gamma;\delta}=-\tfrac{1}{2}D\xi_{\alpha;\delta}\ . \tag{3.13}$$

In what follows we shall also need a number of relations listed below which can be easily verified by using the mentioned properties of ξ^μ:

$$X_{;\mu\nu}=+\frac{D}{X}\xi_\mu\xi_\nu+\frac{1}{2}\frac{X_{,\mu}X_{,\nu}}{X}+2C_{\mu\alpha\nu\beta}\xi^\alpha\xi^\beta\ , \tag{3.14}$$

$$D_{,\mu}=2C_{\alpha\beta\mu\delta}\xi^{\alpha;\beta}\xi^\delta\ , \tag{3.15}$$

$$D_{;\mu;\nu}=2C_{\alpha\beta\mu\delta;\nu}\xi^{\alpha;\beta}\xi^{\delta}+2C^{\alpha\beta}{}_{\nu\lambda}C_{\alpha\beta\mu\delta}\xi^{\lambda}\xi^{\delta}+2C_{\alpha\beta\mu\delta}\xi^{\alpha;\beta}\xi^{\delta}{}_{;\nu}\ . \qquad (3.16)$$

We suppose now that the considered space-time possesses a bifurcate Killing horizon. Then at the two-dimensional surface of the bifurcation of the horizon $\xi^{\mu}=0$ and D is finite and negative.[16]

IV. ISRAEL'S COORDINATES AND EXPANSION OF GEOMETRICAL INVARIANTS NEAR THE HORIZON OF STATIC DISTORTED BLACK HOLES

In order to study the behavior of geometric invariants near the event horizon of static black holes it appears convenient to use the coordinates introduced by Israel.[10] Using the results of his paper one can show that the metric of a static space-time can be written as

$$dS^2=-X\,dt^2+\frac{dX^2}{4\kappa^2X}+h_{ab}d\theta^a d\theta^b\ , \qquad (4.1)$$

where $a,b,\ldots=2,3$, $hab=hab(X,\theta^a)$, and

$$\kappa(X,\theta^a)=(-D/2)^{1/2}\ . \qquad (4.2)$$

We denote by $(\ )_{:a}$ the covariant derivative with respect to the two-dimensional metric

$$dh^2=h_{ab}d\theta^a d\theta^b\ . \qquad (4.3)$$

We also use the three-dimensional metric

$$dq^2=q_{AB}dx^A dx^B\equiv\frac{dX^2}{4\kappa^2X}+h_{ab}\,d\theta^a d\theta^b \qquad (4.4)$$

$(A,B,\ldots=1,2,3)$, and denote the covariant derivative with respect to this metric by $(\ )_{|A}$. Let

$$\partial_X h_{ab}=\kappa^{-1}k_{ab}\ , \qquad (4.5)$$

then $X^{1/2}k_{ab}$ is the external curvature of the two-dimensional surface $X=\text{const}$ embedded in the three-dimensional space dq^2. The condition that the space-time (4.1) is Ricci flat implies the relations

$$\partial_X\kappa=-\tfrac{1}{2}k\ , \qquad (4.6)$$

$$X\partial_X k_a{}^b=-k_a{}^b+\tfrac{1}{2}k\delta_a{}^b-\tfrac{1}{2}(\kappa^{-1})_{:a}{}^{:b}-\frac{X}{4\kappa}[(k_{cd}k^{cd}-k^2)\delta_a{}^b+2kk_a{}^b]\ , \qquad (4.7)$$

$$K=\kappa k-\tfrac{1}{2}X(k_{ab}k^{ab}-k^2)\ , \qquad (4.8)$$

$$\partial_a\kappa=-X(k_{:a}-k_a{}^b{}_{:b})\ . \qquad (4.9)$$

Here K is the Gaussian curvature of the two-dimensional surface $X=\text{const}$ and

$$k=h^{ab}k_{ab}\ . \qquad (4.10)$$

(All the operations with $a,b,\ldots$ indices are performed using the two-dimensional metric h_{ab} and its inverse h^{ab}.) Equations (4.5)–(4.7) play the role of "dynamical" equations and Eqs. (4.8) and (4.9) are "constraints." Given the values κ, h_{ab}, and k_{ab} on the surface $X=X_0$, satisfying the constraint equations, then Eqs. (4.5)–(4.7) allow one to define these quantities for other values of X. The constraints (4.8) and (4.9) will be preserved.

