Feynman Green function inside a Schwarzschild black hole

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A representation is given for the Feynman Green function corresponding to a scalar field that propagates in Schwarzschild spacetime and which is subject to the Hartle-Hawking boundary conditions. This representation is valid for all values of the Schwarzschild radial coordinate r and is used to obtain an expression for $\langle\varphi^2\rangle_{\text{reg}}$ for the region interior to the horizon. This expression is evaluated numerically for the range $0.5M < r < 2M$.

I. INTRODUCTION

Our purpose in this article is to provide an explicit expression for the Hartle-Hawking[1] Green function for a scalar field φ on a Schwarzschild background corresponding to the region inside the event horizon (i.e., valid for coordinate $r<2M$) and to calculate the regularized value of $\langle\varphi^2\rangle$ there.

The Schwarzschild metric is

$$ds^2=-(1-2M/r)dt^2+\frac{dr^2}{1-2M/r}+r^2(d\theta^2+\sin\theta^2\phi^2)\ . \tag{1}$$

The Hartle-Hawking Green function for the massless scalar field is the solution of

$$\Box G(x,x')=-g^{-1/2}\delta(x,x') \tag{2}$$

which is singled out uniquely by the requirement that $G(x,x')$ both tend to zero as the spatial distance $d(\mathbf{x},\mathbf{x}')$ tends to infinity and is periodic in imaginary time. The way in which the periodicity condition picks out a unique Green function is somewhat subtle, as we now indicate.

Consider the metric (1) with t replaced by $i\tau$:

$$ds^2=\left[1-\frac{2M}{r}\right]d\tau^2+\frac{dr^2}{1-2M/r}+r^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{3}$$

or

$$ds^2=\frac{2M}{\kappa^2 r}e^{-r/2M}[\rho^2 d(\kappa\tau)^2+d\rho^2]+r^2(d\theta^2+\sin^2\theta\,d\phi^2)\ , \tag{4}$$

where

$$\rho=\left[\frac{r}{2M}-1\right]^{1/2}e^{r/4M}$$

and

$$\kappa=\frac{1}{4M}\ .$$

Provided $r\geq 2M$, this metric has positive-definite signature and is without conical singularities if τ is interpreted as an angular coordinate with period $2\pi/\kappa=8\pi M$. We schematically graph this manifold in Fig. 1, with θ and ϕ suppressed. Specification of $G(x,x')$ at $r=2M$ and at $r=\infty$ together with Eq. (2) comprises a well-posed problem.

We would like to construct the analogous statement for the region $r\leq 2M$. In this region it is not sufficient to pass to imaginary time, as this would leave our metric with signature $(--++)$. If we make the additional change $\theta\to i\tilde\theta$, the metric becomes

$$ds^2=-\left[\left[\frac{2M}{r}-1\right]d\tau^2+\frac{dr^2}{2M/r-1}+r^2(d\tilde\theta^2+\sinh^2\tilde\theta\,d\phi^2)\right] \tag{5}$$

or

$$ds^2=-\left[\frac{2M}{\kappa^2 r}e^{r/2M}[R^2 d(\kappa\tau)^2+dR^2]+r^2(d\tilde\theta^2+\sinh^2\tilde\theta\,d\phi^2)\right]\ , \tag{6}$$

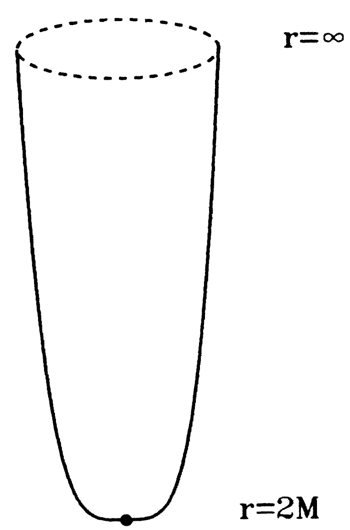

FIG. 1. The Euclidean Schwarzschild manifold for $r>2M$ (θ,ϕ suppressed). As $r\to\infty$, the area grows linearly with r while the circumference $C\to 8\pi M$.

where $R = i\rho$ is a real quantity. This metric has negative-definite signature for $r < 2M$. This manifold is represented by Fig. 2, with θ and ϕ suppressed. The difficulty is this: unlike the case for $r \geq 2M$, the Hartle-Hawking requirements for G do not provide a well-posed problem on this manifold. In particular, it is not clear what boundary conditions G must satisfy on the space-time singularity at $r = 0$.

We may overcome this difficulty by the following argument: if G is uniquely determined on the exterior region, then when it is regarded as a function of well-behaved coordinates such as the Kruskal coordinates, it will be analytic in those coordinates. Values of G for $r < 2M$ can be obtained from those for $r > 2M$ by analytic continuation. This is the procedure that we will follow.

While straightforward in principle, it is complicated in practice. One must inevitably solve the wave equation by separation of variables. One cannot do it in Kruskal coordinates: it is only practical in Schwarzschild coordinates and those simply related to them. Schwarzschild coordinates are, of course, singular at the event horizon and it is this singularity that leads to the problem that the manifolds of Figs. 1 and 2 have "pinched off" and are connected only at the point $r = 2M$.

