Schwarzschild radial functions

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The analytic properties of the scalar radial functions associated with the Schwarzschild metric as functions of the argument, angular momentum, and frequency are discussed.

I. INTRODUCTION

Our purpose in this paper is to establish a number of the analytic properties of the scalar radial functions for the wave equation in Schwarzschild spacetime. Although our immediate interest is to use these properties in calculations involving the quantum propagator on the Schwarzschild background, these functions are also of importance for classical black-hole scattering and perturbation questions. What we present is, in part, a review of results already established in the literature. However, we derive several previously unreported properties of the radial functions, including the existence of solutions which are regular both at the Schwarzschild event horizon and at spatial infinity.

Consider the massless scalar wave equation

$$\Box\varphi=0 \tag{1}$$

for the Schwarzschild metric

$$ds^2=-\left[1-\frac{2M}{r}\right]dt^2+\frac{dr^2}{1-2M/r}+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2 \,. \tag{2}$$

Separable solutions have the form

$$\varphi=e^{\nu\kappa t}Y_l^m(\theta,\phi)R_l^\nu(r)\,, \tag{3}$$

where Y_l^m is a spherical harmonic and R satisfies the equation

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}-l(l+1)-\frac{\nu^2(\xi+1)^4}{16(\xi^2-1)}\right]R(\xi)=0\,, \tag{4}$$

where we have set $\xi=r/M-1$ and $\kappa=(4M)^{-1}$.

For $\nu=0$, the two linearly independent solutions in the region $\xi>1$ $(r>2M)$ are given by the Legendre functions $P_l(\xi)$ and $Q_l(\xi)$.[1] For $\nu\neq0$, the differential equation is a confluent form of the Heun equation with regular singularities at $\xi=\pm1$ and an irregular singularity at $\xi=\infty$ (see Appendix A).

For $\mathrm{Re}\,\nu>0$, let $p_l^\nu(\xi)$ denote the solution that is regular as $\xi\to1$ and $q_l^\nu(\xi)$ the solution that is regular as $\xi\to\infty$. We normalize p and q such that

$$\begin{aligned}p_l^\nu(\xi)&\sim(\xi-1)^{\nu/2}\,,\\ q_l^\nu(\xi)&\sim(\xi-1)^{-\nu/2}\,,\end{aligned}\quad \text{as } \xi\to1^+ \tag{5}$$

and therefore the Wronskian

$$W[p,q]=pq'-qp'=-\frac{2\nu}{\xi^2-1}\,. \tag{6}$$

In the following sections we shall establish the analytic properties of p and q as outlined in Table I. In Sec. IV we find that, for real ν and particular (complex) values of l, there exist solutions which are regular both at the event horizon and at $r=\infty$.

Although we shall only consider the spin-zero scalar mode functions, many of our results have a straightforward generalization to fields of arbitrary spin. The work of Price[2] and Teukolsky[3] has shown that a perturbation of a Schwarzschild black hole by a field of spin s, φ_s, can be written as

$$\varphi_s=e^{\nu\kappa t}{}_sY_l^m(\theta,\phi)R_s(r)\,, \tag{7}$$

where the ${}_sY_l^m(\theta,\phi)$ are the spin-weighted spherical harmonics[4] and R_s is the solution of the Teukolsky equation.

TABLE I. Properties of $p_l^\nu(\xi)$ and $q_l^\nu(\xi)$.

