Gauge-invariant perturbations in a universe with a collisionless gas and a fluid

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Bardeen's formalism for cosmological perturbations is extended to a system of a fluid and a collisionless gas which is the best candidate for the dark matter. For the latter, a kinetic approach is taken and the linearized coupled Einstein-Boltzmann equations are derived. When we solve them, no truncation is performed in our formalism. Because of its gauge invariance, this formalism is especially suited to the study of perturbations of the dark matter in an early stage of the Universe, which eventually grow to form the large-scale structure of the Universe.

I. INTRODUCTION

The linear perturbation theory of spatially homogeneous and isotropic cosmological models, which was pioneered by Lifshitz,[1] has been studied by a number of authors concerned with the formation of galaxies and the large-scale structures of the Universe. As the perturbations of the metric and the energy-momentum tensor generally depend on choices of coordinate gauge, we must impose some conditions to eliminate the gauge ambiguity. Usually the synchronous gauge condition has been chosen for historical reasons, but unfortunately we could not eliminate gauge modes completely in this gauge. So we had to be very careful about the physical interpretation of the perturbations of superhorizon size in which unphysical gauge modes might dominate.

On the other hand, Bardeen[2] formulated the perturbation equations in a completely gauge-invariant way. The advantage of his formalism is that "it is conceptually straightforward and mathematically elegant."[3] His hydrodynamic treatment of matter is not appropriate to a collisionless gas which is the best candidate for the dark matter, and we must take the kinetic approach for them.

Some work has been done on the gauge-invariant formulation of perturbations by the kinetic approach. For example, Moody[4] discussed the gauge-invariant treatment of perturbations of a cold collisionless gas, but he adopted the constraint that the distribution function was isotropic in **p** space, and did not couple the Boltzmann equation with the Einstein equations. Kodama and Sasaki[5] also showed in their comprehensive review[6] a kinetic theory in the gauge-invariant formalism, but because they formulated the equations in terms of the macroscopic quantities which were integrated in **p** space, they needed a nontrivial additive assumption about a higher-moment truncation which is necessary to close the system of equations.

In order to avoid unclear results which may be brought by such an assumption, in this paper we shall take the viewpoint that unphysical gauge modes must be eliminated from the perturbations of the distribution function itself. For this purpose we shall derive the gauge-invariant quantities for them and formulate the gauge-invariant equations for the coupled Einstein-Boltzmann equations.

The plan of the paper is as follows. In Sec. II we summarize the background model and the metric perturbations which are necessary for the description in later sections. In Sec. III we introduce in a single collisionless gas the gauge-invariant quantities for perturbations of the distribution function and derive the linearized Boltzmann equations in the case of scalar, vector, and tensor perturbations. Moreover, in order to consider a more realistic model, we treat in Sec. IV the perturbations in a universe with a fluid and a collisionless gas. Section V contains concluding remarks. Units are chosen as $c=8\pi G=1$. Indices $\lambda,\mu,\nu,\ldots$ and $a,b,c,\ldots$ run from 0 to 3, and $i,j,l,\ldots$ run from 1 to 3.

II. BACKGROUND MODEL AND METRIC PERTURBATIONS

As a background spacetime, we consider the spatially flat Friedmann universe whose line element is expressed as

$$ds^2=S^2(\tau)(-d\tau^2+\delta_{ij}dx^i dx^j)\ . \tag{2.1}$$

Nonzero components of the unperturbed energy-momentum tensor are

$$T^0{}_0=-E(\tau),\quad T^i{}_j=P(\tau)\delta^i_j\ , \tag{2.2}$$

where E and P are the total energy density and pressure. Time evolution of the background is determined by

$$\left[\frac{\dot S}{S}\right]^{\cdot}=-\tfrac{1}{6}(E+3P)S^2\ , \tag{2.3a}$$

$$\left[\frac{\dot S}{S}\right]^2=\tfrac{1}{3}ES^2\ , \tag{2.3b}$$

and

$$\dot E=-3\frac{\dot S}{S}(E+P)\ . \tag{2.4}$$

The metric perturbations are classified into the three types and expressed as follows in Bardeen's notation.[2]

A. Scalar perturbations

These perturbations are represented by

$$\delta g_{00}=-2S^2A(\tau)Q(\mathbf{x}) , \tag{2.5a}$$

$$\delta g_{0i}=-S^2B(\tau)Q_i(\mathbf{x}) , \tag{2.5b}$$

$$\delta g_{ij}=2S^2[H_L(\tau)\delta_{ij}Q(\mathbf{x})+H_T(\tau)Q_{ij}(\mathbf{x})] , \tag{2.5c}$$

where the scalar harmonics in the flat space are

$$Q=\exp(i\mathbf{k}\cdot\mathbf{x}) , \tag{2.6a}$$

$$Q_i=-k^{-1}Q_{,i}=-i\frac{k^i}{k}Q , \tag{2.6b}$$

$$Q_{ij}=k^{-2}Q_{,ij}+\tfrac{1}{3}\delta_{ij}Q=\left[-\frac{k_ik_j}{k^2}+\tfrac{1}{3}\delta_{ij}\right]Q , \tag{2.6c}$$

with $k=|\mathbf{k}|$. These amplitudes of the perturbations are not invariant under the following gauge transformation:

$$\tilde{\tau}=\tau+T(\tau)Q(\mathbf{x}) , \tag{2.7a}$$

$$\tilde{x}^i=x^i+L(\tau)Q^i(\mathbf{x}) . \tag{2.7b}$$

The gauge-invariant quantities for them are

$$\phi_A\equiv A+\frac{1}{k}\left[\dot{B}-\frac{1}{k}\ddot{H}_T\right]+\frac{1\dot{S}}{kS}\left[B-\frac{1}{k}\dot{H}_T\right] , \tag{2.8a}$$

$$\phi_H\equiv H_L+\tfrac{1}{3}H_T+\frac{1\dot{S}}{kS}\left[B-\frac{1}{k}\dot{H}_T\right] . \tag{2.8b}$$

