Gravitational Casimir energy in even dimensions

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A method is described for calculating the one-loop gravitational effective potential (gravitational Casimir energy) in Kaluza-Klein theories with an even number of dimensions. The self-consistency condition that the effective potential vanish at its minimum is used to fix the subtraction point for a renormalization of the graviton wave function. The method avoids the use of a higher-dimensional cosmological constant. The calculation is performed for the background manifold (Minkowski space)⊗(N-sphere), in which case the potential is shown to be real and to possess a single stationary point, which turns out to be a maximum. A higher-dimensional cosmological constant appears to be necessary for global stability of the potential.

Kaluza-Klein theories[1–3] are an interesting attempt to explain the observed gauge symmetries of nature as being a consequence of general relativity applied to a world having more than four dimensions. The extra dimensions are taken to be closed and very small—not much larger than the Planck length.[2] To explain the very small sizes of these extra dimensions it has been suggested that, in analogy with the Casimir effect, quantum fluctuations of the gravitational field might cause the extra dimensions to contract.[4] To test this hypothesis one computes the quantum effective potential of the metric field (the gravitational Casimir energy) on a particular background manifold.[5–7] To obtain a non-Abelian gauge field theory as part of the dimensionally reduced theory requires that the compact internal space have curvature, but when the internal manifold is curved the calculation of this effective potential could previously only be performed in an odd number of dimensions. The purpose of this paper is to describe a method by which this calculation can be performed in an even number of dimensions.

The usual reason offered for why the one-loop gravitational effective potential can only be computed in an odd number of dimensions is that the curvature invariants from which renormalization counterterms can be constructed are all of even order in derivatives, so that one-loop counterterms can only be constructed in an even number of dimensions. In an odd number of dimensions there are therefore no renormalizations that can affect the calculation, while in an even number of dimensions the (unknown) renormalizations destroy the predictive power of the calculation. The calculation of the gravitational Casimir energy to one-loop order is most easily performed using the method of zeta-function regularization,[8,9] and it is useful to see how the difference between even and odd dimensions manifests itself in the zeta-function formalism. Given a general field $\phi(x)$ with action $S[\phi]$ the one-loop effective potential is given by

$$V_{\text{eff}}(\bar{\phi})=\frac{-1}{\Omega_M}\left[S[\bar{\phi}]+i\frac{\hbar}{2}\ln \text{Det}\frac{\delta^2 S}{\delta\phi^2}\right], \tag{1}$$

where the classical field $\bar{\phi}$ is a space-time constant, and Ω_M is the (infinite) volume of Minkowski space. The functional determinant represents a quadratic path integral of the form

$$Z=\int[d\phi]\exp\left[\frac{i\hbar}{2}\int\phi(x)\hat{M}\phi(x)dx\right]=\left|\text{Det}\frac{\hat{M}}{2\pi\mu^2}\right|^{-1/2}, \tag{2}$$

where $\hat{M}$ is the operator $\delta^2 S/\delta\phi^2$ and μ^2 sets the scale of the measure of the path integral,[8] and has the units of a mass squared. The functional determinant is regularized and evaluated by forming the generalized zeta function

$$\zeta(s)=\sum_n \lambda_n{}^{-s}, \tag{3}$$

where λ_n are the eigenvalues of $\hat{M}$. Differentiating this with respect to s and setting $s=0$ then gives[10]

$$\ln\text{Det}\left[\frac{\hat{M}}{2\pi\mu^2}\right]=\left.\frac{-d\zeta}{ds}\right|_{s=0}-\ln(2\pi\mu^2)\zeta(0)\;. \tag{4}$$

At this point the difference between even and odd numbers of dimensions can be seen. In an odd number of dimensions it is found[11,12] that $\zeta(0)=0$, in which case the expression for the functional determinant is independent of the parameter μ^2, as expected. In an even number of dimensions $\zeta(0)$ generally does *not* vanish if there is curvature present, in which case the functional determinant depends upon the unknown μ^2.

