Effects of quantum fields on singularities and particle horizons in the early universe. IV. Initially empty universes

Paul R. Anderson
215 Williamson Hall, University of Florida, Gainesville, Florida 32611
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The behaviors of solutions to the semiclassical back-reaction equations are investigated for conformally invariant free quantum fields and a conformally coupled massive scalar field in spatially flat homogeneous and isotropic spacetimes with no classical radiation or matter. The quantum fields are all initially in their vacuum states. Only solutions beginning with the scale factor equal to zero are considered. The behaviors of solutions depend upon two regularization parameters α and β and on the mass m of the scalar field. For $\beta \geq 3\alpha > 0$, all solutions beginning with the scale factor equal to zero are without particle horizons, while for $3\alpha > \beta > 0$ a one-parameter family of solutions with no particle horizons exists. For $\beta > 0$, $\alpha < 0$ a one-parameter family of solutions with no singularities or particle horizons exists if $m = 0$ and may exist if $m \neq 0$ as well. For $m \neq 0$, $\alpha > 0$ there is evidence that a one-parameter family of solutions which expand like classical matter-dominated Friedmann universes at late times compared to the Planck time exists. For $\alpha < 0$ a two-parameter family of such solutions exists. In both cases particle production due to the massive scalar field fills up the spacetimes.

I. INTRODUCTION

Studies of free quantum fields in homogeneous and isotropic spacetimes allow one to address several issues in cosmology. These include the following questions. Did the Universe begin with an initial singularity? Does it have particle horizons? Where does the matter come from? In Refs. 1 and 2, hereafter referred to as papers I and II, we examined the effects of conformally invariant free quantum fields in homogeneous and isotropic spacetimes containing classical radiation. Fischetti, Hartle, and Hu[3] and Frenkel and Brecher[4] had previously found one solution to the semiclassical back-reaction equations in this case with no particle horizons. We found many more such solutions as well as solutions with no singularities and one solution which undergoes a time-symmetric bounce and has no singularities or particle horizons. In Ref. 5, hereafter referred to as paper III, we added a conformally coupled massive scalar field to the models and found, for spatially flat spacetimes, that in the limit that the scale factor vanishes, the mass of the field has no effect on solutions to the semiclassical back-reaction equations. However, it does significantly affect the late-time behaviors of many solutions.

In each of these papers, classical radiation was included so that at late times compared to the Planck time, when quantum effects are expected to be small, the Universe could expand like a classical radiation-dominated Friedmann universe. However, as might have been expected, we found in paper III that significant particle production occurs for the massive scalar field provided that $m \gtrsim 10^{10}$ GeV. This opens up the possibility that classical radiation might not be needed to support the expansion at late times since the produced particles might do the job as well. In fact, if no classical matter is included and if the quantum fields are initially in their vacuum states, then the resulting models provide a natural explanation for the origin of the matter in the Universe.

In this paper, we investigate the behaviors of initially empty spacetimes which are homogeneous, isotropic, and spatially flat. We first consider models with conformally invariant free quantum fields. In this case no particle production occurs and the spacetimes remain empty.[6] Then we also include a conformally coupled massive scalar field. Particle production can, and usually does, then occur.

As discussed in paper III, we consider spacetimes that are homogeneous, isotropic, and spatially flat because of their simplicity and because the Universe at the time of the decoupling of matter and radiation exhibited these properties, leading one to believe the early Universe may have had them as well. We exclude massive spinor fields but the conformally coupled massive scalar field should provide a great deal of insight into their effects since both are conformally invariant in the limit that either the mass or the scale factor vanishes. Because of the technical difficulties involved, we do not consider gravitons or minimally coupled scalar fields.

Our models should provide a good approximation to the early Universe provided that the fields in the early Universe can be approximated as free fields when the scale factor is small, that most scalar fields are conformally coupled, and that there are a large number of fields so that the effects of gravitons and minimally coupled scalar fields are negligible. However, in the actual limit that the scale factor vanishes the effects of gravitons and minimally coupled scalar fields may dominate.[7]

Empty homogeneous and isotropic spacetimes containing conformally invariant free quantum fields have been previously investigated by Starobinsky[8] and Azuma and Wada.[9] Starobinsky has shown that de Sitter space with an effective cosmological constant on the order of the

Planck scale is a solution to the semiclassical back-reaction equations for all values of α if $\beta>0$, where α and β are the two regularization parameters in the problem. It turns out that $\beta>0$ for all quantum fields. Starobinsky also showed that if $\alpha<0$ then a family of solutions exists which initially expand like the de Sitter solution and thus have no singularities or particle horizons. These solutions along with many others are asymptotically classical solutions (ACS's). For $\alpha>0$, no ACS exist.

In Sec. II we derive and discuss the wave equation for the massive scalar field and the semiclassical back-reaction equations. In Sec. III we complete the investigations begun by Starobinsky and Azuma and Wada and find that for $\beta \geqq 3\alpha>0$ no solutions with particle horizons exist. We also find that the only nonsingular solutions are the de Sitter solution and those solutions for $\alpha<0$ which initially expand like the de Sitter solution.

In Sec. IV we discuss the behavior of solutions when the conformally coupled massive scalar field is also present. We find that the effective cosmological constant for the de Sitter solution is decreased by the presence of this field and that there exists a maximum mass $m_{\max}$ such that for $m>m_{\max} \geqq 24\pi l^{-1}$ the de Sitter solution no longer exists.[10] Here $l \equiv (16\pi G)^{1/2}$ is the Planck length.[11] We find that the rest of the initial types of behavior of solutions are not significantly affected by the mass of the massive scalar field. However, the final behaviors of many solutions are affected by the mass. For $\alpha<0$ a family of stable ACS's still exists while for $\alpha>0$ there is evidence that particle production due to the massive scalar field results in the existence of an unstable family of ACS's.

