PHYSICAL REVIEW D VOLUME 33, NUMBER 6 15 MARCH 1986

On the relic, cosmic abundance of stable, weakly interacting massive particles

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(Received 11 November 1985)

In the context of the expanding Universe, we solve the Boltzmann equation to obtain the relic abundance of a stable, weakly interacting massive particle species with arbitrary mass and interaction strength. We provide approximate analytic formulas for the evolution of the abundance and the final abundance. Our formulas are typically accurate to better than 5%.

INTRODUCTION

Because the Universe was very hot during its earliest history, $T \simeq (t/\text{sec})^{-1/2}$ MeV, all kinds of interesting particles—some known and some yet to be discovered—were present in great abundance. For $T \gg m$ (=the mass of the particle in question), the equilibrium abundance is, to within numerical factors, equal to that of photons. For temperatures $< m$, the equilibrium abundance is less than that of photons by a factor of order

$$(m/T)^{3/2}\exp(-m/T) .$$

Of particular cosmological interest are stable, weakly interacting massive particles, because eventually their interactions cannot keep pace with the changing temperature and equilibrium abundance caused by the expansion of the Universe and their abundance per comoving volume "freezes-out," i.e., becomes constant relative to that of photons. The abundance of such relics can be interesting because they can contribute significantly to the present mass density of the Universe—perhaps even dominate it, or be present in significant enough abundance so as to be detectable today.

If freeze-out occurs when the species is still relativistic ($T > m$), then its relic abundance is simple to compute—it is, up to factors of order unity, that of the relic photons. The case of freeze-out occurring when the species is nonrelativistic ($T < m$) is somewhat more interesting and difficult to handle. In this paper we treat this case in full generality, presenting approximate analytic solutions for the evolution of the relic abundance and for the final abundance. We compare our formulas with numerical integrations of the Boltzmann equation and find that our formulas are quite accurate, typically to within a few percent.

This problem has been addressed previously by many authors for specific particles: heavy neutrinos,[1–7] supersymmetric relics,[8,9] light neutrinos with "stronger than weak" interactions,[10–12] and a variety of stable, weakly interacting particles.[13–17] Our purpose here is to provide approximate solutions of good accuracy to the general case of a relic whose abundance freezes out when it is nonrelativistic. In the main body of the paper we will assume that the species of interest has zero initial chemical potential μ; in the Appendix we will briefly discuss the case of initial $\mu \neq 0$.

THE BOLTZMANN EQUATION

It follows from the Boltzmann equation (which governs the evolution of the phase-space number density of a particle species) that the number density of a species obeys the equation[18]

$$\dot{n} + 3Hn = -\langle \sigma v \rangle (n^2 - n_{\text{EQ}}{}^2) , \tag{1}$$

where an overdot indicates a time derivative, $\langle \sigma v \rangle$ is the thermally averaged annihilation cross section times relative velocity,[19] $H \equiv \dot{R}/R$ is the expansion rate of the Universe, and n and n_{EQ} are, respectively, the actual and equilibrium number density of the species in question. It is useful to measure the abundance relative to a quantity which evolves as R^{-3}; in doing so one is then actually following the number of particles per comoving volume ($\propto nR^3$). Assuming that the expansion of the Universe is isentropic, the entropy density s is such a quantity, since the entropy per comoving volume, $S \propto sR^3$, is constant. The entropy density s is given by the following sum over relativistic species:

$$\begin{aligned} s &= \sum_i (\rho_i + p_i)/T_i , \\ &= (2\pi^2/45)\left[\sum_{\text{Bose}} g_B T_i{}^3 + \tfrac{7}{8} \sum_{\text{Fermi}} g_F T_i{}^3\right] , \\ &\equiv (2\pi^2/45) g_{*s} T_\gamma{}^3 , \end{aligned}$$