The linearized Einstein equations in these variables become

$$-2\frac{k^2}{S^2}\phi_HQ=\delta T^0{}_0-3k^{-2}\frac{\dot{S}}{S}(\delta T^0{}_i)^{,i} , \tag{2.9a}$$

$$-\frac{k^2}{S^2}(\phi_A+\phi_H)Q^i{}_j=\delta T^i{}_j-\tfrac{1}{3}\delta^i{}_j(\delta T^l{}_l) . \tag{2.9b}$$

B. Vector perturbations

In this case, the metric perturbations are expressed as

$$\delta g_{0i}=-S^2B^{(1)}(\tau)Q^{(1)}{}_i(\mathbf{x}) , \tag{2.10a}$$

$$\delta g_{ij}=2S^2H_T^{(1)}(\tau)Q_{ij}^{(1)}(\mathbf{x}) . \tag{2.10b}$$

Here the vector harmonics are

$$Q_i^{(1)}=n_i\exp(i\mathbf{k}\cdot\mathbf{x})=n_iQ , \tag{2.11a}$$

$$Q_{ij}^{(1)}=-\tfrac{1}{2}k^{-1}(Q_{i,j}^{(1)}+Q_{j,i}^{(1)})$$

$$=-\frac{i}{2}k^{-1}(k_in_j+k_jn_i)Q , \tag{2.11b}$$

where n^i is a constant vector with $n^ik_i=0$. Under the gauge transformation

$$\tilde{\tau}=\tau , \tag{2.12a}$$

$$\tilde{x}^i=x^i+L^{(1)}(\tau)Q^{(1)i}(\mathbf{x}) , \tag{2.12b}$$

the above amplitudes are not gauge invariant. The only gauge-invariant variable for the metric perturbations is

$$\psi=B^{(1)}-\frac{1}{k}\dot{H}_T^{(1)} , \tag{2.13}$$

and the Einstein equations reduce to

$$\frac{k^2}{2S^2}\psi Q_i^{(1)}=\delta T_i^0 . \tag{2.14}$$

C. Tensor perturbations

Using tensor harmonics

$$Q_{ij}^{(2)}=s_{ij}\exp(i\mathbf{k}\cdot\mathbf{x}) , \tag{2.15}$$

where s_{ij} is a constant tensor with $s^i{}_i=0$ and $s_{ij}k^j=0$, the perturbations are expressed as

$$\delta g_{ij}=2S^2H_T^{(2)}(\tau)Q_{ij}^{(2)}(\mathbf{x}) . \tag{2.16}$$

The amplitudes $H_T^{(2)}$ are automatically gauge-invariant and the Einstein equations are given by

$$S^{-2}\left[\ddot{H}_T^{(2)}+2\frac{\dot{S}}{S}\dot{H}_T^{(2)}+k^2H_T^{(2)}\right]Q^{(2)i}{}_j=\delta T^i{}_j . \tag{2.17}$$

III. PERTURBATION EQUATIONS FOR A COLLISIONLESS GAS

In this section we derive the gauge-invariant perturbation equation for a system of a single collisionless gas. For that purpose we must analyze the Boltzmann equation based on the relativistic kinetic theory.[7] It is expressed by use of an orthonormal tetrad frame $e^\mu_{(a)}$ as

$$\mathscr{L}(F)\equiv\left[p^{(a)}e^\mu_{(a)}\partial_\mu-\Gamma^i_{bc}p^{(b)}p^{(c)}\frac{\partial}{\partial p^{(i)}}\right]F(x,\mathbf{p})=0 , \tag{3.1}$$

where $p^{(a)}$ is defined by

$$p^{(a)}=e^{(a)}_\mu\frac{dx^\mu}{ds} ,$$

$$\Gamma^i_{bc}=e^\lambda_{(b)}e^{(i)}_\mu e^\mu_{(c);\lambda} \tag{3.2}$$

are the Ricci rotation coefficients, and $F(x,\mathbf{p})$ is the invariant distribution function defined on the sphere bundle. In the background spacetime, a solution of Eq. (3.1) is of a form of

$$F(x,\mathbf{p})=F_B(S(\tau)p) ,$$

where $p=[(p^{(1)})^2+(p^{(2)})^2+(p^{(3)})^2]^{1/2}$.

The energy-momentum tensor is expressed as

$$T^{\mu\nu}=e^\mu_{(a)}e^\nu_{(b)}\int p^{(a)}p^{(b)}F\pi . \tag{3.3}$$

Here π is an invariant $\mathbf{p}$ volume element defined by

$$\pi=\frac{d^3p}{p^{(0)}} , \tag{3.4}$$

where $p^{(0)}=(p^2+m^2)^{1/2}$, and m is the mass of a collisionless particle. The background quantities are given by

$$E=-T^0_0=\int (p^{(0)})^2F_B\pi$$

and (3.5)

$$P=\tfrac{1}{3}T^i_i=\tfrac{1}{3}\int p^2F_B\pi\ .$$

As for tetrad frames we use in this paper orthonormal frames defined in Appendix A.

Now let us consider the perturbation of the distribution function $\delta F\equiv F-F_B$, where F and F_B are the total and unperturbed distribution functions.