II. DERIVATION OF THE BACK-REACTION EQUATIONS

In this section we derive the back-reaction equations appropriate for our models. We use canonical quantization and adiabatic regularization to treat the massive scalar field. These subjects are reviewed in paper III as well as in Ref. 12 and references contained therein. When canonical quantization is used and the fields are in their vacuum states, the semiclassical back-reaction equations are

$$G_{ab}=\frac{l^2}{2}\langle 0 | T_{ab} | 0\rangle \ . \qquad (2.1)$$

Here G_{ab} is the Einstein tensor and T_{ab} is the stress-energy tensor operator for the quantum fields. Note that although the geometry is left classical in this approximation, first-order quantum fluctuations of the gravitational field, gravitons, can be included in $\langle 0 | T_{ab} | 0\rangle$. We shall not do so.

The metric for spatially flat, homogeneous and isotropic spacetimes has the form

$$ds^2=a^2(\eta)(-d\eta^2+d\mathbf{x}^2) \ , \qquad (2.2)$$

where $a(\eta)$ is the scale factor and the proper time of a comoving observer is $dt=a\,d\eta$. G_{ab} for this metric is well known, so we will concentrate our attention on $\langle 0 | T_{ab} | 0\rangle$.

The metric (2.2) is clearly conformal to Minkowski space. For conformally invariant fields the obvious choices of vacuum states are those obtained by conformally transforming the usual Minkowski space modes to curved space. Then, since no particle production occurs for these fields in Minkowski space, none occurs in spacetimes conformal to Minkowski space. The regularized expression $\langle 0 | T_{ab} | 0\rangle$ for conformally invariant fields in conformally flat spacetimes is known to be[13,14]

$$\langle 0 | T_{ab} | 0\rangle=\frac{\alpha}{3}(g_{ab}R^{;c}{}_{;c}-R_{;ab}+RR_{ab}-\tfrac{1}{4}g_{ab}R^2)$$
$$+\beta(\tfrac{2}{3}RR_{ab}-R_a{}^cR_{bc}$$
$$+\tfrac{1}{2}g_{ab}R_{cd}R^{cd}-\tfrac{1}{4}g_{ab}R^2) \ , \qquad (2.3)$$

where g_{ab} is the metric tensor, R_{ab} is the Ricci tensor, R is the scalar curvature, and α and β are regularization parameters that depend on the number and types of fields present and for spin-1 fields on the regularization scheme used. For scalar fields $\beta=\alpha=(2880\pi^2)^{-1}$, for four-component spinor fields $\beta=\frac{11}{6}\alpha=11(2880\pi^2)^{-1}$, and for vector fields dimensional regularization gives $\beta=\frac{31}{6}\alpha=62(2880\pi^2)^{-1}$ while ζ-function regularization gives $\beta=-\frac{31}{9}\alpha=62(2880\pi^2)^{-1}$.

In paper III we derived a regularized expression for $\langle 0 | T_{ab} | 0\rangle$ using adiabatic regularization. We sketch that derivation here.

The starting point is the wave equation which for the conformally coupled massive scalar field is

$$\Box\phi-(m^2+\tfrac{1}{6}R)\phi=0 \ . \qquad (2.4)$$

Expanding the field ϕ in terms of the modes $u_{\mathbf{k}}(x)$, one finds

$$\phi=\int d^3k(a_{\mathbf{k}}u_{\mathbf{k}}+a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}) \ , \qquad (2.5)$$

where $a_{\mathbf{k}}$ is an annihilation operator and

$$u_{\mathbf{k}}=(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}a^{-1}(\eta)\psi_k(\eta) \ . \qquad (2.6)$$

Substitution of (2.5) and (2.6) into (2.4) shows that ψ_k is a solution to the equation

$$\frac{d^2\psi_k}{d\eta^2}+(k^2+m^2a^2)\psi_k=0 \ . \qquad (2.7)$$

Requiring the modes $u_{\mathbf{k}}$ to be orthonormal with respect to the usual Klein-Gordon inner product gives the condition

$$\psi_k\frac{d}{d\eta}\psi_k^{*}-\psi_k^{*}\frac{d}{d\eta}\psi_k=i \ . \qquad (2.8)$$

The vacuum state is defined as the state for which

$$a_{\mathbf{k}}|0\rangle=0 \qquad (2.9)$$

for all $\mathbf{k}$. Thus, choosing initial values for ψ_k for all k is equivalent to choosing a vacuum state.

An unregularized expression for $\langle 0 | T_{ab} | 0\rangle$ can be obtained by substituting Eqs. (2.5) and (2.6) into the classical expression for T_{ab} and taking the vacuum expectation values. The result is[15,16]

$$\langle 0 | T^0{}_0 | 0 \rangle_{\text{unreg}} = (4\pi^2 a^4)^{-1} \times \int_0^\infty dk\, k^2 [\, |\psi'|^2 + (k^2+m^2a^2)|\psi|^2] . \quad (2.10a)$$

$$\langle 0 | T | 0 \rangle_{\text{unreg}} = m^2 (2\pi^2 a^2)^{-1} \int_0^\infty dk\, k^2 |\psi|^2 . \quad (2.10b)$$

Homogeneity and isotropy allow one to deduce the other components of $\langle 0 | T_{ab} | 0 \rangle$ from these.

To regularize $\langle 0 | T_{ab} | 0 \rangle_{\text{unreg}}$ we use adiabatic regularization since this is the scheme most amenable to numerical computations. Birrell has shown that it is equivalent to point splitting for spacetimes with the metric (2.2).[17] Essentially, one solves Eq. (2.7) using a Wentzel-Kramers-Brillouin (WKB) approximation, substitutes this into (2.10), and subtracts the resulting expressions from those for the exact modes. The adiabatic expressions are explicitly given in Eqs. (2.19a) and (2.19b) of paper III. One of the terms in each of these equations is proportional to G^0_0 and R, respectively. Fulling, Parker, and Hu[15] suggest that these terms correspond to a finite renormalization of the gravitational constant. We shall treat them as such and therefore exclude them from the back-reaction equations.

Before writing the explicit form of the back-reaction equations that we shall use, it is useful to define the dimensionless variable

$$b = l^{-1} a . \quad (2.11)$$

Note that since classical radiation is present, this is a different definition of b than that used in papers I—III.