One unusual feature of the problem is not shown in our diagram: while every point in Fig. 1 represents a two-sphere in the suppressed coordinates θ and ϕ, every point in Fig. 2 is a two-hyperboloid. Fortunately the harmonic decomposition on hyperboloids is well understood.[2]

When written in terms of the metric of Eq. (3), the solution of (2) may be expressed in the form[3]

$$G(-i\tau,r,\theta,\phi;-ir',r',\theta',\phi') = \frac{i}{32\pi^2 M^2}\left[\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)P_l(\xi_<)Q_l(\xi_>) + \sum_{n=1}^{\infty}\frac{1}{n}\cos n\kappa(\tau-\tau')\sum_{l=0}^{\infty}(2l+1)P_l(\cos\gamma)p_l^n(\xi_<)q_l^n(\xi_>)\right], \tag{7}$$

where

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$$

and (8)

$$\xi = \frac{r}{M} - 1 .$$

$\xi_<$ and $\xi_>$ denote the lesser and greater of ξ and ξ' on the interval $(1,\infty)$. P_l and Q_l are Legendre functions and p_l^n and q_l^n are the solutions of the radial equation

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi} - l(l+1) - \frac{n^2(\xi+1)^4}{16(\xi^2-1)}\right]R(\xi) = 0 \tag{9}$$

specified by the requirements that, for $n > 0$, $p_l^n(\xi)$ is the solution that remains bounded as $\xi \to 1$ and $q_l^n(\xi)$ the solution that tends to zero as $\xi \to \infty$. These solutions are normalized such that

$$p_l^n(\xi) \sim (\xi-1)^{n/2}$$
$$q_l^n(\xi) \sim (\xi-1)^{-n/2} \quad \text{as } \xi \to 1^+ .$$

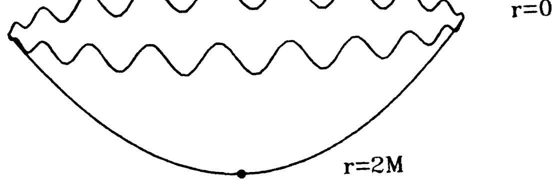

FIG. 2. The Euclidean Schwarzschild manifold for $r < 2M$. As $r \to 0$, the area $A \to 16\pi M^2$ while the circumference diverges as r^{-1}.

Although we expect, on general grounds, that $G(x,x')$ is analytic for all $\xi > -1$ ($r > 0$), this is not apparent from the representation (7), since the functions $p_l^n(\xi)$ and $q_l^n(\xi)$ individually have branch cuts which extend from $\xi = 1$ along the real ξ axis to $\xi = -\infty$.[4] Moreover, the sum, as written, fails to converge for ξ, $\xi' < 1$. Therefore it is not clear how one may unambiguously extend the representation of G by analytic continuation in the complex ξ plane to ξ, $\xi' < 1$. In Sec. II, however, we shall show that although $p_l^n(\xi)$ and $q_l^n(\xi)$ individually have branch cuts in ξ, the sum (7) does not. Furthermore, the expression (7) may be analytically continued to yield a satisfactory, convergent, and unique propagator on the region of the manifold defined by $0 < r \leq 2M$.

Section III provides a numerical calculation of $\langle\varphi^2(x)\rangle_{\text{reg}}$ defined by

$$\langle\varphi^2(x)\rangle_{\text{reg}} = -i\lim_{x'\to x} G(x,x') - \frac{1}{8\pi^2\sigma(x,x)} , \tag{10}$$

where $\sigma(x,x')$ is the geodetic distance.[5] We find that, as in the case for $r > 2M$ (Ref. 6), $\langle\varphi^2(x)\rangle_{\text{reg}}$ separates naturally into two parts:

$$\langle\varphi^2(x)\rangle_{\text{reg}} = \frac{1}{12(8\pi M)^2}\frac{1-(2M/r)^4}{1-2M/r} + \frac{\Delta(r)}{(8\pi M)^2} , \tag{11}$$

where the first term is the approximation given by Whiting[7] and Page[8] and $\Delta(r)$ is a small correction. We present a numerical calculation of $\Delta(r)$ for the range $0.5M \leq r \leq 2M$. We find that the Whiting-Page approximation remains valid for this range but becomes worse as r diminishes.

II. A REPRESENTATION FOR THE GREEN FUNCTION FOR $r<2M$

Our aim in this section is to demonstrate that (a) the Green function of Eq. (7) has no branch point at $\xi(\xi')=1$, and (b) the expression (7) can be analytically continued to a convergent expression which is valid for ξ or $\xi'<1$.

To save writing we present the demonstration for the partial coincidence limit of Eq. (7):

$$\frac{32\pi^2M^2}{i}G(-i\tau,r,\theta,\phi;-i\tau,r',\theta,\phi)$$
$$=G(\xi,\xi')=\sum_{l=0}^{\infty}(2l+1)\sum_{n=1}^{\infty}\left[\frac{1}{n}p_l^n(\xi')q_l^n(\xi)-2P_l(\xi')Q_l(\xi)\right], \quad (12)$$

where we have interchanged the order of the sums. We now take $-1<\xi,\ \xi'<1$ and consider the difference

$$2\,\mathrm{Im}G(\xi,\xi')=G(\xi+i\epsilon,\xi'+i\epsilon)-G(\xi-i\epsilon,\xi'-i\epsilon)$$
$$=\sum_{l=0}^{\infty}(2l+1)\sum_{n=1}^{\infty}\left[\frac{1}{n}[p_l^n(\xi'+i\epsilon)q_l^n(\xi+i\epsilon)-p_l^n(\xi'-i\epsilon)q_l^n(\xi-i\epsilon)]+2\pi i\mathbf{P}_l(\xi')\mathbf{P}_l(\xi)\right]. \quad (13)$$