A. Scalar perturbations

In the case of scalar perturbations, let us expand δF in terms of the scalar harmonics

$$\delta F(x,\mathbf{p})=f(\tau,\mathbf{p})Q(\mathbf{x})\ , \tag{3.6}$$

then straightforward calculation shows that the linearized Boltzmann equation becomes

$$p^{(0)}\dot f-\frac{\dot S}{S}p^{(0)}p^{(i)}\frac{\partial f}{\partial p^{(i)}}+i\mu kpf =\frac{\partial F_B}{\partial p}\left[(\dot H_L+\tfrac{1}{3}\dot H_T)p^{(0)}p +i\mu[(kA+\tfrac{1}{2}\dot B)(p^{(0)})^2+\tfrac{1}{2}Bm^2] +\mu^2\left[\frac{k}{2}B-\dot H_T\right]p^{(0)}p\right]\ , \tag{3.7}$$

where $\mu\equiv k_ip^{(i)}/(kp)$. Under the gauge transformation of Eq. (2.7), it is shown in Appendix A that the amplitude f changes like

$$\tilde f=f-\frac{\partial F_B}{\partial p}\left[\frac{\dot S}{S}Tp-\tfrac{1}{2}i\mu(\dot L-kT)p^{(0)}\right]\ . \tag{3.8}$$

Therefore we can define the following gauge-invariant quantities:

$$I\equiv f+\frac{\partial F_B}{\partial p}\left[\frac{1}{k}\frac{\dot S}{S}\left[B-\frac{1}{k}\dot H_T\right]p +i\mu\left[\tfrac{1}{2}B-\frac{1}{k}\dot H_T\right]p^{(0)}\right] \tag{3.9}$$

and

$$J\equiv f-\frac{\partial F_B}{\partial p}\left[(H_L+\tfrac{1}{3}H_T)p-i\mu\left[\tfrac{1}{2}B-\frac{1}{k}\dot H_T\right]p^{(0)}\right]\ . \tag{3.10}$$

Here these two are connected by the relation

$$I=J+\frac{\partial F_B}{\partial p}p\phi_H\ . \tag{3.11}$$

Next, we consider the physical interpretation of I. Following Bardeen's terminology, an invariant energy density perturbation ϵ_g, which measures the energy density contrast relative to the zero-shear hypersurface, is

$$\epsilon_gQ\equiv-\frac{1}{E}\left[\delta T^0{}_0+3(E+P)\frac{1\dot S}{kS}\left[B-\frac{1}{k}\dot H_T\right]Q\right] =\frac{1}{E}\left[\int (p^{(0)})^2f\pi-3(E+P)\frac{1\dot S}{kS}\left[B-\frac{1}{k}\dot H_T\right]\right]Q\ , \tag{3.12}$$

where the perturbations of the energy-momentum tensor are given in Appendix B. Using (3.5) and (3.10) and doing partial integrations, we obtain

$$\epsilon_g=\frac{1}{E}\int (p^{(0)})^2I\pi=\frac{1}{E}\int 2\pi p^{(0)}I\,d\mu\,p^2dp\ . \tag{3.13}$$

The velocity amplitudes v_s, which expresses the shear of the matter velocity field, is

$$v_sQ^i\equiv-\frac{1}{E+P}\delta T^i{}_0-\frac{1}{k}\dot H_TQ^i =\frac{\iota}{E+P}Q^i\int p^{(0)}p\mu I\pi=\frac{\iota}{E+P}Q^i\int 2\pi p^3\mu I\,d\mu\,dp\ , \tag{3.14}$$

and the traceless anisotropic stress perturbation π_T is

$$\pi_TQ^i{}_j\equiv\frac{1}{P}[\delta T^i{}_j-\tfrac{1}{3}\delta^i{}_j(\delta T^l{}_l)] =-\frac{1}{P}Q^i{}_j\int p^2[\tfrac{1}{2}(3\mu^2-1)]I\pi =-\frac{1}{P}Q^i{}_j\int 2\pi\frac{p^4}{p^{(0)}}[\tfrac{1}{2}(3\mu^2-1)]I\,d\mu\,dp\ . \tag{3.15}$$

Here let us expand I as

$$I(\tau,p,\mu)=\sum_{n=0}^{\infty}a_n(\tau,p)\mathbf{P}_n(\mu)i^n \tag{3.16}$$

in terms of the Legendre polynomials $\mathbf{P}_n(\mu)$. Then we can see that the p integrations of a_0, a_1, and a_2 with appropriate weights [see Eqs. (3.13)–(3.15)] correspond to ϵ_g, v_s, and π_T, respectively.

As for J,

$$\int (p^{(0)})^2J\pi=\int (p^{(0)})^2f\pi+3(E+P)(H_L+\tfrac{1}{3}H_T)\ . \tag{3.17}$$

Therefore the coefficient of $J(\tau,p,\mu)$ for $\mathbf{P}_0(\mu)$ determines the energy density perturbation relative to the "flat" hypersurface,[8] where flat means that the perturbation of the intrinsic curvature on the constant $-\tau$ hypersurface vanishes, i.e., $H_L+\frac{1}{3}H_T=0$. Eliminating f from Eq. (3.7) by Eqs. (3.9) and (3.10), we obtain the linearized Boltzmann equation in terms of gauge-invariant quantities as follows:

$$p^{(0)}\dot{I}-\frac{\dot{S}}{S}p^{(0)}p^{(i)}\frac{\partial I}{\partial p^{(i)}}+i\mu kpI=\frac{\partial F_B}{\partial p}[p^{(0)}p\dot{\phi}_H+i\mu k\phi_A(p^{(0)})^2] \quad (3.18)$$

or

$$p^{(0)}\dot{J}-\frac{\dot{S}}{S}p^{(0)}p^{(i)}\frac{\partial J}{\partial p^{(i)}}+i\mu kpJ=\frac{\partial F_B}{\partial p}i\mu k[(p^{(0)})^2\phi_A-p^2\phi_H]\,. \quad (3.19)$$

Hereafter we use Eq. (3.19) for mathematical convenience. The solution to Eq. (3.19) is expressed as a function of $q\equiv S(\tau)p$, μ, and τ

$$J(q,\mu,\tau)=\exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,\tau_i)]\}J(q,\mu,\tau_i)+i\mu kq\frac{\partial F_B}{\partial q}\int_{\tau_i}^{\tau}dx\left[\frac{q^0(x)}{q}\phi_A(x)-\frac{q}{q^0(x)}\phi_H(x)\right]\exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,x)]\}\,, \quad (3.20)$$

where a subscript i denotes an initial time, $q^0(x)=[q^2+m^2S^2(x)]^{1/2}$ and

$$\zeta(q,\tau)\equiv\int_0^{\tau}\frac{q\,dx}{[q^2+m^2S^2(x)]^{1/2}}\,. \quad (3.21)$$