where T_i is the temperature of species i and g_{*s} is defined to be the effective number of relativistic degrees of freedom which would have the same entropy density at the photon temperature T_γ. If some relativistic species have decoupled from the rest of the Universe and thereby separately conserve their own entropy per comoving volume (e.g., massless neutrinos decouple at $T \simeq$few MeV), one might wish to use only the entropy in the degrees of freedom which are in thermal equilibrium with the photons as the fiducial entropy (denote this entropy density as s_γ). Whenever g_{*s} is constant, the constancy of the entropy per comoving volume implies that $T \propto R^{-1}$, so that the number density of photons also evolves as R^{-3} and could be used as the fiducial quantity to which n is compared. For reference

$$s_\gamma = (2\pi^2/45) g_{*\gamma} T_\gamma{}^3 \simeq 0.439 g_{*\gamma} T_\gamma{}^3 ,$$

$$n_\gamma = [2\zeta(3)/\pi^2] T_\gamma{}^3 \simeq 0.244 T_\gamma{}^3 ,$$

$$s = 1.80 g_{*s} n_\gamma ,$$

$$s(\text{today}) = 1.71 T_\gamma{}^3 = 7.04 n_\gamma ,$$

where s(today) includes the entropy density in the microwave photons and three massless neutrino species. Defining

$$Y \equiv n/s, \quad n/s_\gamma, \text{ or } n/n_\gamma$$

Eq. (1) becomes

$$\dot{Y} = -\langle \sigma v \rangle \begin{bmatrix} s \\ s_\gamma \\ n_\gamma \end{bmatrix} (Y^2 - Y_{\text{EQ}}{}^2) . \qquad (2)$$

The physical meaning of Eq. (2) is manifest: the number of particles per comoving volume decreases due to annihilation, with a rate proportional to $Y - Y_{\text{EQ}}$.

During the epochs of interest the Universe is radiation dominated, so that the expansion rate is

$$H \simeq 1.66 g_*{}^{1/2} T_\gamma{}^2 / m_{\text{Pl}} .$$

As usual, g_* is given by the sum over the relativistic species:

$$g_* = \sum_{\text{Bose}} g_B (T_i/T_\gamma)^4 + \tfrac{7}{8} \sum_{\text{Fermi}} g_F (T_i/T_\gamma)^4 ,$$

where T_i is the temperature of species i. Note that if $T_i = T_\gamma$ for all i, then $g_* = g_{*s}$; if not, then $g_* \neq g_{*s}$. It is very useful to introduce the following dimensionless quantities:

$$x = m/T ,$$

$$\alpha = T/T_\gamma ,$$

$$\langle \sigma v \rangle = (\sigma v)_0 x^{-n} ,$$

$$\lambda = 0.264 (g_{*s}/g_*^{1/2}) m_{\text{Pl}} m (\sigma v)_0 / \alpha \quad (Y = n/s)$$

$$= 0.264 (g_{*\gamma}/g_*^{1/2}) m_{\text{Pl}} m (\sigma v)_0 / \alpha \quad (Y = n/s_\gamma)$$

$$= 0.147 g_*^{-1/2} m_{\text{Pl}} m (\sigma v)_0 / \alpha \quad (Y = n/n_\gamma) ,$$

where m and T are the mass and temperature of the particle species of interest.

We have parametrized the thermally averaged annihilation cross section as

$$\langle \sigma v \rangle = (\sigma v)_0 x^{-n} ,$$

where $(\sigma v)_0$ is temperature independent. We have chosen this form because in the nonrelativistic regime ($x \gg 3$, the regime of interest here), $\langle \sigma v \rangle \propto v^p$, where $p = 0$ for s-wave annihilation, $p = 2$ for p-wave annihilation, and so on. Since $v \propto x^{-1/2}$, this implies $\langle \sigma v \rangle \propto x^{-n}$, where $n = 0$ for s-wave annihilation, $n = 1$ for p-wave annihilation, and so on.