We have used the fact that[9]

$$Q_l(\xi+i\epsilon)-Q_l(\xi-i\epsilon)=-i\pi\mathbf{P}_l(\xi). \quad (14)$$

The expression in the square brackets may be evaluated by writing

$$q_l^\nu(\xi)=p_l^{-\nu}(\xi)+\frac{\nu\alpha_l^\nu p_l^\nu(\xi)}{2\sin\nu\pi} \quad (15)$$

which is valid for all noninteger ν.[4] Momentarily taking $n=\nu$ noninteger, the expression in square brackets in Eq. (13) is

$$\frac{1}{\nu}[p_l^\nu(\xi'_+)q_l^\nu(\xi_+)-p_l^\nu(\xi'_-)q_l^\nu(\xi_-)]$$
$$=\frac{1}{\nu}[p_l^\nu(\xi'_+)p_l^{-\nu}(\xi_+)-p_l^\nu(\xi'_-)p_l^{-\nu}(\xi_-)]$$
$$+\frac{\alpha_l^\nu}{2\sin\pi\nu}[p_l^\nu(\xi'_+)p_l^\nu(\xi_+)-p_l^\nu(\xi'_-)p_l^\nu(\xi_-)]. \quad (16)$$

We define a new function of ξ by

$$\mathbf{p}_l^\nu(\xi)=\lim_{\epsilon\to0}e^{\mp i\nu\pi/2}p_l^\nu(\xi\pm i\epsilon). \quad (17)$$

As $\mathbf{p}_l^\nu(\xi)$ is real on the interval $-1<\xi\le1$, the first term on the right-hand side of Eq. (16) is zero. Our expression is now

$$\frac{1}{\nu}[p_l^\nu(\xi'_+)q_l^\nu(\xi_+)-p_l^\nu(\xi'_-)q_l^\nu(\xi_-)]=i\alpha_l^\nu\mathbf{p}_l^\nu(\xi)\mathbf{p}_l^\nu(\xi'). \quad (18)$$

In order to take ν to the integers, we have this result from the Appendix:

$$\mathbf{p}_l^{-\nu}(\xi)\sim-\frac{\nu\alpha_l^\nu}{2\pi(\nu-n)}\mathbf{p}_l^\nu(\xi)\text{ as }\nu\to n=1,2,3,\dots, \quad (19)$$

or

$$-\frac{2\pi}{\nu}\mathbf{p}_l^\nu(\xi)\mathbf{p}_l^{-\nu}(\xi')\sim\frac{\alpha_l^\nu}{\nu-n}\mathbf{p}_l^n(\xi')\mathbf{p}_l^n(\xi)\text{ as }\nu\to n. \quad (20)$$

Note that, in this limit,

$$p_l^\nu(\xi')p_l^{-\nu}(\xi)=p_l^\nu(\xi)p_l^{-\nu}(\xi')\quad\text{as }\nu\to n. \quad (21)$$

Our expression is

$$2\,\mathrm{Im}G(\xi,\xi')=\sum_l(2l+1)\sum_{n=1}^{\infty}[i\alpha_l^n\mathbf{p}_l^n(\xi')\mathbf{p}_l^n(\xi)+2\pi i\mathbf{P}_l(\xi)\mathbf{P}_l(\xi')]. \quad (22)$$

We convert the sum over n into a contour integral,

$$\sum_{n=1}^{\infty}[\cdots]=-\frac{1}{2\pi i}\oint_\alpha d\nu\left[-\frac{2\pi i}{\nu}\mathbf{p}_l^\nu(\xi')\mathbf{p}_l^{-\nu}(\xi)+2\pi i\cot\pi\nu\mathbf{P}_l(\xi)\mathbf{P}_l(\xi')\right]$$
$$=I(\alpha), \quad (23)$$

using (20) to justify the first term inside the square brackets. α is the contour shown in Fig. 3(a). We are free to deform the contour to those of Fig. 3(b):

$$I(\alpha)=I(\alpha')+I(\alpha_0). \quad (24)$$

But clearly $I(\alpha')=-I(\alpha)$, so we have

$$I(\alpha)=\tfrac{1}{2}I(\alpha_0)$$
$$=\frac{1}{2}\left[-\frac{1}{2\pi i}\right]\oint_{\alpha_0}d\nu\left[\frac{-2\pi i}{\nu}\mathbf{p}_l^\nu(\xi')\mathbf{p}_l^\nu(\xi)+2\pi i\cot\pi\nu\mathbf{P}_l(\xi)\mathbf{P}_l(\xi')\right]$$
$$=0 \quad (25)$$

as $\mathbf{p}_l^0(\xi)=\mathbf{P}_l(\xi)$. Therefore expression (12) has no branch cut in ξ for $-1<\xi<1$.