Let η^{μ} be a unit vector in the direction X^{μ}:

$$n^{\mu}=X^{\mu}/2\kappa X^{1/2}\ . \qquad (4.11)$$

Denote

$$\Pi_{AB}=\frac{1}{2}\left[X_{|AB}-\frac{X_{|A}X_{|B}}{2X}\right]\ ; \qquad (4.12)$$

then we have

$$\Pi_{AB}=X\kappa(k_{ab}\delta_A{}^a\delta_B{}^b-kn_An_B)+X^{1/2}\partial_a\kappa(\delta_A{}^a n_B+\delta_B{}^a n_A),\quad \Pi_A{}^A=0\ . \qquad (4.13)$$

In the Ricci-flat space-time the following relations are valid:[11]

$$C_{\alpha A\beta B}\xi^{\alpha}\xi^{\beta}=\Pi_{AB}\ , \qquad (4.14)$$

$$C_{\alpha ABC}\xi^{\alpha}=0\ , \qquad (4.15)$$

$$C_{AB}{}^{CD}=-\epsilon_{ABM}\epsilon^{CDN}\Pi_N{}^M X^{-1}=X^{-1}(\Pi_A{}^C\delta_B{}^D+\Pi_B{}^D\delta_A{}^C-\Pi_A{}^D\delta_B{}^C-\Pi_B{}^C\delta_A{}^D)\ . \qquad (4.16)$$

These relations allow one to show that

$$C\equiv C_{\alpha\beta\gamma\delta}\xi^{\alpha;\beta}\xi^{\gamma;\delta}=-4\kappa^3k\ , \qquad (4.17)$$

$$E\equiv C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}=8X^{-2}C^{\alpha}{}_{\mu\beta\nu}C^{\beta}{}_{\lambda\alpha\kappa}\xi^{\mu}\xi^{\nu}\xi^{\lambda}\xi^{\kappa}=8X^{-2}\Pi_{AB}\Pi^{AB}=8\kappa^2\left[k_{ab}k^{ab}+k^2+2\frac{\kappa_{:a}\kappa^{:a}}{X\kappa^2}\right]\ . \qquad (4.18)$$

At the event horizon $X=0$ and κ is finite. The regularity of the space-time near the horizon means that the invariants E and K are finite and h_{ab} is regular. Equations (4.8), (4.9), and (4.18) show that k_{ab} and k are finite at the horizon. We can write the following expansions for the quantities defining the geometry near the horizon:

$$\kappa=\sum_{n=0}^{\infty}\kappa_nX^n,\quad k_a{}^b=\sum_{n=0}^{\infty}k_{na}{}^bX^n\ ,\quad k=\sum_{n=0}^{\infty}k_nX^n,\quad h_{ab}=\sum_{n=0}^{\infty}h_{nab}X^n\ ,\quad K=\sum_{n=0}^{\infty}K_nX^n\ . \qquad (4.19)$$

If the quantities h_{0ab} and κ_0 are given, then substituting Eq. (4.19) into (4.9) one can define all the coefficients in the series. The following formulas for the first few coefficients will be used:

$$K_0=\kappa_0k_0\ , \qquad (4.20)$$

$$k_{0a}{}^b=\tfrac{1}{2}k_0\delta_a{}^b\ , \qquad (4.21)$$

$$k_1=-\frac{1}{4\kappa_0}(k_0)^2-\frac{1}{4\kappa_0{}^2}k_{0:a}{}^{:a}\ , \qquad (4.22)$$

$$k_{1a}{}^{b}=-\frac{1}{8\kappa_0}(k_0)^2\delta_a{}^b-\frac{1}{8\kappa_0{}^2}(k_{0:a}{}^{:b}+\tfrac{1}{2}\delta_a{}^b k_{0:c}{}^{:c}) , \tag{4.23}$$

$$h_{1ab}=\frac{1}{2\kappa_0}k_0 h_0 , \tag{4.24}$$

$$h_{2ab}=\frac{1}{4\kappa_0{}^2}k_0{}^2 h_{0ab}+\frac{1}{2\kappa_0}k_{1a}^{c}h_{0cb} , \tag{4.25}$$

$$\kappa_{0,a}=0 , \tag{4.26}$$

$$\kappa_1=-\tfrac{1}{2}k_0 , \tag{4.27}$$

$$\kappa_2=-\tfrac{1}{4}k_1 . \tag{4.28}$$

The quantities h_{0ab} and $\kappa_0=\text{const}$ may be considered as the "initial data" for the dynamical equations (4.5)–(4.7). An arbitrariness of a constant κ_0 reflects the possibility of changing the normalization of ξ^μ. We assume that the normalization of ξ^μ is chosen in such a way that $\xi_\mu\xi^\mu=-1$ at infinity. In this case κ_0 coincides with the surface gravity of the black hole.