Combining Eqs. (2.1)—(2.3), (2.7), (2.10), and (2.11) and Eqs. (2.19) from paper III, we find the following set of coupled equations to be solved:

$$\frac{d^2\psi_k}{d\eta^2} + (k^2 + m^2 l^2 b^2)\psi_k = 0 , \quad (2.12a)$$

$$\frac{b'^2}{b^4} = 2\alpha \left[\frac{b'''b'}{2b^6} - \frac{b''b'^2}{b^7} - \frac{1}{4}\left[\frac{b''}{b^3}\right]^2 \right] + \frac{\beta}{2}\left[\frac{b'}{b^2}\right]^4 + \tfrac{1}{6} I_1 , \quad (2.12b)$$

$$\frac{-b''}{b^3} = 6\alpha \left[-\frac{b''''}{12b^5} + \frac{1}{3}\frac{b'''b'}{b^6} + \frac{1}{4}\left[\frac{b''}{b^3}\right]^2 - \frac{1}{2}\frac{b''b'^2}{b^7} \right] + \beta\left[\left[\frac{b'}{b^2}\right]^4 - \frac{b''b'^2}{b^7} \right] - \tfrac{1}{6} I_2 , \quad (2.12c)$$

where primes denote derivatives with respect to η and

$$I_1 \equiv (4\pi^2 b^4)^{-1} \int_0^\infty dk\, k^2 [\, |\psi'|^2 + (k^2+m^2l^2b^2)|\psi|^2 - (k^2+m^2l^2b^2)^{1/2}] ,$$

$$I_2 \equiv m^2 l^2 (4\pi^2 b^2)^{-1} \times \int_0^\infty dk\, k^2 [\, |\psi|^2 - \tfrac{1}{2}(k^2+m^2l^2b^2)^{-1/2}] .$$

The values of α and β in (2.12b) and (2.12c) are equal to the sum of α and β for each of the conformally invariant fields plus a contribution of $(2880\pi^2)^{-1}$ to both from the conformally coupled massive scalar field. Thus, the massive scalar field adds terms to the back-reaction equations which are the same as those contributed by a conformally invariant scalar field along with the terms I_1 and I_2 which depend on the mass. In the limit $m \to 0$ the massive scalar field becomes conformally invariant and I_1 and I_2 vanish if the field is in a vacuum state which reduces to the conformal vacuum in this limit.

Equations (2.12) are all explicitly independent of η so solutions are invariant under the translation $\eta \to \eta + \eta_0$, with η_0 an arbitrary constant. They are also invariant under the transformation $\eta \to -\eta$, although their solutions, in general, are not. Equation (2.12c) is a fourth-order equation so one expects a four-parameter family of solutions to it. However, the constraint equation (2.12b) accounts for one of these parameters and the invariance of solutions under time translation accounts for another. Thus, one effectively has a two-parameter family of solutions to Eq. (2.12c). When we talk about families of solutions in the following sections it is this effective two-parameter family that we shall be referring to.

III. SOLUTIONS FOR $m = 0$

In this section we discuss solutions to the semiclassical back-reaction equations (2.12) when $m = 0$. Since β is positive for all quantum fields we only consider solutions for $\beta > 0$. If $m = 0$, all the fields are conformally invariant so $I_1 = I_2 = 0$ and we can dispense with Eq. (2.12a). Solutions to Eqs. (2.12b) and (2.12c) have been discussed by Starobinsky[8] for $\alpha < 0$ and Azuma and Wada[9] for both $\alpha > 0$ and $\alpha < 0$. However, their discussions do not explicitly include the initial and final behaviors of all of the solutions and, in particular, the question of which solutions have singularities and particle horizons is only partially answered for $\alpha > 0$. Because of this incompleteness and because we shall be comparing the behaviors of solutions for $m \neq 0$, we give in this section a complete description of the initial and final behaviors of all solutions and how they match up, for $m = 0$.

There are two solutions to (2.12b) and (2.12c) that occur for all α. One is the trivial Minkowski space solution, $b = \text{const}$. The other, which only occurs for $\beta > 0$, is the de Sitter solution

$$b = (\beta/2)^{1/2} (\eta_0 - \eta)^{-1} , \quad (3.1)$$

where η_0 is an arbitrary constant corresponding to the time translation invariance of Eqs. (2.12b) and (2.12c). The value of the actual cosmological constant in our models is zero. However, for the de Sitter solution (3.1) the effective cosmological constant is

$$\Lambda_{\text{eff}} = 6 l^{-2} \beta^{-1} . \quad (3.2)$$

The de Sitter solution begins at $\eta = -\infty$, and a glance at the metric (2.2) shows that for solutions beginning with $\eta = -\infty$, all of the past light cones intersect. Such solutions have no particle horizons. For spacetimes with the metric (2.2) the scalar curvature is

$$R = 6 l^{-2} b^{-3} b'' . \quad (3.3)$$

Substitution of (3.1) into (3.3) shows that R is a constant. Thus, de Sitter space has no curvature singularities.

The behaviors of the rest of the solutions is different for $\alpha>0$, $\alpha=0$, and $\alpha<0$, so we shall discuss these cases separately. For $\alpha=0$, $\beta>0$ there are no other solutions to (2.12).

A. $\alpha>0$

For $\alpha>0$, the qualitative behaviors of solutions can be determined from a phase-plane analysis. With the change of variables

$$w=\ln(|\alpha|^{-3/4}b^3)\,,\quad r=6^{3/4}e^{-w}|b'|^{3/2}\,,\quad s=\frac{dr}{dw}\,,\tag{3.4}$$

Eq. (2.12b) becomes the first-order equation

$$\frac{ds}{dr}=s^{-1}\left[\frac{-\beta r}{12\alpha}+\frac{\alpha}{|\alpha|}r^{-1/3}\right]-1\,.\tag{3.5}$$

A phase-plane analysis of (3.5) shows that for $\alpha=0$, all solutions spiral into the point $s=0$, $r=(12\alpha/\beta)^{3/4}$. This point corresponds to the de Sitter solution (3.1), so all solutions are asymptotically de Sitter in this case.