	$p_l^\nu(x)$	$q_l^\nu(\xi)$
Demonstrated in:		
Section II	$(\xi-1)^{\nu/2}p_l^\nu(\xi)$ analytic on entire ξ plane except for cut $-\infty<\xi<-1$	Analytic on entire ξ plane except for cut $-\infty<\xi<1$
Section III	Analytic on ν plane except for isolated poles at $\nu=-1,-2,-3,\ldots$	Analytic on ν plane except for cut $-\infty<\nu<0$
Section IV	Analytic on entire l plane	Analytic on l plane except for a series of simple poles at $l=-\frac{1}{2}\pm i\lambda_k^\nu, \{\lambda_k^\nu\}$ real

For the identification of φ_s for spin $s=\pm\frac{1}{2},\pm 1,\pm 2$, the reader is referred to Table I of the paper by Teukolsky. The field φ_s for spin $s=\pm\frac{3}{2}$ is treated in the article of Guven.[5]

II. p AND q AS FUNCTIONS OF ξ

$p_l^\nu(\xi)$ can be represented by the series

$$p_l^\nu(\xi)=(\xi-1)^{\nu/2}\sum_{k=0}^{\infty}a_k(\xi-1)^k\ , \quad |\xi-1|<2\ , \tag{8}$$

with the coefficients satisfying the recurrence relation

$$\begin{aligned}-(\nu^2/16)a_{k-2}-(3\nu^2/8)a_{k-1}\\ +[(k+\nu/2)(k+\nu/2+1)-3\nu^2/4-l(l+1)]a_k\\ +[2(k+\nu/2+1)^2-\nu^2/2]a_{k+1}=0\end{aligned} \tag{9}$$

and $a_0=1$. In virtue of the standard theorems relating to differential equations, $(\xi-1)^{-\nu/2}p_l^\nu(\xi)$ extends to an analytic function on the cut plane. We shall choose the cut to extend from $\xi=-\infty$ to $\xi=-1$.

The second solution, $q_l^\nu(\xi)$, is the solution which, for $\mathrm{Re}\nu>0$, tends to zero as $\xi\to\infty$ and is normalized such that $q_l^\nu(\xi)\sim(\xi-1)^{-\nu/2}$ as $\xi\to 1^+$. $q_l^\nu(\xi)$ may be written as a linear combination of $p_l^\nu(\xi)$ and $p_l^{-\nu}(\xi)$:

$$q_l^\nu(\xi)=p_l^{-\nu}(\xi)+\frac{\nu\alpha_l^\nu}{2\sin\nu\pi}p_l^\nu(\xi)\ . \tag{10}$$

We shall assume initially that ν is not an integer and we shall regard Eq. (10) as defining α_l^ν a function whose properties we will study in Sec. V.

If ν is a positive integer n, then

$$\begin{aligned}q_l^n(\xi)=\gamma_l^n\ln(\xi-1)p_l^n(\xi)\\ +(\xi-1)^{-n/2}\sum_{k=0}^{\infty}b_k(\xi-1)^k+\beta_l^n p_l^n(\xi)\ ,\end{aligned} \tag{11}$$

where the b_k's and γ_l^n are given by

$$\begin{aligned}\gamma_l^n[(2k-n+1)a_{k-n}+4(k-n/2+1)a_{k-n-1}]\\ -(n^2/16)b_{k-2}-(3n^2/8)b_{k-1}\\ +[(k-n/2)(k-n/2+1)-3n^2/4-l(l+1)]b_k\\ +[2(k-n/2+1)^2-n^2/2]b_{k+1}=0\ ,\end{aligned} \tag{12a}$$

$$\begin{aligned}\gamma_l^n=(n/32)b_{n-3}+(3n/16)b_{n-2}\\ +(1/2n)[(n/2)(n+1)+l(l+1)]b_{n-1}\ ,\end{aligned} \tag{12b}$$

and $b_0=1$. From these definitions one can see that q is analytic on the ξ plane with a cut $-\infty<\xi<1$.

III. THE ANALYTIC PROPERTIES OF $p_l^\nu(\xi)$ AND $q_l^\nu(\xi)$ AS FUNCTIONS OF ν

To examine $p_l^\nu(\xi)$ as a function of ν we first consider the power-series representation (8). From the recurrence relation for the coefficients (9) we see that the mth coefficient has the form

$$a_m=\frac{1}{\nu+m}f_m(a_{m-1},a_{m-2},a_{m-3},\nu,l)\ . \tag{13}$$

f_m is a polynomial in ν which is generally nonzero (but see Appendix B). From the series representation it is clear that, for $|\xi-1|<2$, $p_l^\nu(\xi)$ is analytic in the entire complex ν plane with the exception of simple poles at the negative integers.