The equilibrium abundance of a particle species of mass m and spin degeneracy g at temperature T is

$$n_{\text{EQ}} = (g/2\pi^2) T^3 \int_0^\infty u^2 du \{\exp[(u^2 + x^2)^{1/2}] + \theta\}^{-1} ,$$

where $\theta = 1$ (Fermi-Dirac), 0 (Maxwell-Boltzmann), or -1 (Bose-Einstein). The relative equilibrium abundance, $Y_{\text{EQ}} = n_{\text{EQ}}/s$, n_{EQ}/s_γ, or n_{EQ}/n_γ, is then, respectively,

$$Y_{\text{EQ}} \simeq \begin{cases} [45\zeta(3)/2\pi^4] \alpha^3 g_{\text{eff}}/g_{*s} = 0.278 (g_{\text{eff}}/g_{*s}) \alpha^3 , \\ [45\zeta(3)/2\pi^4] \alpha^3 g_{\text{eff}}/g_{*\gamma} = 0.278 (g_{\text{eff}}/g_{*\gamma}) \alpha^3 \\ (g_{\text{eff}}/2) \alpha^3 , \end{cases}$$

$$(\text{for } x \ll 3) ,$$

$$Y_{\text{EQ}} \simeq \begin{cases} (45/2\pi^4)(\pi/8)^{1/2} \alpha^3 (g/g_{*s}) x^{3/2} e^{-x} , \\ (45/2\pi^4)(\pi/8)^{1/2} \alpha^3 (g/g_{*\gamma}) x^{3/2} e^{-x} \quad (\text{for } x \gg 3) , \\ (\pi/8)^{1/2} [g/2\zeta(3)] \alpha^3 x^{3/2} e^{-x} \end{cases}$$

$$\equiv a x^{3/2} e^{-x} ,$$

where $g_{\text{eff}} = g$ (for bosons), $3g/4$ (for fermions).

By employing the dimensionless quantities defined above the evolution equation becomes[20]

$$dY/dx = -\lambda x^{-n-2} (Y^2 - Y_{\text{EQ}}{}^2) . \qquad (3)$$

For reference, λ / x^{n+2} is just equal to Γ / xH, where $\Gamma = \langle \sigma v \rangle$ (s, s_γ, or n_γ). Equation (3) is a particular form of the Riccati equation. In general, there are no closed-form solutions to the Riccati equation.

Qualitatively the solution to Eq. (3) is simple to understand. As long as the annihilation rate Γ is greater than the expansion rate H, Y tracks Y_{EQ}. Eventually, Γ becomes equal to and falls below the expansion rate H; say, that $\Gamma \simeq H$ for $x \simeq x_f$ (x_f will be defined more precisely later). Thereafter, $Y(x) \simeq Y_{\text{EQ}}(x_f)$, that is the number of particles per comoving volume has "frozen out." If this occurs when the species is still relativistic, $x \ll 3$, then the final value of Y, Y_∞, is just

$$Y_\infty \simeq \begin{cases} 0.278 (g_{\text{eff}}/g_{*s}) \alpha^3 , \\ 0.278 (g_{\text{eff}}/g_{*\gamma}) \alpha^3 , \\ (g_{\text{eff}}/2) \alpha^3 . \end{cases}$$

This occurs for light neutrinos ($m <$ few MeV) with the ordinary electroweak interactions, and today $n/n_\gamma = \frac{3}{11}$ and $n/s = 0.039$. We will be concerned with the more interesting and complicated case where $x_f \gg 3$—freeze-out occurs when the species is very nonrelativistic. In this case the final abundance is more difficult to calculate.

It is useful to rewrite Eq. (3) instead for the evolution of $\Delta \equiv Y - Y_{\rm EQ}$, the departure of Y from its equilibrium value:

$$d\Delta/dx = -dY_{\rm EQ}/dx - \lambda x^{-n-2}\Delta(2Y_{\rm EQ}+\Delta) \ . \qquad (4)$$