The early-time behaviors of solutions are different for different values of β/α. For $0<\beta/\alpha<3$, Azuma and Wada showed that a two-parameter family of solutions exists with the initial behavior

$$b=\text{const}\times(\eta-\eta_0)^{1/2\sigma},\quad \eta\to\eta_0\tag{3.6}$$

where the constant η_0 is the trivial one corresponding to the time translation invariance of Eqs. (2.12) and $\sigma\equiv\frac{1}{2}(1-\frac{1}{3}\beta/\alpha)^{1/2}$. Substitution into (3.3) shows that these solutions begin with an initial singularity. Since they begin at a finite value of η they have particle horizons.

Azuma and Wada also found a one-parameter family of solutions with the initial behavior

$$b=\text{const}\times(\eta_0-\eta)^{-1/2\sigma},\quad \eta\to-\infty\tag{3.7}$$

when $0<\beta/\alpha<3$. Substitution of (3.7) into (3.3) shows that these solutions also have initial singularities; however, they do not have particle horizons. A phase-plane analysis shows that no other initial behaviors exist for solutions if $0<\beta/\alpha<3$.

For $\beta/\alpha\geqq 3$, the initial behaviors of solutions have not been previously discussed. For $\beta/\alpha=3$ a phase-plane analysis shows that a two-parameter family of solutions has the initial behavior

$$b=\exp[C_1(\eta-\eta_0)^3]\tag{3.8}$$

for a positive but otherwise arbitrary constant C_1. Also a one-parameter family of solutions has the initial behavior

$$b=\exp[C_2(\eta-\eta_0)]\tag{3.9}$$

for a positive but otherwise arbitrary constant C_2. Substitution of (3.8) and (3.9) into (3.3) shows that these solutions begin with initial singularities. They do not have particle horizons.

For $\beta/\alpha>3$, σ is imaginary. If the $(b')^{-2}b^{-4}$ term on the left in Eq. (2.12b) is neglected, then the exact solutions to the resulting equation are

$$b'=cb[\sin^2(3|\sigma|\ln b+\delta)]^{1/3}\,,\tag{3.10}$$

where c and δ are arbitrary constants. These solutions go through an infinite number of inflection points as $b\to 0$ but have no maxima or minima.

Substitution of (3.10) into (2.12b) shows that near $b'=0$, the dominant terms on the right in (2.12b) are proportional to

$$b^{-4}[\sin^2(3|\sigma|\ln b+\delta)]^{1/3}\,.$$

For small enough b, these are clearly much larger than $(b')^2b^{-4}$. For larger values of b' the terms proportional to

$$b^{-4}[\sin^2(3|\sigma|\ln b+\delta)]^{4/3}$$

are also important. Again for small enough b, $(b')^2b^{-4}$ will be much smaller than both these types of terms. Thus, (3.10) describes the approximate behavior of solutions in the limit $b\to 0$.

Substitution of (3.10) into (3.3) shows that these solutions begin with an initial singularity. The curve $b=e^{c\eta}$ serves as a lower bound to these solutions as $b\to 0$ since $b'_{\max}=cb$. Thus, they begin at $\eta=-\infty$ and do not have particle horizons.

This ends our discussion of the behavior of solutions for $\alpha>0$. We have seen that all solutions approach the de Sitter solution (3.1) at late times and that except for the de Sitter solution, all solutions begin with an initial singularity at $b=0$. For $\beta\geqq 3\alpha>0$ no solutions have particle horizons while for $0<\beta<3\alpha$ most, but not all, solutions have particle horizons.

B. $\alpha<0$

For $\alpha<0$, Starobinsky has shown that along with the exact de Sitter solution, there exists a one-parameter family of solutions which initially expand like the de Sitter solution (3.1). Thus, these solutions have no singularities or particle horizons. He has also shown that their late-time behavior is given by

$$b=b_1\eta^2-6^{1/2}\tau\eta^{-1}\cos[6^{1/2}(9\tau)^{-1}b_1\eta^3+\delta]\,,\tag{3.11}$$

where b_1 and δ are arbitrary constants and $\tau\equiv|\alpha/3|^{1/2}$. On time scales $\Delta\eta\gg\tau$, the second term averages to zero, and the solutions expand like classical matter-dominated Friedmann universes. Thus, they are ACS's. However, since no particle production occurs these spacetimes are still empty, even though they expand classically at late times.

The initial behaviors of the remaining solutions for $\alpha<0$ are given by Eq. (3.6), so these solutions have singularities and particle horizons. Starobinsky showed, using a phase-plane analysis, that there are three possible late-time behaviors for these solutions. A two-parameter family has the behavior (3.11) and the solutions are thus ACS. A one-parameter family approaches the de Sitter solution

(3.1) at late times and a two-parameter family has the late-time behavior

$$b=\text{const}\times(\eta_1-\eta)^{-1/2\sigma},\quad \eta\to\eta_1\ , \tag{3.12}$$

where η_1 is an arbitrary constant. These solutions expand so quickly that $R\to\infty$ in the limit $b\to\infty$ and they reach $b=\infty$ in a finite amount of proper time.

This ends our discussion of the behavior of solutions when $\alpha<0$. We have seen that a one-parameter family of nonsingular ACS's with no particle horizons exists. A two-parameter family of ACS's with singularities and particle horizons also exists.

IV. SOLUTIONS FOR $m\neq0$

Having discussed the behavior of solutions for $m=0$, we now turn to their behavior for $m\neq0$. As in paper III, we shall only discuss solutions that begin with $b=0$. However, for $\alpha>0$ it is likely that nonsingular bounce solutions exist and we shall speculate on their probable behavior in Sec. IV A 3. Because the behavior of the solutions is different for $\alpha>0$ and $\alpha<0$, we again discuss these cases separately.