Expression (12) and therefore the full expression (8) for the Feynman Green function is not convergent for ξ, $\xi'<1$. The summand in Eq. (12) has the asymptotic form

$$(2l+1)\left[\frac{1}{n}p_l^n(\xi')q_l^n(\xi)-2P_l(\xi')Q_l(\xi)\right]$$

$$\sim\frac{e^{-(l+1/2)(\zeta-\zeta')}}{(\sinh\zeta\sinh\zeta')^{1/2}}\left\{\frac{1}{(l+\frac{1}{2})}\frac{n^2}{64}\left[\left[\frac{B(\cosh\zeta')}{\sinh\zeta'}-15\zeta'\right]-\left[\frac{B(\cosh\zeta)}{\sinh\zeta}-15\zeta\right]\right]+O((l+\tfrac{1}{2})^{-2})\right\} \tag{26}$$

as $l\to\infty$, where

$$\xi=\cosh\zeta,$$
$$\xi'=\cosh\zeta',\quad 1<\xi'<\xi\ ,$$

and

$$B(\cosh\zeta)=\cosh^3\zeta+8\cosh^2\zeta-17\cosh\zeta-24\ .$$

For ξ, $\xi'<1$ ($\zeta-\zeta'$ purely imaginary), it is necessary to give a small imaginary piece to l to ensure convergence of the l sum. In order to continue (7) to a convergent expression valid for values of ξ less than 1, we are naturally led to rewrite our expression as a contour integral in l. Again taking the partial coincidence limit (12), we have

$$G(\xi,\xi')=\sum_{n=1}^{\infty}\left[-\frac{1}{2i}\right]\oint_{\beta}dl\cot\pi l\,(2l+1)\left[\frac{1}{n}p_l^n(\xi')q_l^n(\xi)-2P_l(\xi')Q_l(\xi)\right]$$
$$=\sum_{n=1}^{\infty}\mathrm{Re}\frac{1}{2i}\int_{\beta+}d(l+\tfrac{1}{2})\tan\pi(l+\tfrac{1}{2})2(l+\tfrac{1}{2})\left[\frac{1}{n}p_l^n(\xi')q_l^n(\xi)-2P_l(\xi')Q_l(\xi)\right], \tag{27}$$

where β,β_+ are the contours shown in Figs. 4(a) and 4(b). We now rotate the contour β_+ by making the substitution

$$(l+\tfrac{1}{2})=e^{i\theta}\lambda,\quad \lambda \text{ real} \tag{28}$$

and taking θ from 0 to $\pi/2$. If we simultaneously make the substitutions

$$\xi=\cosh(e^{-i\theta}\zeta),$$
$$\xi'=\cosh(e^{-i\theta}\zeta'),\quad \zeta,\zeta' \text{ real}\ , \tag{29}$$

then we are assured that the exponent of the integrand in the asymptotic form (26) remains real and negative. As a result, when we rotate the contour the "radial" part of the integral converges while the integral over the arc at infinity vanishes.

Taking θ to $\pi/2$ (Fig. 5), we have

$$G(\xi,\xi')=-\sum_{n=1}^{\infty}\mathscr{P}\int_0^{\infty}d\lambda\tanh\pi\lambda\;2\lambda\left[\frac{1}{n}\mathbf{p}^n_{-1/2+i\lambda}(\xi')\mathbf{q}^n_{-1/2+i\lambda}(\xi)-2\mathbf{P}_{-1/2+i\lambda}(\xi')\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}(\xi)\right], \tag{30}$$

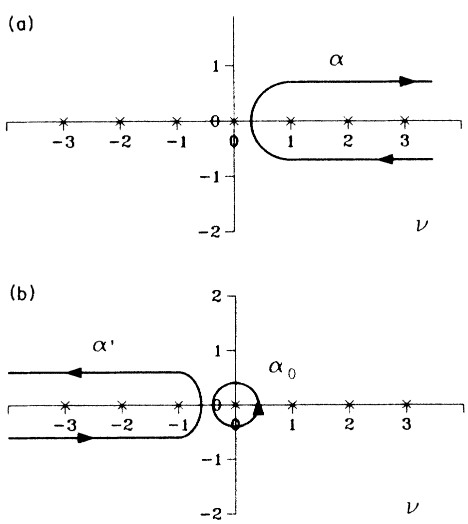

FIG. 3. The contours for Eqs. (23)–(25).

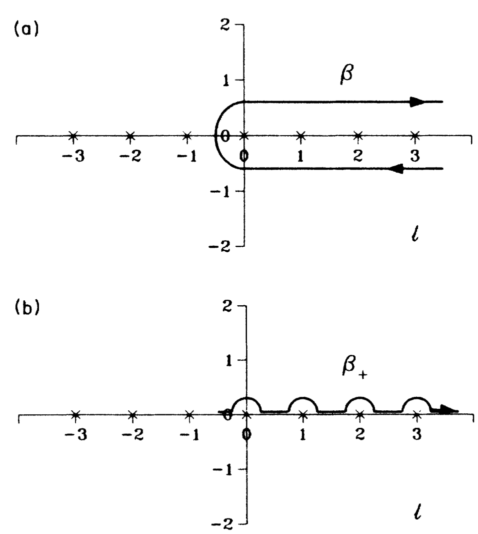

FIG. 4. The contours for Eq. (27).