To remove the restriction on ξ from the above statement we must resort to the techniques of potential scattering analysis.[6] Let us change variables to a Regge-Wheeler-type coordinate $x=\frac{1}{2}(\xi-1)+\ln(\xi-1)-x_0$. Setting $f=(\xi+1)R(\xi)$ and rewriting Eq. (4) in terms of x, we have

$$\frac{d^2f}{dx^2}-\left[\frac{\nu}{2}\right]^2 f=V(x)f\ , \tag{14}$$

where $V(x)$ is defined implicitly by

$$V(x)=4(\xi-1)\left[\frac{l(l+1)}{(\xi+1)^3}+\frac{2}{(\xi+1)^4}\right]\ . \tag{15}$$

If one writes $V(x)$ in terms of its Laplace transform, and noting that $V(x+2\pi ik)=V(x)$ for k an integer, it can be shown that[7]

$$V(x)=\sum_{m=1}^{\infty}C_m e^{mx}\ \text{ for } x<0\ . \tag{16}$$

If $f_+(x)$ is the solution of (14) with the asymptotic form

$$f_+(x)\sim e^{\nu x/2}\ \text{ as } x\to-\infty\ , \tag{17}$$

then clearly $p_l^\nu(\xi)=2e^{x_0}(\xi+1)^{-1}f_+$. We may rewrite Eq. (14) for $f_+(x)$ as the integral equation

$$f_+(x)=e^{\nu x/2}+\int_{-\infty}^{x}dx'G(x,x')V(x')f_+(x')\ , \tag{18}$$

where $G(x,x')$ is the Green function for Eq. (14) with $V=0$,

$$G(x,x')=\frac{\sinh[(\nu/2)(x-x')]}{\nu/2}\ . \tag{19}$$

We solve Eq. (18) by iteration:

$$\begin{aligned}f_+(x)=e^{\nu x/2}+\int_{-\infty}^{x}dx_1G(x,x_1)V(x_1)e^{\nu x_1/2}\\ +\int_{-\infty}^{x}dx_1G(x,x_1)V(x_1)\\ \times\int_{-\infty}^{x_1}dx_2G(x_1,x_2)V(x_2)e^{\nu x_2/2}\\ +\cdots\ .\end{aligned} \tag{20}$$

Employing (16) and (19) and performing the integrations, we have

$$f_+(x)=e^{\nu x/2}\left[1+(-1)\sum_{m_1=1}^{\infty}\frac{C_{m_1}e^{m_1x}}{m_1(m_1+\nu)}+(-1)^2\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{C_{m_1}C_{m_2}e^{(m_1+m_2)x}}{m_2(m_1+m_2)(m_2+\nu)(m_1+m_2+\nu)}+\cdots\right]\ . \tag{21}$$

Using methods exactly analogous to those employed by Chandrasehkar and Hartle[7] in a similar proof, one may show that this series converges for all ν such that ν is not a negative integer. From this representation for $f_+(x)$ we see explicitly that f_+ is analytic in ν except at $\nu=-1,-2,-3,\ldots$ where it has simple poles. Although we must restrict the validity of (21) to those values of ξ for which $x<0$, we may extend this range to arbitrary ξ by the choice of the constant x_0. Therefore we conclude that $p_l^\nu(\xi)=2e^{x_0}(\xi+1)^{-1}f_+[x(\xi)]$ is analytic in ν for arbitrary real ξ, except for the isolated simple poles already noted.