By use of Eqs. (2.9a), (2.9b), and (B2)—(B4), the linearized Einstein equations are expressed in a form without time derivative of ϕ_H as

$$\left[2\frac{k^2}{S^2}+3(E+P)\right]\phi_H=\int(p^{(0)})^2J\pi+3\frac{i\dot{S}}{kS}\int p^{(0)}p\mu J\pi\,, \quad (3.22)$$

and

$$\frac{k^2}{S^2}(\phi_A+\phi_H)=\tfrac{1}{2}\int p^2(3\mu^2-1)J\pi\,. \quad (3.23)$$

The coupled equations (3.20)—(3.23) can be solved as follows. We give first the initial perturbation of the distribution function $J(q,\mu,\tau_i)$, and solve the coupled Volterra-type integral equations for ϕ_A and ϕ_H with respect to time, which are derived by substituting Eq. (3.20) into Eqs. (3.22) and (3.23). Finally we obtain $J(q,\mu,\tau)$ from Eq. (3.20). In the process of these calculations no truncation is necessary.

B. Vector perturbations

A vector-type perturbation for δF can be expressed as

$$\delta F(x,\mathbf{p})=f^{(1)}(\tau,p,\mu)\frac{p^{(i)}}{p}Q_i^{(1)}(\mathbf{x})\,. \quad (3.24)$$

Then the linearized Boltzmann equation becomes

$$p^{(0)}\dot{f}^{(1)}-\frac{\dot{S}}{S}p^{(0)}p^{(i)}\frac{\partial f^{(1)}}{\partial p^{(i)}}+i\mu kpf^{(1)}=-\frac{\partial F_B}{\partial p}\left[\tfrac{1}{2}\dot{B}^{(1)}(p^{(0)})^2+\frac{\dot{S}}{2S}B^{(1)}m^2-i\mu k\left[\tfrac{1}{2}B^{(1)}-\frac{1}{k}\dot{H}_T^{(1)}\right]p^{(0)}p\right]\,. \quad (3.25)$$

Under the gauge transformation of Eq. (2.12), it is shown in Appendix A that $f^{(1)}$ changes like

$$\tilde{f}^{(1)}=f^{(1)}-\frac{\partial F_B}{\partial p}\tfrac{1}{2}\dot{L}^{(1)}p^{(0)}\,. \quad (3.26)$$

Therefore we can introduce the gauge-invariant quantity

$$I^{(1)}\equiv f^{(1)}+\frac{\partial F_B}{\partial p}\tfrac{1}{2}B^{(1)}p^{(0)}\,. \quad (3.27)$$

Then the linearized Boltzmann equation reduces to

$$p^{(0)}\dot{I}^{(1)}-\frac{\dot{S}}{S}p^{(0)}p^{(i)}\frac{\partial I^{(1)}}{\partial p^{(i)}}+i\mu kpI^{(1)}=\frac{\partial F_B}{\partial p}i\mu kp^{(0)}p\psi\,. \quad (3.28)$$

The solution of this equation is given in terms of q, μ, and τ as

$$I^{(1)}(q,\mu,\tau)=\exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,\tau_i)]\}I^{(1)}(q,\mu,\tau_i)+i\mu kq\frac{\partial F_B}{\partial q}\int_{\tau_i}^{\tau}d\tau'\psi(\tau')\exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,\tau')]\}\,, \quad (3.29)$$

and the Einstein equation becomes

$$\frac{k^2}{S^2}\psi = \int (1-\mu^2)p^{(0)}pI^{(1)}\pi \ . \qquad (3.30)$$

The invariant velocity amplitude $v_c^{(1)}$ is

$$v_c^{(1)}Q_i^{(1)} \equiv (E+P)^{-1}\delta T_i^0 = \tfrac{1}{2}(E+P)^{-1}\int (1-\mu^2)p^{(0)}pI^{(1)}\pi Q_i^{(1)} \qquad (3.31)$$

and the traceless anisotropic stress perturbation $\pi_T^{(1)}$ is

$$\pi_T^{(1)}Q_j^{(1)i} \equiv P^{-1}\delta T_j^i = Q_j^{(1)i}\int (\mu-\mu^3)p^2I^{(1)}\pi \ . \qquad (3.32)$$

From the conservation law $\delta T^{\mu}_{i;\mu}=0$ we can obtain the equation of motion expressed by these quantities:

$$\dot v_c^{(1)} = \frac{\dot S}{S}\left[\frac{3\dot P}{\dot E}-1\right]v_c^{(1)} - \frac{kP}{E+P}\pi_T^{(1)} \ . \qquad (3.33)$$

This equation shows that in a collisionless gas, rotational motion is produced spontaneously in contrast with ideal fluids ($\pi_T=0$).

C. Tensor perturbations

For tensor perturbations, δF can be expanded as

$$\delta F = f^{(2)}(\tau,p,\mu)\frac{p^{(i)}p^{(j)}}{p^2}Q_{ij}^{(2)}(\mathbf{x}) \ , \qquad (3.34)$$

where $f^{(2)}$ is gauge invariant by itself. In terms of it, we obtain

$$\dot f^{(2)} - \frac{\dot S}{S}p^{(i)}\frac{\partial f^{(2)}}{\partial p^{(i)}} + i\mu kpf^{(2)}/p^{(0)} = \frac{\partial F_B}{\partial p}p\dot H_T^{(2)} \qquad (3.35)$$

for the Boltzmann equation, and

$$S^{-2}\left[\ddot H_T^{(2)} + 2\frac{\dot S}{S}\dot H_T^{(2)} + k^2H_T^{(2)}\right] = \tfrac{1}{4}\int (1-2\mu^2+\mu^4)p^2f^{(2)}\pi \qquad (3.36)$$

for the Einstein equation. The solution of Eq. (3.35) is expressed as a function of q, μ, and τ:

$$f^{(2)}(q,\mu,\tau) = \exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,\tau_i)]\}f^{(2)}(q,\mu,\tau_i) + i\mu kq\frac{\partial F_B}{\partial q}\int_{\tau_i}^{\tau}d\tau'\dot H_T^{(2)}(\tau')\exp\{-i\mu k[\zeta(q,\tau)-\zeta(q,\tau')]\} \ . \qquad (3.37)$$