where now $-1<\xi<\xi'<1$ and

$$\mathbf{q}_l^n(\xi)\equiv\lim_{\epsilon\to 0}\tfrac{1}{2}[e^{in\pi/2}q_l^n(\xi+i\epsilon)+e^{-in\pi/2}q_l^n(\xi-i\epsilon)]\ . \tag{31}$$

We are careful to write the integral as a Cauchy principal value since $\mathbf{q}^n_{-1/2+i\lambda}(\xi)$ has simple poles for discrete values of λ.[4]

Note that in deriving Eq. (30) we could have just as easily started with the real part of the integral over the lower half of the contour in Fig. 4(a). We also need not have chosen the phases of the substitution (29) to exactly cancel those of (28). The vital point has been to continue to integrand in ξ in such a way as to render a finite result at every stage. This will be true as long as the conditions of the Hartog theorem hold: i.e., that the integrand is analytic separately in ξ and l in the domains of interest.

The full expression for the Feynman Green function for $r<2M$ is

$$\begin{aligned}G(-i\tau,r,i\tilde\theta,\phi;-i\tau',r',\tilde\theta',\phi')&\\ =\frac{i}{16\pi^2M^2}\sum_{n=1}^{\infty}\cos n\kappa(\tau-\tau')\mathscr{P}\int_0^\infty d\lambda\,2\lambda\tanh\pi\lambda P_{-1/2+i\lambda}(\cosh\tilde\gamma)&\\ \times\left[\frac{1}{n}\mathbf{p}^n_{-1/2+i\lambda}(\xi_>)\mathbf{q}^n_{-1/2+i\lambda}(\xi_<)-2\mathbf{P}_{-1/2+i\lambda}(\xi_>)\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}(\xi_<)\right]&\ ,\end{aligned} \tag{32}$$

where

$$\cosh\tilde\gamma=\cosh\tilde\theta\cosh\tilde\theta'+\sinh\tilde\theta\sinh\tilde\theta'\cos(\phi-\phi')\ .$$

Equation (32) is the principal result of this paper.

III. THE CALCULATION OF $\langle\varphi^2(x)\rangle$ FOR $r<2M$

In this section we shall use our expression for $G(x,x')$ to calculate the regularized value of $\langle\varphi^2\rangle$ for $r<2M$. We regularize by the geodesic point-splitting technique of DeWitt[10] and Christensen,[5] defining

$$\langle H\mid\phi^2(x)\mid H\rangle_{\text{reg}}=\lim_{x'\to x}\left[-iG(x,x')-\frac{1}{8\pi^2\sigma(x,x')}\right]\ , \tag{33}$$

where $\mid H\rangle$ denotes the Hartle-Hawking state and $\sigma(x,x')$ is the geodetic interval between x and x'. As in the case for $r>2M$, we set $x=(-i\tau,r,\theta,\phi)$, $x'=(-i\tau+\epsilon,r,\theta,\phi)$, and expand

$$\frac{1}{8\pi^2\sigma(x,x')}=\frac{1}{4\pi^2}\left[\frac{\xi+1}{\xi-1}\frac{1}{\epsilon^2}+\frac{1}{12M^2(\xi+1)^2(\xi^2-1)}\right]+O(\epsilon^2) \tag{34}$$

or

$$\frac{1}{8\pi^2\sigma(x,x')}=-\frac{\kappa^2}{4\pi^2}\frac{\xi+1}{\xi-1}\left\{\sum_{n=1}^{\infty}n\cos n\kappa\epsilon+\frac{1}{12}\left[1-\left[\frac{2}{\xi+1}\right]^4\right]\right\}+O(\epsilon^2)\ , \tag{35}$$

where we have employed the identity

$$\epsilon^{-2}=-\kappa^2\sum_{n=1}^{\infty}\cos n\kappa\epsilon-\kappa^2/12+O(\epsilon^2)\ . \tag{36}$$

Inserting Eqs. (35) and (32) into (33) and taking the limit $\epsilon\to 0$, we have

$$\langle\varphi^2\rangle_{\text{reg}}=\frac{1}{(8\pi M)^2}\frac{1}{12}\frac{1-(2M/r)^4}{1-(2M/r)}+\frac{\Delta(r)}{(8\pi M)^2}\ . \tag{37}$$

We recognize the first term as the Whiting-Page approximation $\langle\varphi^2\rangle_{\text{WP}}$. The second term is given by

$$\Delta(r)=2\sum_{n=1}^{\infty}\left[-\mathscr{P}\int_0^\infty d\lambda\tanh\pi\lambda\,2\lambda\left[\frac{1}{n}\mathbf{p}^n_{-1/2+i\lambda}(\xi)\mathbf{q}^n_{-1/2+i\lambda}(\xi)-2\mathbf{P}_{-1/2+i\lambda}(\xi)\,\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}(\xi)\right]+\frac{n}{2}\frac{\xi+1}{\xi-1}\right]\ . \tag{38}$$