In order to examine $q_l^\nu(\xi)$ *qua* function of ν, we first consider the asymptotic forms of the solutions to Eq. (4) at large ξ:

$$G_\pm(\xi)\sim e^{\pm\nu\xi/4}\xi^{\pm\nu/2-1} \text{ as } \xi\to\infty . \tag{22}$$

As $q_l^\nu(\xi)$ is a solution which is regular at infinity, it is clearly proportional to $G_-(\xi)$. We may write $G_-(\xi)$ as the linear combination

$$G_-(\xi)=C_1(\nu,l)p_l^{-\nu}(\xi)+C_2(\nu,l)p_l^\nu(\xi) , \tag{23}$$

where C_1 and C_2 are functions of ν and l. They can be found from computing the Wronskians:

$$C_1(\nu,l)=-\frac{\xi^2-1}{2\nu}W[p_l^\nu(\xi),G_-(\xi)] , \tag{24a}$$

$$C_2(\nu,l)=-\frac{\xi^2-1}{2\nu}W[p_l^{-\nu}(\xi),G_-(\xi)] . \tag{24b}$$

Comparing Eqs. (23) and (10) we see that $q_l^\nu(\xi)=C_1^{-1}G_-(\xi)$ and that

$$\frac{\nu\alpha_l^\nu}{2\sin\nu\pi}=\frac{C_2(\nu,l)}{C_1(\nu,l)}=\frac{W[p_l^{-\nu}(\xi),G_-(\xi)]}{W[p_l^\nu(\xi),G_-(\xi)]} . \tag{25}$$

If we exclude the integers, the analytic properties of $q_l^\nu(\xi)$ in the ν plane are completely determined by those of α_l^ν. From the above equation it is clear that, for ν not an integer, the analytic properties of α_l^ν are in turn determined from those of $G_-(\xi)$. We therefore turn our attention to this quantity.

It is in fact more convenient to examine the related function

$$g_-(x)=(\xi+1)G_-(\xi) , \tag{26}$$

where now $x=\frac{1}{2}(\xi-1)+\ln(\xi-1)$. $g_-(x)$ is the solution of Eq. (14) with the asymptotic form

$$g_-(x)\sim e^{-\nu x/2} \text{ as } x\to\infty . \tag{27}$$

In analogy with Eq. (18) we write the integral equation

$$g_-(x)=e^{-\nu x/2}+\int_x^\infty dx'G(x,x')V(x')g_-(x') , \tag{28}$$

where $G(x,x')$ and $V(x)$ are defined as above. Again, we may solve this equation by iterating

$$g_-(x)=e^{-\nu x/2}\left[1+\frac{(-1)}{\nu}\int_0^\infty dy_1(1-e^{-\nu y_1})V(x+y_1)+\frac{(-1)^2}{\nu^2}\int_0^\infty dy_1(1-e^{-\nu y_1})V(x+y_1)\int_0^\infty dy_2(1-e^{-\nu y_2})V(x+y_1+y_2)+\cdots\right] \tag{29}$$

or

$$g_-(x)=e^{-\nu x/2}\left[1+\sum_{k=1}^\infty A_k(x)\right] , \tag{30}$$

where

$$A_k(x)=(-1/\nu)^k\int_0^\infty dy_1(1-e^{-\nu y_1})V(x+y_1)\cdots\int_0^\infty dy_k(1-e^{-\nu y_k})V(x+y_1+\cdots+y_k) . \tag{31}$$

In these equations we have effected the change of integration variable $y_i=x_i-x_{i-1}$. However we cannot find a representation for $V(x)$ analogous to Eq. (16) that we may insert into the integrals above. Despite this difficulty, we can determine much of the analytic properties of $g_-(x)$ without solving the integral equation explicitly.

Employing standard techniques, it may be shown that

$$|A_k(x)|\le\frac{e^{\nu_R x/2}}{k!}[I(x)]^k , \tag{32}$$

where

$$I(x)=\int_0^\infty dy\,|V(x+y)|\,\frac{2y}{1+\nu y/2} . \tag{33}$$

Here we have imposed the restriction $\mathrm{Re}\nu=\nu_R>0$. The series (30) converges for all such ν.