IV. A COUPLED SYSTEM OF A FLUID AND A COLLISIONLESS GAS

In Sec. III we have derived gauge-invariant perturbation equations for a single collisionless gas. But actually our Universe is filled with a multicomponent matter which consists of, e.g., radiation, baryonic matter, and probably a collisionless gas as the dark matter. Kodama and Sasaki[5] extended Bardeen's formalism to a multifluid system. In order to consider the perturbations in a more realistic model universe, in this section, we treat the invariant perturbations in a system of a fluid and a collisionless gas. In spite of different treatment, much of our notation is consistent with Kodama and Sasaki.

In the present model, the total energy-momentum tensor is decomposed into two parts,

$$T^{\mu}{}_{\nu} = T^{\mu}_{(f)\nu} + T^{\mu}_{(c)\nu} \equiv \sum_{\alpha} T^{\mu}_{(\alpha)\nu} \ , \qquad (4.1)$$

where α takes f and c. Since the energy-momentum tensor for a collisionless gas also can formally be described as that of an imperfect fluid, let us express the components of two parts parallel as follows:

$$T^{\mu}_{(\alpha)\nu} = \mathrm{diag}(-E_\alpha,P_\alpha,P_\alpha,P_\alpha) + \delta T^{\mu}_{(\alpha)\nu} \ . \qquad (4.2)$$

A. Scalar perturbations

For scalar perturbations we have

$$\delta T^0_{(\alpha)0} = -E_\alpha\delta_\alpha(\tau)Q(\mathbf{x}) \ , \qquad (4.3a)$$

$$\delta T^0_{(\alpha)i} = (E_\alpha+P_\alpha)(v_\alpha-B)Q_i(\mathbf{x}) \ , \qquad (4.3b)$$

and

$$\delta T^i_{(\alpha)j} = P_\alpha[\pi_{L(\alpha)}\delta^i{}_jQ(\mathbf{x}) + \pi_{T(\alpha)}Q^i{}_j(\mathbf{x})] \ . \qquad (4.3c)$$

For a fluid there is an equation of state constraining the fluid quantities, while for a collisionless gas the fluid quantities are specified by a distribution function satisfying the Boltzmann equation, as can be seen from Eqs. (4.3) and Appendix B. The gauge-invariant variables for the above amplitudes are

$$v_{s(\alpha)} \equiv v_\alpha - \frac{1}{k}\dot H_T \ , \qquad (4.4a)$$

$$\epsilon_{m(\alpha)} \equiv \delta_\alpha + 3(1+w_\alpha)\frac{1\dot S}{kS}(v_\alpha-B) \ , \qquad (4.4b)$$

$$\eta_\alpha \equiv \pi_{L(\alpha)} - \frac{c_\alpha^2}{w_\alpha}\delta_\alpha \ , \qquad (4.4c)$$

where

$$w_\alpha = \frac{P_\alpha}{E_\alpha}$$

and (4.5)

$$c_\alpha{}^2=\frac{\dot{P}_\alpha}{\dot{E}_\alpha} .$$

The invariant quantities for $\alpha=c$ are

$$(E_c+P_c)v_{s(c)}=i\int p^{(0)}p\mu J\pi , \quad (4.6a)$$

$$E_c\epsilon_{m(c)}=\int (p^{(0)})^2 J\pi+\frac{3i}{k}\frac{\dot{S}}{S}\int p^{(0)}p\mu J\pi -3(E_c+P_c)\phi_H , \quad (4.6b)$$

$$P_c\eta_c=\tfrac{1}{3}\int p^2 J\pi-c_c{}^2\int (p^{(0)})^2 J\pi . \quad (4.6c)$$

Now let us derive the equations to be satisfied by these quantities. First the Einstein equations are given by

$$2(k/S)^2\phi_H=E\epsilon_m , \quad (4.7a)$$

$$-(k/S)^2(\phi_A+\phi_H)=P\pi_T . \quad (4.7b)$$

Here the total quantities, E, P, ϵ_m, v_s, π_T, and η are defined by

$$E=\sum_\alpha E_\alpha,\quad P=\sum_\alpha P_\alpha,\quad E\epsilon_m=\sum_\alpha E_\alpha\epsilon_{m\alpha} ,$$

$$P\eta=\sum_\alpha P_\alpha\eta_\alpha,\quad (E+P)v_s=\sum_\alpha (E_\alpha+P_\alpha)v_{s(\alpha)}$$

and

$$P\pi_T=\sum_\alpha P_\alpha\pi_{T(\alpha)} .$$

The conservation law for the total energy-momentum tensor is the same as that for a single fluid with the total fluid quantities. From the above conservation law, therefore, we obtain the equation for ϕ_H

$$\dot{\phi}_H+\frac{\dot{S}}{S}\phi_H=-\tfrac{1}{2}(E+P)S^2k^{-1}v_s-S\dot{S}k^{-2}P\pi_T \quad (4.8)$$

in the same way as Eq. (4.7) in Ref. 2.