At early times ($x \ll x_f$, x_f to be defined below), $\Delta \ll Y_{\rm EQ}$ and $|\Delta'| \ll -dY_{\rm EQ}/dx$ and a simple approximate solution for Δ can be obtained:

$$\begin{aligned}\Delta &\simeq -(dY_{\rm EQ}/dx)\lambda^{-1}x^{n+2}/(2Y_{\rm EQ}+\Delta)\ ,\\ &\simeq (1-3/2x)x^{2+n}/[\lambda(2+\beta)] \simeq x^{n+2}/2\lambda\ ,\end{aligned} \qquad (5)$$

where $\beta \equiv \Delta/Y_{\rm EQ}$. When Δ becomes of order $Y_{\rm EQ}$ (β of order unity), both approximations (Δ' negligible and $\beta \ll 1$) break down. Define freeze-out ($x = x_f$) by the condition

$$\begin{aligned}&\Delta(x_f) = cY_{\rm EQ}(x_f)\ ,\\ &e^{x_f} \simeq (2+c)acx_f^{\,-n-1/2}\lambda\ ,\\ &x_f \simeq \ln[(2+c)\lambda ac] - (n+\tfrac{1}{2})\ln\ln[(2+c)\lambda ac]\ .\end{aligned} \qquad (6)$$

Note that x_f depends only logarithmically on the yet undefined constant of order unity c (as will the final abundance Y_∞).

For $x \gg x_f$, both the terms $-2\lambda x^{-n-2}\Delta Y_{\rm EQ}$ and $-dY_{\rm EQ}/dx$ in Eq. (4) become negligible, so that

$$d\Delta/dx \simeq -\lambda x^{-n-2}\Delta^2\ . \qquad (7)$$

During this time, particle creation is essentially not occurring, but annihilations are still somewhat important, causing a slight reduction in Y compared to its value at $x = x_f$. Integrating this approximation to Eq. (4) from $x = x_f$ to $x = \infty$, we find that the final abundance is[21]

$$\begin{aligned}&Y_\infty \simeq \frac{(n+1)}{\lambda}x_f^{\,n+1}\ ,\\ &x_f \simeq \ln[(n+1)a\lambda] - (n+\tfrac{1}{2})\ln\{\ln[(n+1)a\lambda]\}\ .\end{aligned} \qquad (8)$$

Note that the final abundance depends only logarithmically on the match point (through c). We have chosen $c(2+c) = (n+1)$ to obtain the best agreement with our numerical results. Since λ depends upon g_* it is not really constant. However, it is the value of λ at $x \simeq x_f$ that is crucial, and so in most cases it should be sufficient to set $\lambda = \lambda(x_f)$. Also note that Eq. (8) can be written in terms of the annihilation cross section at freeze-out, $\langle \sigma v\rangle_f$,

$$Y_\infty = \frac{3.79(n+1)ag_*^{\,1/2}g_{*s}}{m_{\rm Pl}m\langle\sigma v\rangle_f}x_f\ ,$$

here for $Y = n/s$. For $n = 0$, this means that the relic abundance Y_∞ determines the present annihilation cross section.

In integrating Eq. (4) numerically, we used the exact formulas for $Y_{\rm EQ}$, normalized to correspond to $a = 1$ for $x \gg 1$, for Fermi-Dirac, Maxwell-Boltzmann, and Bose-Einstein statistics. Our numerical results are shown in Figs. 1 and 2. In Fig. 1 the evolution of Δ is shown. The solid curves are numerical results, for the indicated values of λ and n, and the dashed curves are the analytic approximations given by Eq. (5) and the exact integration of Eq. (7). The values of x_f calculated from Eq. (8) are indicated by the arrows. The approximate analytic solution agrees quite well with the numerical results. Figure 2 shows the final abundance Y_∞ calculated from our analytic formula

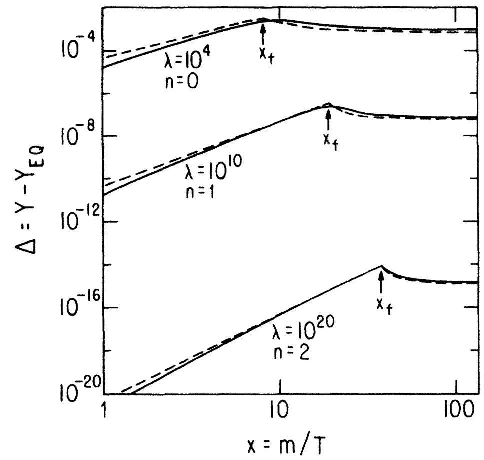

FIG. 1. Evolution of $\Delta \equiv Y - Y_{\rm EQ}$, the departure from equilibrium abundance. The solid curves are the numerical results for the indicated values of λ and n, and the dashed curves are the analytic approximation. The values of x_f calculated from Eq. (8) are indicated by the arrows.