We may relax the restriction on ν by contour rotation. If we allow y_m to be complex, setting $y_m=|y_m|e^{i\theta}$ in the mth integral of Eq. (31), and let $\nu=|\nu|e^{i\phi}$, then the integrals remain convergent if we choose θ such that

$$|\theta+\phi|<\pi/2 . \tag{34}$$

By choice of θ we may extend the domain of analyticity for $g_-(x)$ to any complex ν. We cannot, however, exclude the very likely possibility of a branch point for $g_-(x)$ at $\nu=0$. Therefore we conclude that $G_-(\xi)=(\xi+1)^{-1}g_-(x)$ is analytic on the complex ν plane with cut $-\infty<\nu\le0$.

By determining the analytic properties of $G_-(\xi)$ and $p_l^\nu(\xi)$, we determine the properties of the individual

Wronskians that appear in Eq. (25). Until now we have excluded the case for which ν is an integer. Since $q_l^n(\xi)$ for $n=1,2,3,\ldots$ can be written explicitly [Eq. (11)], it is clear that, although $p_l^{-\nu}(\xi)$ has poles at the positive integers, $q_l^\nu(\xi)$ does not. The negative integers are excluded from the domain of ν by the position of the branch cut.

A remaining possible source of poles for $q_l^\nu(\xi)$ is the fact that $W[p_l^\nu(\xi),G_-(\xi)]$ may have zeros for certain choices of l and ν. In that case, there would exist a solution which tends to zero as both $\xi\to\infty$ and as $\xi\to 1$. That no such solution exists for l real has been proven by Zerilli.[8] (Note: an elegant summary of this proof appears in the first of the celebrated papers by Press and Teukolsky.[9]) However, we shall see in the following section that there exist discrete complex values of l for which such a solution *does* exist.

In conclusion, we state that for ξ and l real, $q_l^\nu(\xi)$ is analytic on the ν plane cut $-\infty<\nu\leq 0$.

VI. THE ANALYTIC PROPERTIES OF $p_l^\nu(\xi)$ AND $q_l^\nu(\xi)$ AS FUNCTIONS OF l

The differential equation (4) is unchanged by the replacement $l\to -l-1$. Since the boundary condition

$$p_l^\nu(\xi)\sim(\xi-1)^{\nu/2} \quad \text{as } \xi\to 1^+ \tag{35}$$

determines the solution uniquely for $\mathrm{Re}\,\nu>0$, it is immediate that $p_l^\nu(\xi)$ satisfies the relation

$$p_l^\nu(\xi)=p_{-l-1}^\nu(\xi)\ . \tag{36}$$

The function $q_l^\nu(\xi)-q_{-l-1}^\nu(\xi)$ satisfies the differential equation and is of order $(\xi-1)^{\nu/2}$ as $\xi\to 1^+$. It also tends to zero as $\xi\to\infty$ and hence is identically zero. Thus

$$q_l^\nu(\xi)=q_{-l-1}(\xi)\ . \tag{37}$$

More generally, from Eq. (20) it is clear that since $V(x)$ is a simple polynomial in l, each term in the infinite series for $f_+(x)$ is an analytic function in l. As the series also converges uniformly for any l in the complex plane, we may conclude that $f_+(x)$ and therefore $p_l^\nu(\xi)$ are analytic functions of l.[10]

As in Sec. III, the case for $q_l^\nu(\xi)$ is the more troublesome one. If we divide the Wronskian equation (6) by $[p_l^\nu(\xi)]^2$ and integrate, however, we obtain a convenient expression for $q_l^\nu(\xi)$ in terms of $p_l^\nu(\xi)$:

$$q_l^\nu(\xi)=2\nu p_l^\nu(\xi)\int_\xi^\infty \frac{d\xi'}{(\xi'^2-1)[p_l^\nu(\xi')]^2} \quad \text{for } \xi>1\ . \tag{38}$$