Next let us consider the components separately. The energy-momentum tensor of each part $T^\mu_{(f)\nu}$ and $T^\mu_{(c)\nu}$ is conserved independently, because of

$$T^{\mu\nu}_{(c);\nu}=\int p^\mu \mathscr{L}(F)\pi$$

and $\mathscr{L}(F)=0$ for a collisionless gas.[7] For the collisionless gas Eqs. (3.18) and (3.19) hold. For the fluid part we get after straightforward calculations of $\delta(T^\mu_{(f)0;\mu})=0$ and $\delta(T^\mu_{(f)i;\mu})=0$ the following equations:

$$(S^3E_f\epsilon_{m(f)})^{\cdot}-3S^3(E_f+P_f)\frac{1}{k}\left[\frac{\dot{S}}{S}v_{s(f)}\right] +S^3(E_f+P_f)kv_{s(f)}+3S^3(E_f+P_f)\dot{\phi}_H=0 , \quad (4.9a)$$

and

$$\dot{v}_{s(f)}+\frac{\dot{S}}{S}v_{s(f)} =k\phi_A+\frac{k}{E_f+P_f}(c_f{}^2E_f\epsilon_{m(f)}+P_f\eta_f-\tfrac{2}{3}P_f\pi_{T(f)}) . \quad (4.9b)$$

Equation (4.9a) reduces with the help of (4.7b), (4.8), and (2.3) to

$$(S^3E_f\epsilon_{m(f)})^{\cdot}=-S^3(E_f+P_f)kv_{s(f)} +\tfrac{3}{2}S^5(E_f+P_f)(E_c+P_c)\frac{1}{k}(v_{s(c)}-v_{s(f)}) -2S^2\dot{S}P_f\pi_{T(f)} . \quad (4.10)$$

Eliminating $v_{s(f)}$ from Eqs. (4.9b) and (4.10) we finally obtain a second-order equation for $\epsilon_{m(f)}$

$$(S^3E_f\epsilon_{m(f)})^{\cdot\cdot}+[(1+3c_f{}^2)+W_2(1+3c_c{}^2)]\frac{\dot{S}}{S}(S^3E_f\epsilon_{m(f)})^{\cdot} +[W_1k^2c_f{}^2-\tfrac{1}{2}(E_f+P_f)S^2](S^3E_f\epsilon_{m(f)}) =s(f)+s(c) , \quad (4.11)$$

where

$$W_1k^2=k^2+\tfrac{3}{2}(E_c+P_c)S^2 ,$$

and

$$W_2=\tfrac{3}{2}(E_c+P_c)S^2/(W_1k^2) .$$

The "source terms" of Eq. (4.11) are

$$s(f)=-W_1k^2(S^3P_f\eta_f) +[\tfrac{2}{3}k^2+2(P-Ec_f{}^2)S^2](S^3P_f\pi_{T(f)}) -2\dot{S}(S^2P_f\pi_{T(f)})^{\cdot} , \quad (4.12)$$

$$s(c)=\tfrac{1}{2}(1+3c_c{}^2)(E_f+P_f)S^5(E_c\epsilon_{m(c)}-2W_2S^{-3}\dot{S}kv_{s(c)}) +\tfrac{3}{2}(E_f+P_f)S^5P_c\eta_c , \quad (4.13)$$

where $\epsilon_{m(c)}$, $v_{s(c)}$, and η_c can be replaced by their integral forms in Eqs. (4.6a)–(4.6c). Note that $\pi_{T(c)}$, the anisotropic stress perturbation for a collisionless gas, does not appear as a source term. Finally, the Einstein equations (4.7a) and (4.7b) are expressed as

$$\left[2\frac{k^2}{S^2}+3(E+P)\right]\phi_H =E_f\epsilon_{m(f)}+\int (p^{(0)})^2J\pi+3\frac{i\dot{S}}{kS}\int p^{(0)}p\mu J\pi , \quad (4.14)$$

and

$$\frac{k^2}{S^2}(\phi_A+\phi_H)=P_f\pi_{T(f)}+\tfrac{1}{2}\int (3\mu^2-1)p^2J\pi . \quad (4.15)$$

Equations (3.20) and (4.11)–(4.15) form the complete set of scalar perturbation equations for a fluid plus collisionless gas system.

B. Vector perturbations

The perturbations of the energy-momentum tensor are

$$\delta T^0_{(\alpha)i}=(E_\alpha+P_\alpha)(v^{(1)}_\alpha-B^{(1)})Q^{(1)}_i \,, \tag{4.16a}$$

$$\delta T^i_{(\alpha)j}=P_\alpha \pi_{T(\alpha)}^{(1)} Q^{(1)i}_{\ \ j} \,. \tag{4.16b}$$

The corresponding gauge-invariant variables are $\pi_{T(\alpha)}^{(1)}$ themselves and

$$v_{c(\alpha)}^{(1)}\equiv v^{(1)}_\alpha-B^{(1)} \,. \tag{4.17}$$

For $\alpha=c$ we have

$$(E_c+P_c)v_{c(c)}^{(1)}=\tfrac{1}{2}\int(1-\mu^2)p^{(0)}pI^{(1)}\pi \,, \tag{4.18a}$$

$$P_c\pi_{T(c)}^{(1)}=-i\int\mu(\mu^2-1)p^2I^{(1)}\pi \,. \tag{4.18b}$$

For $I^{(1)}$ Eq. (3.28) holds. The Einstein equation in Eq. (2.14) is

$$\frac{k^2}{2S^2}\psi=\sum_\alpha (E_\alpha+P_\alpha)v_{c(\alpha)}^{(1)} \,, \tag{4.19}$$

and the equations of motion are

$$\dot v_{c(\alpha)}^{(1)}=\frac{\dot S}{S}(3c_\alpha{}^2-1)v_{c(\alpha)}^{(1)}-\frac{kP_\alpha}{(E_\alpha+P_\alpha)}\pi_{T(\alpha)}^{(1)} \,. \tag{4.20}$$

The equations to be solved are Eqs. (3.29), (4.18a), (4.19), and (4.20) with $\alpha=f$, while Eqs. (4.18b) and (4.20) with $\alpha=c$ explain the production of rotation in a collisionless gas. If the fluid is ideal ($\pi_{T(f)}=0$) and initially nonrotating, it is always nonrotating (i.e., the law of circulation holds), while the rotation in the collisionless gas appears independently of the fluid because of nonvanishing $\pi_{(T(c)}$.