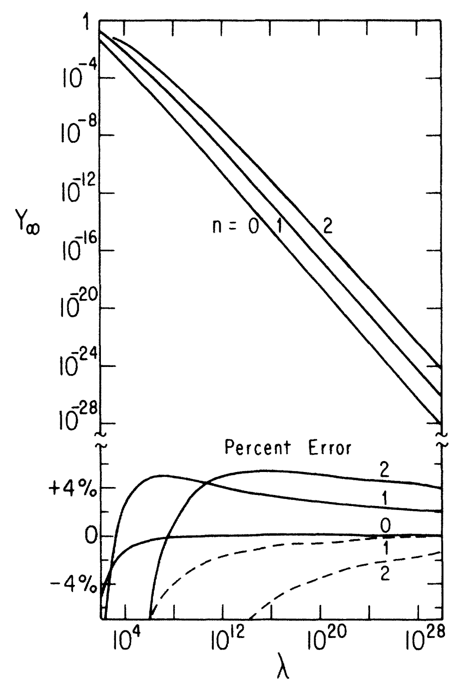

FIG. 2. Final abundance Y_∞ as a function of λ (with $a = 1$) for $n = 0$, 1, and 2 calculated from our analytic fit Eq. (8). Also shown are the errors made in using the analytic fit: solid curves for $c(2+c) = n+1$, dashed curves for $c(2+c) = 1$. Taking $c(2+c) = n+1$ gives the best overall fit, while $c(2+c) = 1$ converges more rapidly to the numerical results as $\lambda \to \infty$. The final abundance is very insensitive to the statistics of the particle (differences $< 1\%$). Our analytic results are valid only for $x_f \geq 3$, corresponding to $\lambda \geq 38$ ($n = 0$), $\lambda \geq 130$ ($n = 1$), and $\lambda \geq 1300$ ($n = 2$).

[Eq. (8)] for $a=1$, $\lambda=10^2$–10^{30}, and $n=0,1,2$. Our analytic results are only valid for $x_f \geq 3$, which corresponds to $\lambda \geq 38$ ($n=0$), $\lambda \geq 130$ ($n=1$), $\lambda \geq 1300$ ($n=2$), so our graphs have been truncated where necessary. Then for the range of λ given in this graph, x_f varies from 4–67 ($n=0$), 3–63 ($n=1$), and 3–60 ($n=2$). The percent difference between our analytic values for Y_∞ and the numerical values is given in Fig. 2 (solid lines). The agreement between the analytic results and the numerical results is quite good: for $x_f > 3$, typically to 5% or better, and no worse than 5% for $n=0$, 10% for $n=1$, and 50% for $n=2$, with the largest errors occurring for the smallest values of λ. The dashed curves give the errors for an alternate analytic fit: $c(c+2)=1$. This fit results in faster convergence to the numerical result as $\lambda \to \infty$, while $c(2+c)=n+1$ ensures smaller errors over a wider range in λ. The final abundances are very insensitive to the statistics of the particle; the values of Y_∞ for Fermi-Dirac, Maxwell-Boltzmann, and Bose-Einstein statistics differ by $<1\%$ over our range of interest.

From the relic abundance it is straightforward to calculate the species' contribution to the present mass density

$$(\Omega h^2/T_{2.7}{}^3) \simeq 267(m/\text{keV})(n/s) \tag{9a}$$

$$\simeq 38(m/\text{keV})(n/n_\gamma)\ , \tag{9b}$$

where $\Omega=\rho/\rho_{\text{crit}}$ is the fraction of critical density contributed by the species, $\rho_{\text{crit}}=1.88h^2\times 10^{-29}$ g cm^{-3} is the critical density, $H_0=100h$ km sec^{-1}Mpc^{-1} is the present value of the Hubble parameter, and $2.7T_{2.7}$ K is the current photon temperature. In Eq. (9), s is the total entropy density, and n/s has been multiplied by s(today) to calculate the mass density. Of course, if any entropy has been produced since freeze-out, the relic abundance today must be reduced by the factor γ that the entropy has increased; $\gamma = S(\text{today})/S(\text{freeze-out})$.