V. $\langle\phi^2\rangle^{\text{ren}}$ NEAR DISTORTED BLACK HOLES

In order to describe the behavior of quantities $\langle\phi^2\rangle_B^P$ and $\langle\phi^2\rangle_H^P$ given by Eqs. (2.10) and (2.11) near distorted black holes we express them in terms of the invariant functions which enter in the Israel metric (4.1). Namely, using Eqs. (3.6) and (4.2) we have

$$\langle\phi^2\rangle_B^P=\frac{-\kappa^2}{48\pi^2 X} \tag{5.1}$$

and

$$\langle\phi^2\rangle_H^P=\frac{\kappa_0{}^2-\kappa^2}{48\pi^2 X} . \tag{5.2}$$

The expectation value $\langle\phi^2\rangle_B^P$ for the Boulware vacuum state diverges at the horizon

$$\langle\phi^2\rangle_B^P=\frac{-\kappa_0{}^2}{48\pi^2 X}+\frac{K_0}{48\pi^2}+O(X) , \tag{5.3}$$

where κ_0 is the surface gravity and K_0 is the Gaussian curvature[17] of the black hole. The divergent part of this expression correctly reproduces the behavior of $\langle\phi^2\rangle_B$ near the Schwarzschild black hole.

The value of $\langle\phi^2\rangle_H^P$ is finite at the horizon. Using the relations (4.20) and (4.27) one can verify that at the horizon $\langle\phi^2\rangle_H^P$ depends only on the Gaussian curvature K_0 of the two-dimensional surface of the horizon:

$$\langle\phi^2\rangle_H^P=\frac{1}{48\pi^2}K_0 . \tag{5.4}$$

It could be noted that this formula reproduces the exact value of $\langle\phi^2\rangle_H$ at the horizon of a Schwarzschild black hole[1] and $\langle\phi^2\rangle_H$ at the pole of an axisymmetric distorted black hole.[13]

It is worthwhile emphasizing that the same formula (5.4) gives the exact value of $\langle\phi^2\rangle_H$ at the pole of the event horizon of a rotating black hole.[18] It means that one can expect that the Page approximation must be rather good in an axisymmetric stationary (not necessary static) Ricci-flat space-time for the points located near the axis of symmetry.

VI. $\langle T_\mu{}^\nu\rangle^{\text{ren}}$ NEAR DISTORTED BLACK HOLES

We consider now the asymptotic behavior of $\langle T_\mu{}^\nu\rangle_B^P$ and $\langle T_\mu{}^\nu\rangle_H^P$ near the event horizon.

In a static space-time the tensors $A_{\mu\nu}^{(i)}$ and $B_{\mu\nu}^{(i)}$ ($i=0,1,2$) possess the following property:

$$A_{0A}^{(i)}=B_{0A}^{(i)}=0 . \tag{6.1}$$

In order to calculate $\langle T_{\mu\nu}\rangle_H^P$ and $\langle T_{\mu\nu}\rangle_B^P$ one needs to know only AB components of these quantities; 00 components can be fixed by using the expression for conformal anomalies:

$$\begin{aligned}\langle T_0^0\rangle_H^P+\langle T_A{}^A\rangle_H^P&=\langle T_0^0\rangle_B^P+\langle T_A{}^A\rangle_B^P=(\alpha+\beta)C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}\\&=8(\alpha+\beta)\kappa^2\left[k_{ab}k^{ab}+k^2+\frac{2\kappa_{:a}\kappa^{:a}}{X\kappa^2}\right] .\end{aligned} \tag{6.2}$$