C. Tensor perturbations

The perturbations of the energy momentum tensor are

$$\delta T^i_{(\alpha)j}=P_\alpha \pi_{T(\alpha)}^{(2)} Q^{(2)i}_{\ \ j} \,. \tag{4.21}$$

For $\alpha=c$ we have

$$\pi_{T(c)}^{2}=\tfrac{1}{4}\int(1-2\mu^2+\mu^4)p^2f^{(2)}\pi \,,$$

where $f^{(2)}$ is given by Eq. (3.37). The Einstein equation is expressed as

$$S^{-2}\left[\ddot H_T^{(2)}+2\frac{\dot S}{S}\dot H_T^{(2)}+k^2H_T^{(2)}\right]=\sum_\alpha \pi_{T(\alpha)}^{(2)} \,. \tag{4.22}$$

V. CONCLUDING REMARKS

We have derived the gauge-invariant perturbation equations for a collisionless gas by introducing the gauge-invariant quantities which correspond to the perturbation of the distribution function. Our equations for scalar perturbation equations seem a little simpler than others, e.g., Bond and Szalay's[9] in synchronous gauge, because our fundamental equation (3.21) deriving the perturbations does not include any time derivative of gravitational perturbations such as $\dot\phi_A$ and $\dot\phi_H$. The equations for vector and tensor perturbations also are interesting, because in a collisionless gas rotation and gravitational waves are generated owing to its dissipative character ($\pi_T^{(1)}$ and $\pi_T^{(2)}$ are nonzero). In our formalism, unphysical gauge modes are automatically excluded, whereas in other gauge-specifying methods they may not be.[10] It will, therefore, be a useful tool for the study of linear perturbations of the dark matter in the early stage of the Universe, which grow to eventually form the large-scale structure of the Universe. Numerical solutions for the perturbation equations in some interesting cases will be shown in a separate paper.

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APPENDIX A

In this appendix, we analyze the gauge dependence of the tetrad components of momenta and the perturbations of the distribution function. The perturbed metric has the form

$$ds^2=S^2(\tau)(\eta_{\mu\nu}+h_{\mu\nu})dx^\mu dx^\nu \,. \tag{A1}$$

Let us introduce a tetrad frame

$$e^\mu_{(a)}=\frac{1}{S}(\delta^\mu_a-\tfrac{1}{2}h^\mu_a)$$

and (A2)

$$e^{(a)}_\mu=S(\delta^a_\mu+\tfrac{1}{2}h^a_\mu) \,.$$

Then it satisfies the relations $e^\mu_{(a)}e^{(b)}_\mu=\delta^b_a$ and $e^\mu_{(a)}e^{(a)}_\nu=\delta^\mu_\nu$, and is orthonormal, i.e., $g_{\mu\nu}e^\mu_{(a)}e^\nu_{(b)}=\eta_{ab}$. The tetrad indices a and b run from 0 to 3. It is shown below that for arbitrary gauge transformations the tetrad frames have a vector transformation followed by a Lorentz transformation.

1. Scalar perturbations

Under the gauge transformation of Eq. (2.7), the tetrad frame changes as

$$\tilde e^\mu_{(a)}=\tilde S^{-1}(\delta^\mu_a-\tfrac{1}{2}\tilde h^\mu_{\ a})$$

and (A3)

$$\tilde e^{(a)}_\mu=\tilde S(\delta^a_\mu+\tfrac{1}{2}\tilde h^{\ a}_{\mu})$$

and the metric perturbations are

$$\tilde h^\mu_\nu=h^\mu_\nu+\Delta h^\mu_\nu \,,$$

where

$$\Delta h^i_0=-(\dot L+kT)Q^i$$

and

$$\Delta h^i_j=-2\left[\left[\frac{k}{3}L+\frac{\dot S}{S}T\right]\delta^i_j Q-kLQ^i_j\right].$$

If we consider a matrix defined by

$$L^a_b\equiv\frac{\partial\tilde x^\mu}{\partial x^\nu}e^\nu_{(b)}\tilde e^{(a)}_\mu\ ,\tag{A4}$$

it is found that generally L^a_b is not δ^a_b but a Lorentz matrix, because the relation $\eta_{ab}L^a_cL^b_d=\eta_{cd}$ can be derived using the orthonormality condition. The change in the tetrad components $p^{(i)}\equiv e^{(i)}_\mu dx^\mu/ds$ is

$$\tilde p^{(i)}-p^{(i)}=\tilde e^{(i)}_\mu\frac{\partial\tilde x^\mu}{\partial x^\nu}\frac{dx^\nu}{ds}-e^{(i)}_\mu\frac{dx^\mu}{ds}$$

$$=(L^i_a-\delta^i_a)p^{(a)}$$

$$=\tfrac{1}{2}(\dot L-kT)p^{(0)}Q^i\ .\tag{A5}$$

This difference comes from the Lorentz transformation $\tilde p^{(i)}=L^i_ap^{(a)}$.

The distribution function $F(x,\mathbf{p})$ is a scalar, but

$$\tilde F(\tilde x,\mathbf{p})=F_B(S(\tilde\tau)\tilde p)+\tilde fQ$$

$$=F_B(S(\tau)p)$$

$$+\frac{\partial F_B}{\partial p}\left[\frac{\dot S}{S}pTQ+\frac{1}{2}\frac{p^{(0)}p^{(i)}}{p}(\dot L-kT)Q_i\right]+\tilde fQ\ .$$

Thus the perturbation f does change as

$$\tilde f=f-\frac{\partial F_B}{\partial p}\left[\frac{\dot S}{S}Tp-\frac{i}{2}\mu(\dot L-kT)p^{(0)}\right].\tag{A6}$$