ACKNOWLEDGMENTS

We are grateful to J. Bernstein, L. Brown, E. Kolb, and G. Steigman for useful comments and discussions. We also want to thank E. W. Kolb for assistance with the numerical calculations. We acknowledge support of this work by the U.S. Department of Energy (at Chicago and Fermilab), NASA (at Fermilab), the McCormick Foundation (R.J.S.), and the Alfred P. Sloan Foundation (M.S.T.).

APPENDIX

Throughout we have assumed that $\mu=0$ (or that μ is very small) for the species of interest. Suppose $\mu \neq 0$, which corresponds to a nonzero value for a conserved quantum number (q) associated with the particle species. For $x \geq 3$ and $(m/T - |\mu|/T) \gtrsim 1$,

$$Y_{\text{EQ}}^{+} = e^{\xi} Y_{\text{EQ}}\ ,$$

$$Y_{\text{EQ}}^{-} = e^{-\xi} Y_{\text{EQ}}\ ,$$

$$Q = qY_{\text{EQ}}^{+} - qY_{\text{EQ}}^{-}$$

$$\simeq 2q \sinh(\xi) Y_{\text{EQ}}\ ,$$

where $Y_{\text{EQ}}^{\pm}$ are the equilibrium number of particles and antiparticles per comoving volume, respectively, $\xi \equiv \mu/T$, and Q is the net quantum number per comoving volume, which remains constant. In the limits $\xi \ll 1$ and $\xi \gg 1$, we have

$$\xi \simeq (Q/q)/2Y_{\text{EQ}} \quad (\xi \ll 1)\ ,$$

$$\xi \simeq \ln[(Q/q)/Y_{\text{EQ}}] \quad (\xi \gg 1)\ .$$

The value $\xi \simeq 1$ occurs for $x \simeq x_Q$:

$$x_Q \simeq \ln(2.35aq/Q) + \tfrac{3}{2}\ln[\ln(2.35aq/Q)]\ .$$

For $\xi \leq 1$ (corresponding to $x \leq x_Q$),

$$Y_{\text{EQ}}^{\pm} \simeq Y_{\text{EQ}}(1 \pm \xi)$$

$$\simeq Y_{\text{EQ}} \pm \tfrac{1}{2}(Q/q)$$

and the correction to $Y_{\text{EQ}}^{\pm}$ due to $\mu \neq 0$ is small. For $\xi \geq 1$ (corresponding to $x \geq x_Q$),

$$Y_{\text{EQ}}^{+} \simeq (Q/q) \geq Y_{\text{EQ}}\ ,$$

$$Y_{\text{EQ}}^{-} \simeq Y_{\text{EQ}}{}^2/(Q/q) \leq Y_{\text{EQ}}\ ,$$

the effect of $\mu \neq 0$ is very significant.

If x_f [defined in the usual way, cf. Eq. (6)] is smaller than x_Q, then freeze-out occurs before the effect of $\mu \neq 0$ is significant, and the final abundances are given by the usual formula

$$Y_\infty^{\pm} \simeq (n+1)x_f{}^{n+1}/\lambda\ .$$

In this case, the usual freeze-out abundance is much greater than the abundance needed to conserve the initial charge Q per comoving volume, so the final abundance is not affected by the nonzero value of μ.