Using the identity established in the Appendix

$$\int_0^\infty d\lambda\left[\tanh\pi\lambda\ 2\lambda\mathbf{P}_{-1/2+i\lambda}(x)\,\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}(x)-\frac{1}{(1-x^2)^{1/2}}\right]=0 \tag{39}$$

we rewrite Δ as

$$\Delta(r)=2\sum_{n=1}^{\infty}\left[-\mathscr{P}\int_0^{\infty}d\lambda\left[\tanh\pi\lambda\; 2\lambda\frac{1}{n}\mathbf{p}^{n}_{-1/2+i\lambda}(\xi)\mathbf{q}^{n}_{-1/2+i\lambda}(\xi)-\frac{1}{(1-\xi^2)^{1/2}}\right]+\frac{n}{2}\frac{\xi+1}{\xi-1}\right]. \tag{40}$$

We see from Eq. (26) that the integrand above is of order λ^{-2} for large λ and so the λ integral is clearly convergent. Because of several convenient cancellations, the summand in the large square brackets can be shown to be of order n^{-5} for large n, ensuring convergence of the n sum.[6]

We turn now to the numerical calculation of $\Delta(r)$. In order to speed convergence of the λ integral we add and subtract the first three WKB approximants to the function $(1/n)\mathbf{p}^{n}_{-1/2+i\lambda}(\xi)\mathbf{q}^{n}_{-1/2+i\lambda}(\xi)$ (by a slight abuse of notation we write W_λ for $W_l=W_{-1/2+i\lambda}$):

$$\Delta(r)=-2\sum_{n=1}^{\infty}\mathscr{P}\int_0^{\infty}d\lambda\tanh\pi\lambda\; 2\lambda\left[\frac{1}{n}\mathbf{p}^{n}_{-1/2+i\lambda}(\xi)\mathbf{q}^{n}_{-1/2+i\lambda}(\xi)-W1^n_\lambda(\xi)-W2^n_\lambda(\xi)-W3^n_\lambda(\xi)\right]$$
$$+(-2)\sum_{n=1}^{\infty}[U_n(\xi)+V_n(\xi)]\,, \tag{41}$$

where

$$U_n(\xi)=\int_0^{\infty}d\lambda\left[2\lambda\tanh\pi\lambda W1^n_\lambda(\xi)-\frac{2}{(1-\xi^2)^{1/2}}\right]+\frac{n}{2}\frac{1+\xi}{1-\xi}\,, \tag{42a}$$

$$V_n(\xi)=\int_0^{\infty}d\lambda\, 2\lambda\tanh\pi\lambda[W2^n_\lambda(\xi)+W3^n_\lambda(\xi)]\,. \tag{42b}$$

The WKB approximants are given in Table II of Ref. 6. When replaced in expression (41) they render the integrand of order λ^{-6} for large λ, ensuring rapid convergence.

For actual numerical calculation it is best to deform our contour in l to steer wide of the poles in the function $\mathbf{q}^{n}_{-1/2+i\lambda}(\xi)$. We found it convenient to rewrite Eq. (41) as

$$\Delta(r)=2\sum_{n=1}^{\infty}\mathrm{Re}\frac{1}{i}\int_\gamma dl\tan\pi(l+\tfrac{1}{2})\,2(l+\tfrac{1}{2})\left[\frac{1}{n}\mathbf{p}^n_l(\xi)\mathbf{q}^n_l(\xi)-W1^n_l-W2^n_l-W3^n_l\right]-2\sum_{n=1}^{\infty}(U_n+V_n)\,. \tag{43}$$

γ is the contour shown in Fig. 6.

The function $p^n_l(\xi)$ [and thus $p^n_l(\xi)$] was easily generated by either summing its series representation (for $\xi<3.0$) or by a fourth-order Runge-Kutta integration routine.[11] $\mathbf{q}^n_l(\xi)$ was generated by the integral of the Wronskian relation

$$q^n_l(\xi)=2np^n_l(\xi)\int_\infty^{\xi}\frac{d\xi'}{(\xi'^2-1)[p^n_l(\xi')]^2}\,. \tag{44}$$

For $\xi<1$, the path of integration in ξ was chosen in the complex ξ plane to avoid the pole at $\xi=1$. $\mathbf{q}^n_l(\xi)$ then follows from Eq. (31).

The results of our numerical evaluation of $\Delta(r)$ are given in Table I and Fig. 7. In Table I (n) is the number of terms in the n sum needed for 3-figure accuracy. We found that as ξ becomes close to -1 more terms must be included in the n sum. This problem can be explained by the breakdown of the WKB approximation to the product $(1/n)\mathbf{p}^n_l(\xi)\mathbf{q}^n_l(\xi)$. For example, the first WKB approximant is

$$W1^n_l(\xi)=\left[(l+\tfrac{1}{2})^2(\xi^2-1)+\frac{n^2}{16}(1+\xi)^4\right]^{-1/2}. \tag{45}$$

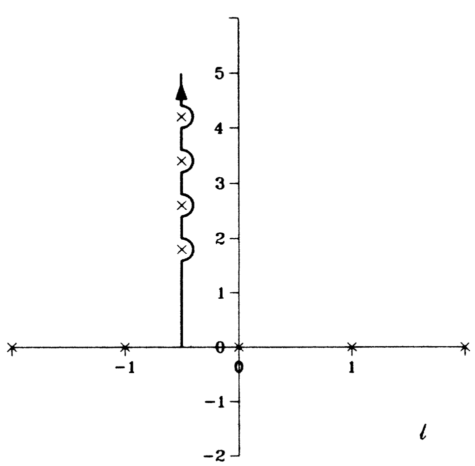

FIG. 5. The rotated l contour.