2. Vector perturbations

Under the gauge transformation of Eq. (2.10), we have

$$\tilde h^\mu{}_\nu=h^\mu{}_\nu+\Delta h^\mu{}_\nu\ ,$$

$$\tilde e^{(i)}_\mu=S(\tau)(\delta^i{}_\mu+\tfrac{1}{2}\tilde h^i{}_\mu)=e^{(i)}_\mu+\tfrac{1}{2}S\Delta h^i{}_\mu\ ,$$

where $\Delta h^i{}_0=-\dot LQ^{(1)i}$ and $\Delta h^i{}_j=2kLQ^{(1)i}_j$. The change in the component $p^{(i)}$ is, therefore, given by

$$\tilde p^{(i)}-p^{(i)}=(L^i_a-\delta^i_a)p^{(a)}$$

$$=\tfrac{1}{2}\dot Lp^{(0)}Q^{(1)i}+kLp^{(j)}Q^{(1)i}_j+Lp^{(j)}Q^i{}_{,j}\ .\tag{A7}$$

Moreover we obtain

$$F_B(S(\tilde\tau)\tilde p)=F_B(S(\tau)p)$$

$$+\frac{\partial F_B}{\partial p}\frac{p^{(i)}}{p}(\tfrac{1}{2}\dot Lp^{(0)}Q^{(1)i}+kLp^{(j)}Q^{(1)i}_j$$

$$+Lp^{(j)}Q^{(1)}{}_{,j}^{i})\ .\tag{A8}$$

Therefore the perturbation $f^{(1)}$ changes like

$$\tilde f^{(1)}=f^{(1)}-\frac{1}{2}\frac{\partial F_B}{\partial p}\dot Lp^{(0)}\ .\tag{A9}$$

APPENDIX B

In this appendix, we calculate the perturbations of the energy-momentum tensor for a collisionless gas. Here, the subscript c is omitted. From (3.3), the perturbations are expressed as

$$\delta T^\mu{}_\nu=\delta^\mu{}_a\delta^b{}_\nu\int p^{(a)}p_{(b)}\delta F\pi$$
$$+\tfrac{1}{2}(\delta^\mu{}_ah^b{}_\nu-h^\mu{}_a\delta^b{}_\nu)\int p^{(a)}p_{(b)}F_B\pi\ .\tag{B1}$$

1. Scalar perturbations

From Eqs. (3.5) and (3.11)

$$\delta T^0{}_0=Q\int p^{(0)}p_{(0)}f\pi$$
$$=Q\left[\int p^{(0)}p_{(0)}J\pi+3(E+P)(H_L+\tfrac{1}{3}H_T)\right],\tag{B2}$$

$$\delta T^0{}_i=Q\int p^{(0)}p_{(i)}f\pi-\tfrac{1}{2}(E+P)BQ_i$$
$$=-i\frac{k^i}{k}Q\int ip^{(0)}p\mu f\pi-\tfrac{1}{2}(E+P)BQ_i$$
$$=Q_i\left[i\int p^{(0)}p\mu J\pi-(E+P)(B-1/kH_T)\right],\tag{B3}$$

and

$$\delta T^i{}_j=Q\int p^{(i)}p_{(j)}f\pi$$
$$=\tfrac{1}{3}\delta^i_jQ\int p^2f\pi-\left[\frac{1}{3}\delta^i_j-\frac{k^ik_j}{k^2}\right]$$
$$\times Q\int p^2\tfrac{1}{2}(3\mu^2-1)f\pi$$
$$=\tfrac{1}{3}\delta^i_jQ\left[\int p^2J\pi-3c^2(E+P)(H_L+\tfrac{1}{3}H_T)\right]$$
$$-Q^i_j\int p^2\tfrac{1}{2}(3\mu^2-1)J\pi\ .\tag{B4}$$

2. Vector perturbations

$$\delta T^0{}_i=Q^{(1)}_j\int p^{(0)}p_{(i)}\frac{p^{(j)}}{p}f^{(1)}\pi$$
$$=Q^{(1)}_j\left[\delta^j{}_i\int\tfrac{1}{2}(1-\mu^2)p^{(0)}pI^{(1)}\pi\right.$$
$$\left.+\frac{k^jk_i}{k^2}\int\tfrac{1}{2}(3\mu^2-1)p^{(0)}pI^{(1)}\pi\right].\tag{B5}$$

By use of $Q^{(1)}_jk^j=n_jk^jQ=0$,

$$\delta T^0{}_i=Q^{(1)}_i\int\tfrac{1}{2}(1-\mu^2)p^{(0)}pI^{(1)}\pi\ .\tag{B6}$$

Moreover,

$$\delta T^i_j = Q^{(1)}_m \int p^{(i)} p_{(j)} p^{(m)} p^{-1} f^{(1)} \pi$$
$$= Q^{(1)i}_{\ \ j} \int \mu(\mu^2-1) p^2 I^{(1)} \pi \, . \quad \text{(B7)}$$

3. Tensor perturbations

$$\delta T^i{}_j = Q^{(2)}_{lm} \delta_{jn} \int p^{(i)} p^{(n)} \frac{p^{(l)} p^{(m)}}{p^2} f^{(2)} \pi \, . \quad \text{(B8)}$$

The integral of Eq. (B8) can be decomposed as

$$\int p^{(i)} p^{(n)} p^{(l)} p^{(m)} p^{-2} f^{(2)} \pi$$
$$= X k^{(i} k^n k^l k^{m)} + Y \delta^{(in} k^l k^{m)} + Z \delta^{(in} \delta^{lm)} \, . \quad \text{(B9)}$$

Since $Q^{(2)i}_{\ \ i}=0$ and $Q^{(2)i}_{\ \ j} k^j = 0$, only the third term in the right-hand side of Eq. (B9) is relevant, and

$$Z \delta^{(in} \delta^{lm)} = \tfrac{1}{8} (\delta^{in} \delta^{lm} + \delta^{il} \delta^{nm} + \delta^{im} \delta^{nl})$$
$$\times \int (1 - 2\mu^2 + \mu^4) p^2 f^{(2)} \pi \, . \quad \text{(B10)}$$

Then

$$\delta T^i{}_j = \tfrac{1}{4} Q^{(2)i}_{\ \ j} \int (1 - 2\mu^2 + \mu^4) p^2 f^{(2)} \pi \, . \quad \text{(B11)}$$

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