For $m\neq0$, Eqs. (2.12b) and (2.12c) depend upon the solutions to Eq. (2.12a) so we shall consider solutions to this equation first. For each k there exists a two-parameter family of solutions to (2.12a). One of these parameters is effectively used up by the orthonormality condition (2.8). The other corresponds to the choice of vacuum state. In the limit $b\to0$, the general solution to (2.12a) is

$$\psi_k=Ae^{-ik\eta}+Be^{ik\eta}\ . \tag{4.1}$$

Substitution into (2.8) gives the condition

$$|A|^2-|B|^2=(2k)^{-1}\ . \tag{4.2}$$

In paper III we chose the vacuum state

$$A=(2k)^{-1/2},\quad B=0 \tag{4.3}$$

because it is the only state which reduces to the conformal vacuum in both the limits $m\to0$ and $b\to0$. We showed that it is also the only state for which $\langle 0|T_{ab}|0\rangle$ has no piece that acts like classical radiation in the limit $b\to0$. Further, this choice gives the usual de Sitter—invariant vacuum for de Sitter space[12] and the Chitre-Hartle[18] vacuum for the spacetimes with $a=a_0e^{\eta}$. For all these reasons we choose (4.3) as the vacuum state in this paper as well.

With this choice of vacuum, the solution to (2.12a) for a spacetime beginning with $b=0$ is

$$\psi_k=(2k)^{-1/2}e^{-ik\eta}+(2k)^{-1/2}\sum_{n=1}^{\infty}(-1)^n(ml)^{2n}k^{-n}\times\int_w^{\eta}du_1\cdots\int_w^{u_{n-1}}du_n[b^2(u_1)\cdots b^2(u_n)\sin k(\eta-u_1)\sin k(u_1-u_2)\cdots\times\sin k(u_{n-1}-u_n)e^{-iku_n}]\ , \tag{4.4}$$

where $b(w)=0$. We showed in paper III that a sufficient condition for the convergence of this series is

$$\int_w^{\eta}du\, b^2(u)<\infty\ . \tag{4.5}$$

This is satisfied by all of the solutions that we find.

With the choice of vacuum (4.3) there is one solution that occurs for all α if $\beta>0$. This is the de Sitter solution

$$b=(3/\Lambda_{\text{eff}})^{1/2}(\eta_0-\eta)^{-1}\ , \tag{4.6}$$

where, as usual, η_0 is an arbitrary constant corresponding to the time translation invariance of Eqs. (2.12). For de Sitter space $\langle 0|T_0^0|0\rangle$, and therefore I_1, are well known for the massive scalar field.[12] Substituting these expressions and (4.6) into (2.12b) one finds the following equation for Λ_{eff}:

$$2\Lambda_{\text{eff}}l^2=\tfrac{1}{3}\beta\Lambda_{\text{eff}}^2l^4+\frac{m^4l^4}{64\pi^2}[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)-\ln(3m^2/\Lambda_{\text{eff}})]\ , \tag{4.7}$$

where $\nu^2\equiv\frac{1}{4}-3m^2/\Lambda_{\text{eff}}$ and $\psi(\frac{3}{2}\pm\nu)$ is a digamma function. For $m=0$, $\Lambda_{\text{eff}}=6l^{-2}\beta^{-1}$.

For $12m^2/\Lambda_{\text{eff}}\ll1$, one can expand the ψ's in a Taylor series about $m=0$ with the result that, to lowest order,

$$\Lambda_{\text{eff}}l^2=6\beta^{-1}\left[1+\frac{m^4l^4\beta}{4608\pi^2}[1-2\gamma-\ln(\tfrac{1}{2}m^2l^2\beta)]\right]^{-1}\ , \tag{4.8}$$

where γ is Euler's constant. Thus, the presence of a massive scalar field decreases Λ_{eff} for small m. For the values of β one expects in the early Universe, (4.8) should be a good approximation for $ml\lesssim1$.

If m^2/Λ_{eff} is large, one can also expand the ψ's with the result

$$\Lambda_{\text{eff}}l^2=6\left[1-\frac{m^2l^2}{576\pi^2}\right][\beta-(2880\pi^2)^{-1}]^{-1}\ ,\quad \beta\neq(2880\pi^2)^{-1} \tag{4.9a}$$

$$\Lambda_{\text{eff}}l^2=\left[13\,608\left[\frac{m^2l^2}{576\pi^2}-1\right]\right]^{1/2}\ ,\quad \beta=(2880\pi^2)^{-1}\ . \tag{4.9b}$$

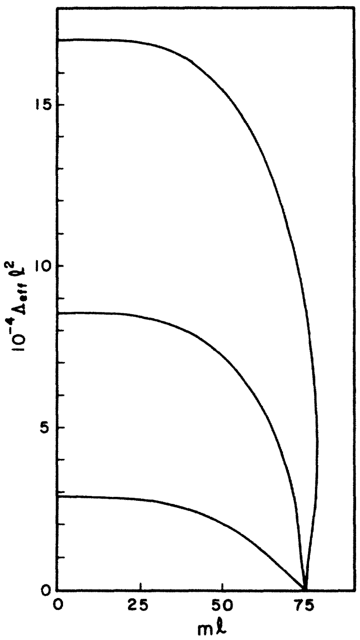

FIG. 1. In this figure the value of the effective cosmological constant Λ_{eff} for the de Sitter solution is plotted as a function of the mass of the conformally coupled massive scalar field. From top to bottom the curves correspond to $2880\pi^2\beta = 1,2,6$. Note that the curve for $\beta = (2880\pi^2)^{-1}$ is double valued for $ml \geq 24\pi$. In each case, the curves end at $ml = 24\pi$ and have a maximum value $m_{max}l \geq 24\pi$. For larger m, the de Sitter solution does not exist.

Thus, for $\beta > (2880\pi^2)^{-1}$ it is likely that the maximum value of ml is 24π. For $\beta \leq (2880\pi^2)^{-1}$ the maximum value of m must be larger than this which implies that Λ_{eff} is a double-valued function of m for $ml > 24\pi$. In both cases for large enough m, de Sitter space is not a solution to Eqs. (2.12). In Fig. 1 we show a plot of Λ_{eff} vs m for $\beta(2880\pi^2) = 1,2,6$.

The only other solution that occurs for all α (and β) is the trivial Minkowski-space solution. For $\alpha = 0$, there are no other solutions to (2.12) with the choice of vacuum (4.3) if $ml < 24\pi$. For $\alpha \neq 0$, there are many other solutions and we discuss them next beginning with the case $\alpha > 0$.