In order to evaluate $A^{(0)A}{}_B$ and $B^{(0)A}{}_B$ [given by Eqs. (2.14) and (2.15)] near a horizon we use Eqs. (3.14)–(3.16), (4.14)–(4.18), and the following relations:

$$\begin{aligned}
&X_\alpha=2\kappa X^{1/2}n_\alpha ,\\
&\omega_\alpha=\kappa X^{-1/2}n_\alpha ,\\
&\omega_{\alpha\beta}=\frac{-\kappa^2}{X}n_\alpha n_\beta-\frac{\kappa^2}{X^2}\xi_\alpha\xi_\beta+\frac{1}{X}\Pi_{AB}\delta_\alpha{}^A\delta_\beta{}^B ,\\
&(\omega_\gamma\omega^\gamma)_{;\alpha}=\frac{2\kappa}{X^{3/2}}(-\kappa^2 n_\alpha-X\kappa k n_\alpha+X^{1/2}\partial_a\kappa\delta^a{}_\alpha) ,\\
&\omega^{\alpha\gamma}\omega_{\gamma\beta}=\frac{1}{X^2}\left[-\frac{\kappa^4}{X}\xi^\alpha\xi_\beta+\kappa^4 n^\alpha n_\beta+4\kappa^2(n^\alpha n^\gamma\Pi_{BC}\delta_\beta{}^B\delta_\gamma{}^C+n_\beta n^\gamma\Pi_B{}^C\delta_\gamma{}^B\delta_C{}^\alpha)+\Pi^{AC}\Pi_{CB}\delta_A{}^\alpha\delta_\beta{}^B\right] ,\\
&(\omega_\gamma\omega^\gamma)_{;\alpha\beta}=\frac{8\kappa^4}{X^2}n_\alpha n_\beta-\frac{\kappa^2 X_{\alpha\beta}}{X}+\frac{D_{,\alpha}X_\beta+D_{,\beta}X_\alpha}{2X^2}-\frac{D_{\alpha\beta}}{2X} .
\end{aligned} \tag{6.3}$$

These relations allow one to show that the leading term of $\langle T_B{}^A\rangle_B^P$ divergent at the horizon is of the form

$$\langle T_B{}^A\rangle_B^P \approx \frac{2\kappa_0{}^4}{X^2}(\beta-\tfrac{1}{3}\alpha)\delta_B{}^A . \tag{6.4}$$

The trace $\langle T_\mu{}^\mu\rangle_B^P$ is finite at the horizon; thus

$$\langle T_\mu{}^\nu\rangle_B^P \approx \frac{2\kappa_0{}^4}{X^2}(\beta-\tfrac{1}{3}\alpha)\operatorname{diag}(-3,1,1,1)_\mu^\nu , \tag{6.5}$$

when $\operatorname{diag}(a_1,a_2,a_3,a_4)$ means the diagonal matrix with the diagonal elements a_i.

It is instructive to compare these results with the exact asymptotics of $\langle T_\mu{}^\nu\rangle_B$ for conformal massless fields of spin $s=0$, $\frac{1}{2}$, and 1 near the Schwarzschild black hole:[19]

$$\langle T_\mu{}^\nu\rangle_B \approx \frac{-h(s)}{6\pi^2X^2}\kappa_0{}^4F(s)\operatorname{diag}(-3,1,1,1)_\mu^\mu , \tag{6.6}$$

where

$$F(s)=\int_0^\infty \frac{dx\,x(x^2+s^2)}{\exp(2\pi x)-(-1)^{2s}}=\begin{cases}\frac{1}{240}, & s=0 ,\\ \frac{17}{1920}, & s=\frac{1}{2} ,\\ \frac{37}{480}, & s=1 .\end{cases} \tag{6.7}$$

The asymptotics (6.4) and (6.5) coincide for $s=0$ and $s=\frac{1}{2}$ and are different for $s=1$. It means that the Brown-Ottewill formula (2.13) does not give the correct result for the electromagnetic energy-momentum tensor in the Boulware vacuum state.

The expressions for $\langle T_\beta{}^\nu\rangle_H^P$ components are rather cumbersome. We present here only the expression for $\langle T_0^0\rangle_H^P$ which describes the energy density of the vacuum polarization. The straightforward calculations give

$$\langle T_0^0\rangle_H^P=\alpha A_0^0+\beta B_0^0 ,$$

$$A_0^0=-2\frac{(\kappa_0{}^2-\kappa^2)(\kappa^2+3\kappa_0{}^2)}{X^2}-2\frac{C}{X}+\tfrac{4}{3}E , \tag{6.8}$$

$$B_0^0=-\frac{6}{X^2}(\kappa_0{}^2-\kappa^2)^2+\tfrac{3}{2}E .$$

[Here C and E are given by Eqs. (4.17) and (4.18).] Using the expansions (4.19)—(4.28) we obtain at the event horizon

$$\epsilon=-\langle T_0^0\rangle_H^P=-(7\alpha+12\beta)K_0{}^2+\alpha K_{0:a}{}^{:a} . \tag{6.9}$$

The coefficients α,β, taking into account the dependence on the spin, are given by Eq. (2.5).