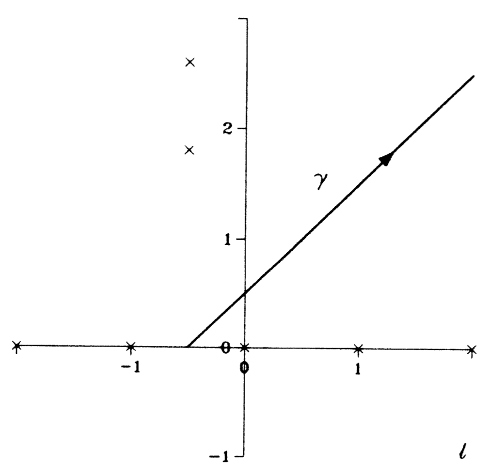

FIG. 6. The contour for Eq. (43).

TABLE I. Numerical evaluation of $\Delta(r)$.

ξ	$\Delta(\xi)$	(n)
1.0	0.0	
0.9	−0.000 93	1
0.8	−0.002 28	1
0.7	−0.004 12	1
0.6	−0.007 11	2
0.5	−0.011 6	2
0.4	−0.018 5	2
0.3	−0.028 8	3
0.2	−0.043 7	3
0.1	−0.065 5	4
0.0	−0.094 3	5
−0.1	−0.155	6
−0.2	−0.250	8
−0.3	−0.415	10
−0.4	−0.706	10
−0.5	−1.150	10

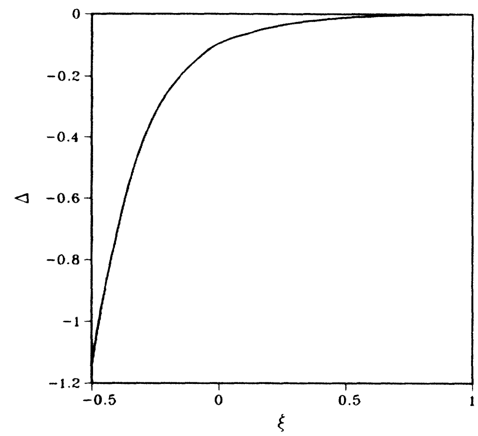

FIG. 7. $\Delta(\xi)$.

It is the overall factor of $(\xi+1)^{-1/2}$ which causes difficulties as ξ approaches -1. If ξ is near -1 the above is a good approximation to $(1/n)\mathbf{p}_l^n(\xi)\mathbf{q}_l^n(\xi)$ only for large n. The large amount of computing time necessary for evaluating each n term prohibits us from taking ξ smaller than -0.5.

A graph of $\langle\varphi^2\rangle$ and $\langle\varphi^2\rangle_{\rm WP}$ is provided by Fig. 8. As one might expect, the Whiting-Page approximation becomes worse as r approaches the spacetime singularity. The difficulties in evaluating $\Delta(r)$ for small r prevent us from gaining a clear picture of the form of $\langle\varphi^2\rangle$ as r approaches zero. The poor convergence of the n sum in Δ is, in some sense, a measure of the violence of the vacuum fluctuations at small r. We expect that a calculation of $\langle T_\mu^\nu\rangle$ using the mode sum definition analogous to that performed[12] for $r>2M$ would show similar results.

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APPENDIX

In this appendix we shall establish Eqs. (19) and (39).

To establish the asymptotic form (19) we consider the equation

$$q_l^\nu(\xi)=p_l^{-\nu}(\xi)+\frac{\nu\alpha_l^\nu}{2\sin\nu\pi}p_l^\nu(\xi)\ . \qquad \text{(A1)}$$

It has been shown in Ref. 4 that while $p_l^{-\nu}(\xi)$ has poles at positive integer values of ν, $q_l^\nu(\xi)$ does not. Thus as ν approaches a positive integer $p_l^{-\nu}(\xi)$ has the form

$$p_l^{-\nu}(\xi)\sim\frac{(-1)^{n+1}n\alpha_l^n}{2\pi(\nu-n)}p_l^n(\xi)\ \text{ as } \nu\to n=1,2,3\ldots\ . \qquad \text{(A2)}$$

Applying Eq. (17) to (A2) results in Eq. (19).

We now establish the identity

$$\int_0^\infty d\lambda\left\{2\lambda\tanh\pi\lambda(-1)^n\mathbf{P}_{-1/2+i\lambda}^{-n}(x)\,\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}^n(x)-\frac{1}{(1-x^2)^{1/2}}\right\}=\frac{n}{1-x^2}\ \text{ for } x<1 \qquad \text{(A3)}$$

of which Eq. (39) is a special case. We begin by taking the Fourier transform of the standard identity[13]

$$\mathbf{Q}_l[xx'+(1-x^2)^{1/2}(1-x'^2)^{1/2}\cos\psi]=\mathbf{P}_l(x_>)\mathbf{Q}_l(x_<)+2\sum_{n=1}^\infty(-1)^n\mathbf{P}_l^{-n}(x_>)\mathbf{Q}_l^n(x_<)\cos n\psi\ . \qquad \text{(A4)}$$