A. $\alpha > 0$

The techniques used to determine the behavior of solutions for $m \neq 0$ are different for the initial, intermediate, and final behavior of the solutions. Therefore, we shall discuss these types of behavior separately.

1. Initial behavior

One of our main results is that the initial behavior of solutions other than the de Sitter solution are no different for $m \neq 0$ than for $m = 0$. To show this we consider each initial type of behavior found in Sec. III and show that I_1 in Eq. (2.12b) is negligible compared to other terms in the limit $b \to 0$.

Substituting (4.4) into the expression for I_1 below Eqs. (2.12), the first thing one notices is that all of the integrals over time vanish in the limit $b \to 0$. Thus, I_1 always diverges less quickly than b^{-4} as $b \to 0$. This alone is enough to prove that I_1 is negligible compared to other terms in (2.12b) for the initial types of behavior (3.6), (3.8), and (3.9).

If (4.4) is substituted into I_1, then the first few terms are

$$I_1 = \frac{-m^4 l^4}{4\pi^2}\left[\tfrac{1}{8}\gamma + \tfrac{1}{32} + \tfrac{1}{16}\ln(m^2 l^2 b^2) + b^{-4}\int_w^\eta du_1 b(u_1) b'(u_1) \int_w^{u_1} du_2 b(u_2) b'(u_2) \ln|u_1 - u_2|\right]. \quad (4.10)$$

Substituting the initial behavior (3.7) into (4.10) one sees that to this order

$$I_1 \to \frac{m^4}{64\pi^2}\left[\frac{1}{\sigma} - 2\right]\ln(-\eta), \quad \eta \to -\infty . \quad (4.11)$$

The most divergent terms in (2.12b) have the form $\text{const} \times (-\eta)^{2/\sigma - 4}$ as $\eta \to -\infty$. Thus, to this order I_1 is negligible as $b \to 0$. Given that the convergence of the series (4.4) for ψ_k implies the convergence of the series in (4.10) for I_1, it seems unlikely that higher-order terms will diverge more rapidly than those explicitly calculated. Thus, the initial behavior given in (3.7) should still occur for $m \neq 0$.

The last type of initial behavior found for solutions when $\alpha > 0$, $m = 0$ is given by Eq. (3.10). Substitution of (3.10) into (2.12b) shows that terms are proportional to

$$b^{-4}[\sin^2(3|\sigma|\ln b + \delta)]^{4/3}$$

and

$$b^{-4}[\sin^2(3|\sigma|\ln b + \delta)]^{1/3} .$$

Thus, except near points where b' and b'' are small, these solutions diverge more quickly than I_1 as $b \to 0$. Near points where b' and b'' are small, I_1 will be important. However, since I_1 will be much smaller than other terms for larger values of b' and b'', the change it makes in the behavior of solutions will be insignificant as $b \to 0$. Numerical work bears this out and shows that the effect of I_1 is to keep the value of b' from dropping all the way to zero as b'' goes through zero. Thus, (3.10) is a good approximation to the initial type of behavior of solutions when $m \neq 0$, $\beta > 3\alpha > 0$.

2. Intermediate values of b

The biggest difference between solutions at intermediate values of b for $m = 0$ and $m \neq 0$, if $\alpha > 0$, is that the former must expand or contract monotonically while the latter may have maxima and minima. To see this, note that at $b'=0$ Eq. (2.12b) implies

$$b''=\pm b^3(2I_1/\alpha)^{1/2} . \quad (4.12)$$

Now, all of our numerical calculations of I_1 show it to be always positive for the choice of vacuum (4.3). This is also true of de Sitter space for which I_1 is known analytically. Thus, for $m=0$, there are no maxima and minima while for $m\neq 0$, maxima and minima are allowed if $\alpha>0$ and not otherwise.

To solve Eqs. (2.12) for intermediate values of b, we numerically integrated them beginning at some small value of b and using the results of Sec. IV A 2 to obtain starting values for b' and b''. Equation (4.4) was used to obtain starting values for ψ_k and ψ'_k and Eq. (2.12b) was used to obtain a starting value for b'''. Our numerical methods were discussed in some detail in paper III and we refer the interested reader to that paper.

Some of our results for $ml=20$ are shown in Figs. 2 and 3 for $\beta=6\alpha=6(2880\pi^2)^{-1}$ and $\beta=\alpha=(2880\pi^2)^{-1}$, respectively. In each figure a one-parameter family of solutions is shown in which the parameter that is varied is the starting value of b'. In general, we find there are always many solutions that are essentially indistinguishable from those which occur for $m=0$. For $\beta>3\alpha>0$ we find that some other solutions undergo multiple bounces for $m\neq 0$. For $0<\beta\leqq 3\alpha$ some solutions reach a maximum value of b after which they monotonically decrease.

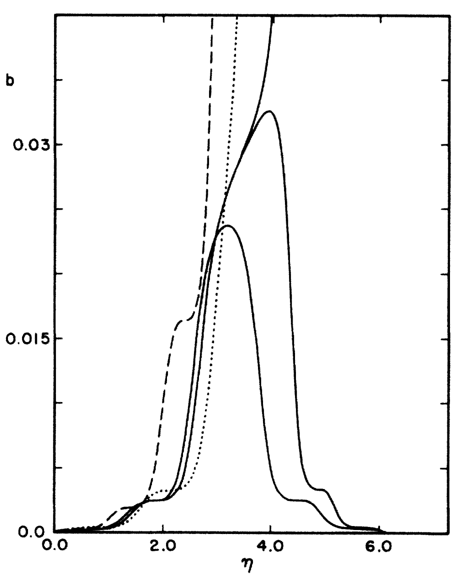

FIG. 2. In this figure a one-parameter family of solutions to Eqs. (2.12) is plotted for $ml=20$, $\beta=6\alpha=6(2880\pi^2)^{-1}$. The starting values of η,b,b'' are the same for each solution and the starting value of b' is different. The dotted curve on the left corresponds to the solution with the smallest starting value of b' and the dashed curve corresponds to the solution with the largest starting value of b'. A solution is hinted at, though not explicitly shown, which comes arbitrarily close to turning over but does not quite do so. This solution would appear to expand less rapidly than the diverging solutions and is probably an ACS. In the region where solutions are either not quite turning over or are just barely turning over, the accuracy of the curves shown is suspect due to the extreme instability of the solutions in this regime.