It is apparent that $q_l^\nu(\xi)$ has a pole for those values of l, $\{l_\kappa\}$, for which

$$p_{l_\kappa}^\nu(\xi)\to 0 \quad \text{as } \xi\to\infty\ . \tag{39}$$

As discussed in the previous section, it has been shown that there are no solutions to (4) which satisfy both (35) and (39) for l real. There are, however, discrete values of (real) λ for which $p_{-1/2+i\lambda}^\nu(\xi)$ *does* satisfy both boundary conditions. Standard theorems relating to Stürm-Liouville theory assure us that,[11] for l set to $-\frac{1}{2}+i\lambda$, λ, and ν real, Eq. (4) will have real solutions which are zero at $r=2M$ and $r=\infty$ for discrete values of λ; $\{\lambda_k^\nu\}$, $k=1,2,3,\ldots$. At these values $q_{-1/2+i\lambda}^\nu(\xi)$ has a pole.

TABLE II. Some radial eigenvalues λ_k^n.

k	λ_k^1	λ_k^2
1	1.8110	3.1057
2	2.6665	4.0202
3	3.4622	4.8840

Let us define a new set of functions

$$u_k^\nu(\xi)=N_k^\nu p_{-1/2+i\lambda_k}^\nu(\xi)\ . \tag{40}$$

The u_k's satisfy the boundary conditions (35) and (39) and form a complete set on the open interval $\xi\in(1,\infty)$. The constants N_k^ν are chosen such that

$$\sum_{k=1}^\infty u_k^\nu(\xi)u_k^\nu(\xi')=\delta(\xi-\xi')\ . \tag{41}$$

The u_k's are the "exterior" counterpart of the radial eigenfunctions which are zero at $\xi=\pm 1$ described by Matzner and Zamorano.[12] Some values of the eigenvalues λ_k^ν are given in Table II.[13] Graphs of the first three u_k^ν's for $\nu=1$ are shown in Fig. 1.

It has been shown[14] that the unique solution of

$$\left[\frac{d}{d\xi}(\xi^2-1)\frac{d}{d\xi}-l(l+1)-\frac{\nu^2(1+\xi)^4}{16(\xi^2-1)}\right]g_l^\nu(\xi,\xi') = -\delta(\xi-\xi') \tag{42}$$

which is regular as $\xi\to 1$ and which tends to zero as $\xi\to\infty$ is given by

$$g_l^\nu(\xi,\xi')=\begin{cases} P_l(\xi_<)Q_l(\xi_>) & \text{for } \nu=0\ ,\\ \frac{1}{\nu}p_l^\nu(\xi_<)q_l^\nu(\xi_>) & \text{for } \nu>0\ ,\end{cases} \tag{43}$$

where $\xi_<$ ($\xi_>$) is the lesser (greater) of (ξ,ξ'). For $\nu\neq 0$ an alternative representation for the Green function g_l^ν is furnished by the functions $u_k^\nu(\xi)$:

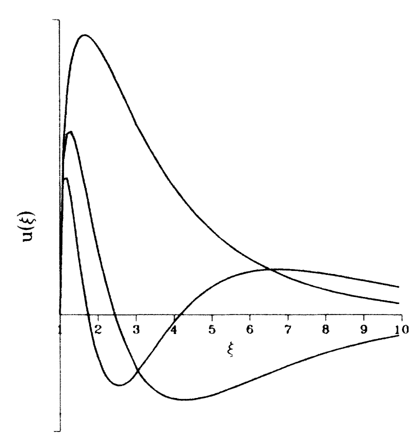

FIG. 1. The functions u_k^1 for $k=1,2,3$.

$$g_l^\nu(\xi,\xi')=\sum_{k=1}^{\infty}\frac{u_k^\nu(\xi)u_k^\nu(\xi')}{(l+\frac{1}{2})^2+(\lambda_k^\nu)^2}\ . \tag{44}$$

Comparing Eqs. (43) and (44), we find the interesting relation

$$p_l^\nu(\xi_<)q_l^\nu(\xi_>)=\sum_{k=1}^{\infty}\frac{u_k^\nu(\xi)u_k^\nu(\xi')}{(l+\frac{1}{2})^2+(\lambda_k^\nu)^2} \tag{45}$$

from which we can see explicitly that $q_l^\nu(\xi)$ has simple poles when $l=-\frac{1}{2}\pm i\lambda_k^\nu$.