Taking the inverse Fourier transform gives the relation

$$(-1)^n\mathbf{P}_l^{-n}(x_>)\mathbf{Q}_l^n(x_<)=\frac{1}{2\pi}\int_0^{2\pi}d\psi\cos n\psi\,\mathbf{Q}_l[xx'+(1-x^2)^{1/2}(1-x'^2)^{1/2}\cos\psi]\ . \qquad \text{(A5)}$$

This equation is true for general l. Letting $l=-\frac{1}{2}+i\lambda$, multiplying by $[2\lambda\tanh\pi\lambda P_{-1/2+\lambda}(\cosh\gamma)]$, taking the real part and integrating over λ, we have

$$\int_0^\infty d\lambda\,2\lambda\tanh\pi\lambda(-1)^n\mathbf{P}_{-1/2+i\lambda}^{-n}(x_>)\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}^n(x_<)P_{-1/2+i\lambda}(\cosh\gamma)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}d\psi\cos n\psi\int_0^\infty d\lambda\,2\lambda\tanh\pi\lambda P_{-1/2+i\lambda}(\cosh\gamma)\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}[xx'+(1-x^2)^{1/2}(1-x'^2)^{1/2}\cos\psi]\ . \qquad \text{(A6)}$$

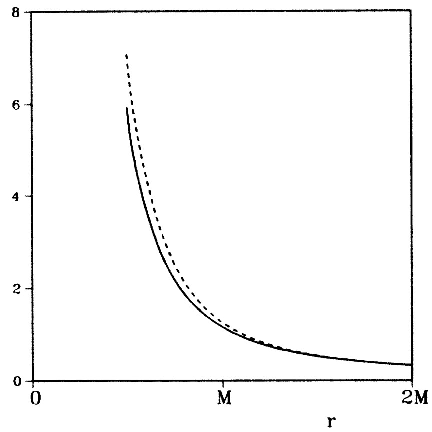

FIG. 8. $(8\pi M^2)\langle\phi^2(r)\rangle$ (solid curve) and $(8\pi M^2)\langle\phi^2(r)\rangle_{\mathrm{WP}}$ (dashed curve) for $0.5M<r<2M$.

Employing the identities

$$\mathrm{Re}\mathbf{Q}_{-1/2+i\lambda}(z)=\frac{\pi}{2\cosh\pi\lambda}\mathbf{P}_{-1/2+i\lambda}(-z) \tag{A7}$$

and

$$\pi\int_0^\infty d\lambda\frac{\lambda\tanh\pi\lambda}{\cosh\pi\lambda}P_{-1/2+i\lambda}(y)\mathbf{P}_{-1/2+i\lambda}(-z)=\frac{1}{y-z} \tag{A8}$$

we may write the left-hand side of (A6) as

$$\frac{1}{2\pi}\int_0^{2\pi}d\psi\frac{\cos n\psi}{\cosh\gamma-[xx'+(1-x^2)^{1/2}(1-x'^2)^{1/2}\cos\psi]}=\frac{2e^{-nu}}{(x^2+x'^2-2xx'\cosh\gamma+\sinh^2\gamma)^{1/2}}\ , \tag{A9}$$

where

$$u=\cosh^{-1}\left[\frac{\cosh\gamma-xx'}{(1-x^2)^{1/2}(1-x'^2)^{1/2}}\right] .$$

Taking the limit of small γ and letting $x'\to x$, we have

$$\lim_{\gamma\to 0}\int_0^\infty d\lambda\, 2\lambda\tanh\pi\lambda(-1)^n\mathbf{P}^{-n}_{-1/2+i\lambda}(x)\times\mathrm{Re}\mathbf{Q}^{n}_{-1/2+i\lambda}(x)P_{-1/2+i\lambda}(\cosh\gamma)=\frac{1}{\gamma(1-x^2)^{1/2}}+\frac{n}{1-x^2}+O(\gamma)\ . \tag{A10}$$

To remove the γ^{-1} term we begin by writing the Mehler-Dirichlet formula

$$P_l(\cos\alpha)=\frac{\sqrt{2}}{\pi}\int_0^\alpha\frac{\cos(l+\frac{1}{2})\phi\, d\phi}{(\cos\phi-\cos\alpha)^{1/2}}\ . \tag{A11}$$

Setting $l=\frac{1}{2}+i\lambda$, $\alpha=i\gamma$, $\phi=-it$, we have

$$P_{-1/2+i\lambda}(\cosh\gamma)=\frac{\sqrt{2}}{\pi}\int_0^\gamma\frac{\cos\lambda t\, dt}{(\cosh\gamma-\cosh t)^{1/2}}\ . \tag{A12}$$

We recognize (A12) as the Fourier cosine transform of the function $\theta(\gamma-t)(\cosh\gamma-\cosh t)^{-1/2}$, where θ is the step function. Taking the inverse transform and letting t go to zero leads to the identity

$$\int_0^\infty P_{-1/2+i\lambda}(\cosh\gamma)d\lambda=\frac{1}{\sqrt{2}(\cosh\gamma-1)^{1/2}}\ . \tag{A13}$$

As γ approaches zero this is

$$\int_0^\infty P_{-1/2+i\lambda}(\cosh\gamma)d\lambda=\frac{1}{\gamma}+O(\gamma)\ . \tag{A14}$$

By multiplying this result by $(1-x^2)^{-1/2}$, subtracting it from (A10), and taking $\gamma\to 0$ we arrive at the desired relation (A3).

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