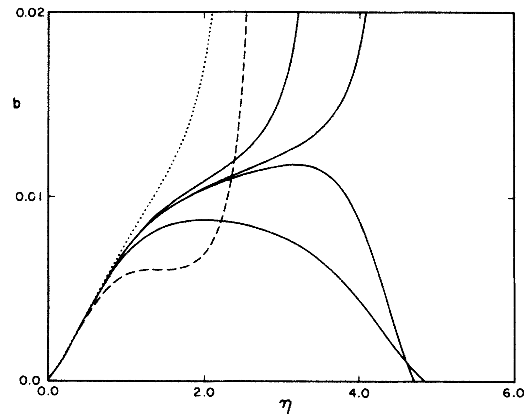

FIG. 3. In this figure a one-parameter family of solutions to Eqs. (2.12) is plotted for $ml=20$, $\beta=\alpha=(2880\pi^2)^{-1}$. The starting values of η,b,b'' are fixed and the starting value of b' is allowed to vary. The dotted curve on the left corresponds to the solution with the smallest starting value of b' and the dashed curve corresponds to the solution with the largest starting value of b'. A solution is hinted at which comes arbitrarily close to turning over. This solution would appear to expand more slowly than the other diverging solutions and is probably an ACS. In the region where solutions are either not quite turning over or are just barely turning over the accuracy of the curves shown is suspect due to the extreme instability of the solutions in this regime.

As in papers I—III the cutoff point for multiple bounce solutions appears to be $\beta=3\alpha$. The plots in Figs. 2 and 3 show all of the generic types of solutions that we find for $m\neq 0$.

3. Late-time behavior of solutions

For $m=0$, all solutions approach the de Sitter solution (3.1) at late times. For $m\neq 0$, we shall argue that solutions can approach the de Sitter solution (4.6), have some other type of divergent behavior, be asymptotically classical, or approach $b=0$ at late times. We shall discuss these possibilities separately.

In Figs. 2 and 3 there are solutions that diverge very rapidly at intermediate times. For intermediate values of b, I_1 and I_2 have no real influence on these solutions so we expect that they will begin expanding like the de Sitter solution (4.6), particularly for $ml \ll 24\pi$. It was shown in Sec. IV C 2 of paper III that once solutions begin expanding like the de Sitter solution they tend to continue to do so despite particle production. Thus, we expect that for $m<m_{\max}$, where $m_{\max}\geqq 24\pi l^{-1}$ is the maximum value of m for which the de Sitter solution exists, a two-parameter family of asymptotically de Sitter solutions will exist. For $m\geqq m_{\max}$ our numerical work shows that solutions still diverge rapidly at intermediate values of b, but we do not know their specific late-time behaviors.

There is another type of solution hinted at in Figs. 2 and 3. That is an ACS. In both figures a one-parameter family of solutions is plotted and some of these turn over while others do not. Continuity implies the existence of a solution which comes arbitrarily close to turning over but does not quite do so. From the figures it appears that this solution expands more slowly than those which are asymptotically de Sitter just as an ACS would. In paper I it was proven that for universes with classical radiation and only conformally invariant fields such solutions are always ACS's. In paper III we showed that when classical radiation is present, a one-parameter family of ACS's exists for many choices of the initial vacuum state of the conformally coupled massive scalar field. However, we could not specify for certain whether all reasonable choices led to the existence of ACS's. That argument is trivially modified to include universes with no classical radiation. So this result coupled with the numerical evidence just described seems to imply the existence of a one-parameter family of ACS's for the choice of vacuum (4.3).

From Figs. 2 and 3 it is clear that the third type of late-time behavior solutions can have is that they can turn over and collapse to $b=0$. Since particle production will occur for these solutions and since for small b the mass term in Eq. (2.12a) becomes negligible, the stress energy of the created particles has the same form as that for classical radiation as $b \to 0$. Thus, the final behavior of these solutions should be the same as the initial behavior of solutions in universes containing classical radiation. These were discussed in papers I and III. This means that for $\beta > 3\alpha > 0$ solutions will undergo an infinite number of bounces, approaching $b=0$ at $\eta = \infty$. For $\beta = 3\alpha$ we expect a two-parameter family of solutions to have the final behavior

$$b = \text{const} \times \exp[c(\eta_1 - \eta)^3], \quad \eta \to \infty \tag{4.13}$$

with c and η_1 constants. We also expect that a one-parameter family may have the final behavior

$$b = \text{const} \times \exp[-c(\eta_1 - \eta)^2], \quad \eta \to \infty \ . \tag{4.14}$$

For $0 < \beta < 3\alpha$, a two-parameter family of solutions will have the final behavior

$$b = \text{const} \times (\eta_1 - \eta)^{1/2\alpha}, \quad \eta \to \eta_1 \tag{4.15}$$

and a one-parameter family should have the final behavior

$$b = \text{const} \times \exp(-c\eta), \quad \eta \to \infty \ . \tag{4.16}$$

Note that in all cases solutions collapse to a final singularity in a finite amount of proper time $dt = lb\, d\eta$.

Although we are restricting ourselves to spacetimes that begin with $b=0$, there is one other type of initial behavior for $\alpha > 0$. That is a solution which begins at $b = \infty$ and initially collapses like the time reverse of the de Sitter solution (4.6). Such solutions certainly occur for $m < m_{\max}$. For $m=0$, they all collapse to $b=0$ and so are not of interest. However, a nonzero value of I_1 for $m \neq 0$ allows for the possibility that some solutions bounce. In paper I we found, for universes containing classical radiation, that a two-parameter family of such initially collapsing de Sitter solutions bounced at nonzero values of b and that a one-parameter family of these are ACS's. It is reasonable to expect that this is the case for initially empty universes with $m \neq 0$ as well. Of course, we have no proof of this.