V. POSTSCRIPT ON THE PROPERTIES OF α_l^ν

In light of the established properties of $p_l^\nu(\xi)$ and $q_l^\nu(\xi)$ we record here some properties of the coefficients α_l^ν and γ_l^ν which appear in Eqs. (10) and (11).

In writing

$$q_l^\nu(\xi)=p_l^{-\nu}(\xi)+\frac{\nu\alpha_l^\nu}{2\sin\nu\pi}p_l^\nu(\xi) \tag{46}$$

we first notice that, as $p_l^{-\nu}(\xi)$ has poles in ν at the positive integers and $q_l^\nu(\xi)$ does not, the coefficient of $(\nu-n)^{-1}$ in $p_l^{-\nu}(\xi)$ as ν approaches $n=1,2,3,\ldots$ must cancel that of the second term to yield a finite remainder, given by Eq. (11). For certain values of (complex) l, however, $p_l^{-\nu}(\xi)$ has no pole at $\nu=n$. For these values of l, $q_l^\nu(\xi)$ is given by Eq. (11) with γ_l^n set to zero. These choices of $l(n)$ correspond to those yielding the "exact solutions" described in Appendix B.

From Eq. (45) and the fact that $p_l^\nu(\xi)$ is analytic in l we conclude that α_l^ν has simple poles for $l=-\frac{1}{2}\pm i\lambda_k^\nu$.

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APPENDIX A

Heun[15] has given the general form for a differential equation with four regular singular points:

$$z(z-1)(z-a)\frac{d^2y}{dz^2}+\{(\alpha+\beta+1)z^2-[\alpha+\beta-\delta+1+(\gamma+\delta)a]z+a\gamma\}\frac{dy}{dz}+\alpha\beta(z-q)y=0\ . \tag{A1}$$

Heun's equation may be summarized by the Riemann scheme:

$$P\left\{\begin{matrix}0 & 1 & a & \infty & \\ 0 & 0 & 0 & \alpha & z\\ 1-\gamma & 1-\delta & 1-\epsilon & \beta & \end{matrix}\right\},$$

where $\epsilon=\alpha+\beta-\gamma-\delta+1$. Let $\alpha=\kappa a$. The equation can be written

$$\frac{d^2y}{dz^2}+\left[\frac{\delta}{z-1}+\frac{\kappa a}{z-a}+\frac{\gamma}{z}\right]\frac{dy}{dz}+\frac{\kappa a\beta(z-q)}{z(z-1)(z-a)}y=0\ . \tag{A2}$$

Taking the limit $a\to\infty$, we have

$$\frac{d^2y}{dz^2}+\left[\frac{\delta}{z-1}-\kappa+\frac{\gamma}{z}\right]\frac{dy}{dz}-\kappa\beta\left[\frac{1-q}{z-1}+\frac{q}{z}\right]y=0\ . \tag{A3}$$

We are left with an equation with regular singularities at $z=0$, 1, and an irregular singularity at $z=\infty$. This is the confluent Heun equation.