On the other hand, if $x_Q \geq x_f$, then the effect of $\mu \neq 0$ on the final abundance is very significant. In this case it is clear that we must have

$$Y_\infty^{+} \simeq Q/q\ .$$

The abundance of the antiparticle can track its equilibrium value as long as its annihilation rate, $\Gamma_{\text{ann}} \simeq \langle \sigma v \rangle (Q/q)s$, is greater than the expansion rate H. Freeze-out of the annihilations of antiparticles on the few excess (over Q/q) particles occurs when

$$\Gamma_{\text{ann}}(x_f') \simeq H(x_f')\ ,$$

$$x_f' \simeq (\lambda Q/q)^{1/(n+1)}\ ,$$

where we have used a prime on x_f to distinguish it from the usual x_f. The relic abundance of antiparticles is just

$$Y_\infty^{-} \simeq Y_{\text{EQ}}^{-}(x_f') \simeq Y_{\text{EQ}}{}^2(x_f')/(Q/q)$$

$$\simeq a^2(q/Q)(\lambda Q/q)^{3/(n+1)}$$

$$\times \exp[-2(\lambda Q/q)^{1/(n+1)}]\ .$$

Note that the relic abundance of antiparticles depends exponentially upon λ, rather than as a power. The fact that the abundance of particles remains nearly equal to (Q/q), rather than decreasing exponentially as $x^{3/2}e^{-x}$ in the $\mu \neq 0$ case, accounts for this fact.

Consider the case of baryons. Here $Q/q \simeq 10^{-10}$ and $\langle \sigma v \rangle \simeq \pi / m_\pi{}^2$, so that $\lambda \simeq 10^{21}$, $a \simeq 1$, $n \simeq 0$, and $x_f' \simeq 10^{11}$ (corresponding to $T_f \simeq 10^{-2}$ eV). This results in a negligible relic abundance of antiprotons:

$$Y_\infty^- \simeq 10^{43} \exp(-2 \times 10^{11}) .$$

This is an incredibly small number, so one should not take it too literally, as other effects could allow the survival of more antiprotons than this.

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[18]Throughout we use units where $\hbar = c = k_B = 1$, so that $G_N \equiv m_{\rm Pl}{}^{-2}$, $m_{\rm Pl} = 1.22 \times 10^{19}$ GeV. The assumptions made in going from the Boltzmann equation, which governs the phase-space number density $n(p)$, to Eq. (1) for the evolution of the number density, $n \equiv (2\pi)^{-3} \int n(p) d^3p$, are (i) the species in question remains in kinetic equilibrium—as long as annihilations and inverse annihilations are occurring rapidly this will necessarily be true, (ii) the species into which the particle annihilates are in thermal equilibrium, (iii) the temperature of the Universe is less than m, the mass of the particle or the species can be treated as a Maxwell-Boltzmann particle, and (iv) the initial chemical potential of the species is very small—if not, the abundance is determined by the initial excess of particles over antiparticles (or vice versa), as is the case with baryons and antibaryons (see the Appendix). Bernstein, Brown, and Feinberg (Ref. 6) give a careful discussion of the derivation of Eq. (1) and the assumptions involved. They also derive an approximate solution similar to ours for the case of $n = 0$.

[19]The quantity $\langle \sigma v \rangle$ is the thermally averaged annihilation cross section times relative velocity, summed over final states and averaged over initial spin states. As usual, special care must be taken when there are identical particles in the final or initial states.

[20]In deriving Eq. (3) we have assumed that the Universe is radiation dominated during the epoch of interest ($x \sim x_f$). If instead it is matter dominated, then Eq. (3) is $dY/dx = -\lambda' x^{-n-5/2}(Y^2 - Y_{\rm EQ}{}^2)$, where $\lambda' = \lambda / r^{1/2}$ and r is the ratio of energy density in nonrelativistic particles to that in relativistic particles when $x = 1$.

[21]If $n \leq -1$ (or $n \leq -1.5$ in the matter-dominated case, see Ref. 20), then annihilations never cease and the relic abundance is very small: for $n < -1$, $Y \simeq -(n+1) x^{n+1} / \lambda$; for $n = -1$, $Y \simeq [\lambda \ln(x/x_f)]^{-1}$. Since $\langle \sigma v \rangle \propto v^{2n}$, the case of $n \leq -1$ corresponds to $\langle \sigma v \rangle$ decreasing as v^{-2} or faster.