This ends our discussion of the behavior of solutions for $\alpha > 0$. We have seen that the initial types of behavior of solutions are the same as for $m=0$ while the intermediate and late-time behavior are the same for some solutions and very different for others. Most importantly, there is evidence for the existence of a one-parameter family of ACS's for $m \neq 0$ while no ACS's exist for $m=0$.

B. $\alpha < 0$

1. Initial behavior

For $m=0$, a two-parameter family of solutions exists with the initial type of behavior (3.6). For these types of behavior the dominant terms in (2.12b) diverge faster than b^{-4} as $b \to 0$. It was argued in Sec. IV A 1 that I_1 diverges less quickly than b^{-4} in this limit, so these are still solutions for $m \neq 0$.

For $m=0$, a one-parameter family of solutions, which initially expand like the de Sitter solution (3.1), also exists. Since the de Sitter solution (4.3) is an exact solution for $m < m_{\max}$, we expect that a one-parameter family of solutions exists in this case which has the initial types of behavior given by (4.3). This seems particularly likely for $12m^2/\Lambda_{\text{eff}} \ll 1$, since Λ_{eff} is almost unchanged from its value for $m=0$ in this case. However, we have no proof that such solutions exist.

2. Intermediate behavior

As for $\alpha > 0$, the intermediate types of behavior of solutions must be determined numerically. A glance at (4.12) shows that if $I_1 > 0$ (as it is always observed by us to be) then there are no extrema for solutions if $\alpha < 0$; that is, solutions either expand or contract monotonically.

In Fig. 4 we show the behavior of selected solutions for $\beta = -\alpha = (2880\pi^2)^{-1}$, $ml = 20$. Note that one of the solu-

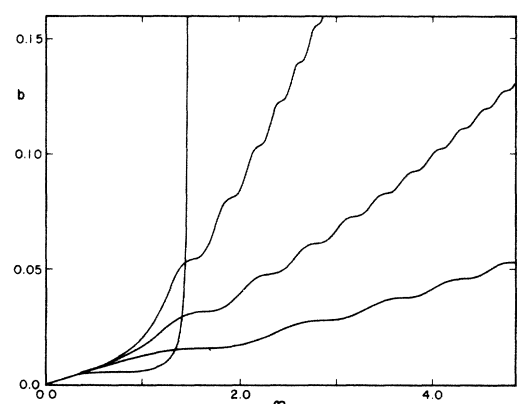

FIG. 4. In this figure selected solutions for $ml = 20$, $\beta = -\alpha = (2880\pi^2)^{-1}$ are shown. The solution on the left diverges rapidly and does not significantly differ from the corresponding solution for $m=0$. The other solutions undergo phase-plane oscillations and are thus ACS's. They differ from their $m=0$ counterparts in that they expand somewhat more rapidly at intermediate times and in that b' stays positive throughout the oscillation cycle instead of approaching zero when b'' does so.

tions diverges very rapidly, while the others expand more slowly. For the former, I_2 is always negligible compared to other terms in (4.12c) so this solution differs insignificantly from its counterpart for $m=0$.

The solutions which expand more slowly undergo phase-plane oscillations such as occur for $m=0$. For most of these solutions I_1 and I_2 are important at intermediate values of b and there is some evidence that particle production (which is automatically taken into account in the terms I_1 and I_2) dissipates the oscillations. Nonzero values of I_1 and I_2 do keep b' from vanishing when $b''=0$, so the solutions do not go through an infinite number of inflection points as they do for $m=0$.

3. *Final behavior*

For $m=0$, there are three types of final behavior solutions can have: A two-parameter family has the very divergent behavior (3.12); a one-parameter family approaches the de Sitter solution (3.1); and a two-parameter family has the behavior (3.11). For $m\neq 0$, our numerical work shows that a two-parameter family of solutions rapidly diverges at intermediate values of b. Thus, we expect that they will continue to do so, but we do not know whether they will expand like the solutions in (3.12). Since the de Sitter solution (4.3) is an exact solution for $m\neq 0$, we expect a one-parameter family of solutions to be asymptotically de Sitter solutions at late times, but we have no proof that such a family exists.

For $m\neq 0$, a two-parameter family of solutions have late-time behaviors similar to those in (3.11). They are given by

$$b=b_c+B\tau b_c^{-1/2}\cos\left[\left(\tfrac{2}{3}\right)^{1/2}\tau^{-1}\int^{\eta} b_c\,d\eta\right],\quad b\to\infty \tag{4.17a}$$

where $\tau\equiv|\alpha/3|^{1/2}$, B is an arbitrary constant, and b_c is a solution to the equation

$$(b_c' b_c^{-2})^2=\tfrac{2}{3}B^2 b_c^{-3}+\tfrac{1}{6}I_1 \,. \tag{4.17b}$$

It was shown in paper III that for the behavior (4.17a), the leading-order terms in I_1 for large b are

$$I_1=c b_c^{-3}+m^2 l^2(96\pi^2)^{-1}(b_c' b_c^{-2})^2 \,. \tag{4.18}$$

A third term was shown to be bounded by

$$m l 6^{1/2}(8\pi)^{-1} b_c^{-3} b_c'(b-b_c) \,.$$

Thus, to a good approximation (4.17b) is just the equation for a classical matter-dominated Friedmann universe. This makes the solutions (4.17a) ACS's. As mentioned in Sec. IV B 3, our numerical computations show some evidence that the phase-plane oscillations are dissipated by particle production as the Universe expands.

This ends our discussion of the behavior of solutions for $\alpha<0$. We have seen that the initial types of behavior of most, and possibly all, solutions remain the same as for $m=0$. The intermediate types of behavior of solutions which diverge rapidly are for the most part unchanged for $m\neq 0$, but their late-time behavior is unknown. The intermediate and late-time behavior of the solutions (4.17) are somewhat different from (3.11). This is primarily because the solutions in (3.11) go through an infinite number of inflection points where $b'=b''=0$, while nonzero values of I_1 and I_2 keep b' from vanishing for the solutions in (4.17).

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