If we rewrite Eq. (4) in terms of the variable $z=(\xi+1)/2=r/2M$ and set

$$R(\xi)=(z-1)^{\nu/2}e^{\nu(z-1)/2}y(z) \tag{A4}$$

the equation for y is

$$\frac{d^2y}{dz^2}+\left[\frac{\nu+1}{z-1}+\nu+\frac{1}{z}\right]\frac{dy}{dz}+\left[\frac{\nu-l(l+1)}{z-1}+\frac{l(l+1)}{z}\right]y=0\ . \tag{A5}$$

We see that y is a confluent Heun function with $\beta=1$, $\delta=\nu+1$, $\gamma=1$, $k=-\nu$, and $q=l(l+1)/\nu$ (Ref. 16).

APPENDIX B (Ref. 17)

Although Eq. (4) is not solvable in terms of familiar functions for general ν and l, Whiting[18] has shown that there exist such solutions to (4) for particular values of l and ν. We give here a method of constructing such solutions.

Let us restrict our attention to $\nu=n=1,2,3,\ldots$. Let

$$R(\xi)=(\xi-1)^{-n/2}e^{n(\xi-1)/4}g(\xi) \tag{B1}$$

in Eq. (4). One solution for g has the form

$$g(\xi)=\sum_{k=0}^{\infty}c_k(\xi-1)^k\ , \tag{B2}$$

where the c_k's satisfy

$$[2(k+1-n)(k+1)]c_{k+1}+[k(k+1)-n^2-l(l+1)]c_k+[(k-n)n/2]c_{k-1}=0\ . \tag{B3}$$

A moment's examination reveals that if l is chosen so that $l(l+1)=-1$ and $n=1$ the power series terminates after one term. Therefore a solution to Eq. (4) with $\nu=1$ and $l=-\frac{1}{2}\pm i\sqrt{3}/2$ is

$$f_1(\xi)=\frac{e^{\xi/4}}{(\xi-1)^{1/2}}\ . \tag{B4}$$

A second solution may be constructed using standard methods:

$$f_2(\xi)=\frac{e^{\xi/4}}{(\xi-1)^{1/2}}E_1[(\xi+1)/2]\ , \qquad \text{(B5)}$$

where E_1 denotes the first exponential integral

$$E_1(x)=\int_x^\infty \frac{e^{-t}}{t}dt\ . \qquad \text{(B6)}$$

p and q are given by

$$p^1_{-1/2\pm i\sqrt{3}/2}(\xi) = -2\frac{e^{(\xi+3)/4}}{(\xi-1)^{1/2}}\{E_1[(\xi+1)/2]-E_1(1)\}\ ,$$

$$q^1_{-1/2\pm i\sqrt{3}/2}(\xi)=\frac{e^{(\xi-1)/4}}{(\xi-1)^{1/2}}\frac{E_1[(\xi+1)/2]}{E_1(1)}\ . \qquad \text{(B7)}$$

For $n=2$, the constructed solutions are

$$f_1(\xi)=\frac{e^{(\xi-1)/2}}{\xi-1}\{1-\tfrac{1}{2}[l(l+1)+4](\xi-1)\} \qquad \text{(B8)}$$

and

$$f_2(\xi)=f_1(\xi)I(\xi)\ , \qquad \text{(B9)}$$

where

$$I(\xi)=\int_1^\xi \frac{(t-1)e^{-(t-1)}dt}{(t+1)\{1-\frac{1}{2}[l(l+1)+4](t-1)\}^2} \qquad \text{(B10)}$$

and $l(l+1)=-3\pm\sqrt{3}$ or $l=-\frac{1}{2}\pm i(11\pm4\sqrt{3})^{1/2}/2$. p and q are given by

$$p_l{}^2(\xi)=4f_2(\xi)\ , \qquad \text{(B11a)}$$

$$q_l{}^2(\xi)=f_1(\xi)-[I(\infty)]^{-1}f_2(\xi)\ . \qquad \text{(B11b)}$$

This construction generalizes. For each integral $n\geq1$, there are n choices of l and l^* which give pairs of exact solutions.

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[17]Appendix B is based on conversations with B. Whiting. After this work was completed it came to the authors' attention that similar results have been published in W. E. Couch, J. Math. Phys. **22**, 1457 (1981).

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