When the black hole is not distorted K_0 is a constant and only the first term in the right-hand side of Eq. (6.7) survives. For the scalar ($s=0$), spinor (two-components) ($s=\frac{1}{2}$), and electromagnetic fields ($s=1$), the values of $\langle T_\mu{}^\nu\rangle_H^{P(s)}$ at the horizon of the Schwarzschild black hole are

$$\langle T_\mu{}^\nu\rangle_H^{P(s=0)}=\frac{1}{1920\pi^2(2M)^4}\operatorname{diag}(3,3,1,1)_\mu^\nu ,$$

$$\langle T_\mu{}^\nu\rangle_H^{P(s=1/2)}=\frac{1}{960\pi^2(2M)^4}\operatorname{diag}(-1,-1,8,8)_\mu^\nu , \tag{6.10}$$

$$\langle T_\mu{}^\nu\rangle_H^{P(s=1)}=\frac{1}{480\pi^2(2M)^4}\operatorname{diag}(-41,-41,28,28)_\mu^\nu .$$

Candelas and Howard[5,6] have shown that $\langle T_\mu{}^\nu(x)\rangle_H^{P(s=0)}$ reproduces the behavior of the exact values $\langle T_\mu{}^\nu(x)\rangle_H^{\mathrm{ren}(s=0)}$ with high accuracy. The deflection of the components of $\langle T_\mu{}^\nu\rangle_H^{P(s=0)}$ in (t,r,θ,ϕ) coordinates from the exact ones at the event horizon does not exceed 20%.

For the electromagnetic field the value of $\langle T_\mu{}^\nu\rangle_H^{P(s=1)}$ at the event horizon of the Schwarzschild black hole coincides identically with the exact value $\langle T_\mu{}^\nu\rangle_H^{\mathrm{ren}}$.[11,12] It is a rather astonishing fact because neither $\langle T_\mu{}^\nu\rangle_H^P$ at large distances nor $\langle T_\mu{}^\nu\rangle_B^P$ at the horizon reproduces the correct result for the electromagnetic field.

VII. DISCUSSION

The general property of Page and Brown's ansatz is the possibility to express in Ricci-flat static space-times the quantities $\langle T_\mu{}^\nu\rangle^{\mathrm{ren}}$ and $\langle\phi^2\rangle^{\mathrm{ren}}$ in terms of the Weyl tensor $C_{\alpha\beta\gamma\delta}$, the Killing vector ξ^μ, and their derivatives. All the derivatives of ξ^μ which are of higher than first order can be eliminated by using the relation (3.9). The first-order derivative $\xi_{\alpha;\beta}$ can be expressed in terms of ξ_α and of the gradient $X_{,\mu}=(\xi^\alpha\xi_\alpha)_{,\mu}$ [Eq. (3.11)]. It means that the components of $\langle T_\mu{}^\nu\rangle_H^P$ and $\langle T_\mu{}^\nu\rangle_B^P$ can be expressed in terms of the Weyl tensor, its derivatives and two vectors ξ^α and X^α. The explicit form of these expressions is greatly simplified in Israel's coordinates. The main advantage of these coordinates is the possibility of expressing the quantities under consideration in terms of the internal and external curvature of two-dimensional (equipotential) surfaces $t=$const, $\xi_\mu\xi^\mu=$const. The external curvature of the surface of the static black hole appears to be equal to zero and at this surface all the expressions are greatly simplified. In particular, $\langle\phi^2\rangle_H^P$ is simply proportional to the Gaussian curvature K_0 of the black-hole surface [Eq. (5.4)] while the expression for the "energy density" $\epsilon=-(1/|\xi_\alpha\xi^\alpha|)T_{\mu\nu}\xi^\mu\xi^\nu$ contains $K_0{}^2$ and ${}^{(2)}\Delta K_0$.

It should be stressed once more that we restricted ourselves by studying the expression for $\langle\phi^2\rangle^{\mathrm{ren}}$ and $\langle T_\mu{}^\nu\rangle^{\mathrm{ren}}$ only in the framework of Page and Brown's ansatz. We have shown that in a number of cases these values coincide with the exact value for these quantities. In particular, it happens with the components of $\langle T_\mu{}^\nu\rangle_H^P$ at the event horizon of the Schwarzschild black hole in the case of the electromagnetic field. The reason for this as well as the reason for the remarkable accuracy of Page's approximation in the case of the scalar field still remains unknown.

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