A seemingly unrelated problem in previous calculations of the gravitational Casimir energy is that of the four-dimensional cosmological constant. Given a solution to the quantum-corrected equations of motion (i.e., a minimum of the effective potential) the value of the effective potential represents a constant vacuum energy density, which may be interpreted as an effective cosmological constant in four dimensions. But to match observation the four external dimensions are taken to be Poincaré invariant, and the presence of a cosmological constant is inconsistent with this assumption. Another way to say this is that the Minkowski metric is a solution of the four-dimensional components of the quantum-corrected Einstein equations only if this cosmological constant vanishes.[13] One therefore requires of the quantum effective potential the two conditions

$$\left.\frac{dV_{\text{eff}}}{dr}\right|_{r_0}=0 \quad \text{(solves the one-loop equations of motion)}\,, \tag{5a}$$

$$V_{\text{eff}}(r_0)=0 \quad \text{(no observed cosmological constant)}\,. \tag{5b}$$

These are two independent equations in only one unknown, so solving them simultaneously requires the introduction of an additional parameter into the theory. In all previous attempts to solve these equations[13,12,7,6] a cosmological constant has been introduced into the original higher-dimensional action, with the value of this cosmological constant tuned so that Eqs. (5) are satisfied. Given the discussion above about the difficulties encountered in calculating the gravitational Casimir energy in an even number of dimensions it should be possible to solve Eqs. (5) in an even number of dimensions using the unknown mass scale μ as the second parameter, without a higher-dimensional cosmological constant. Since μ sets the scale of the measure of the path integral, changing μ may be viewed as a multiplicative renormalization of the wave function. The condition $V_{\text{eff}}(r_0)=0$ can then be viewed as a renormalization condition that fixes μ. This is similar to what happens in the renormalization of a scalar field theory, where the renormalization conditions on the renormalized one-particle-irreducible n-point functions become conditions on derivatives of the effective potential and μ is the subtraction point.[14]

Of course when the underlying theory is gravitation rather than scalar field theory one must approach any discussion of renormalization with caution, because general relativity is not a renormalizable theory. Still, in these kinds of calculations one may adopt the view that although general relativity is not a consistent quantum theory it is the correct classical limit of the "true" quantum theory of gravity, whatever that may be. One then expects the quantum corrections to this classical limit to be a good reflection of the "true" theory provided that the perturbation expansion parameter is small. In the calculation of the Kaluza-Klein Casimir energy this requires that the size of the internal dimensions be larger than the Planck length.[12] The determination of r_0 and of the subtraction point μ^2 may both be considered reliable when this condition is satisfied.

As an example of the strategy described above the gravitational Casimir energy has been calculated and solutions sought for Eqs. (5) on the background manifold $M^m \otimes S^N$ with an even number of dimensions. The mathematical techniques needed for this calculation are described in Ref. 12. For the gravitational field the one-loop effective potential is given by

$$V_{\text{eff}}(g)=\frac{1}{\Omega_M}\left[S_{\text{E}}(g)+\tfrac{1}{2}\ln \text{Det}\frac{\delta^2(S_{\text{E}}+S_{\text{GF}})}{\delta g_{\mu\nu}\delta g_{\alpha\beta}} - \ln \text{Det}\Delta_{\text{FP}}\right], \tag{6}$$

where $S_E(g)$ is the dimensionally reduced Euclidean continuation of the Einstein-Hilbert action, now *without* the cosmological constant, S_{GF} is a gauge-fixing term, Δ_{FP} is the appropriate Faddeev-Popov ghost matrix for this choice of gauge, and Ω_M is the (infinite) volume of the m-dimensional external space M^m. The evaluation of the functional determinants using the zeta-function method will be described shortly, but at this point we note that because the effective potential must have the units of an m-dimensional energy density the potential can be written as

$$V_{\text{eff}}(r)=\frac{-\Omega_M r^N}{16\pi G_D}\frac{N(N-1)}{r^2}+\frac{F_1+F_2\ln(2\pi\rho^2)}{(2\pi r)^m}\,. \tag{7}$$

Here r is the radius of the N-sphere and ρ is the dimensionless radius $\rho^2=\mu^2 r^2$; Ω_N is the volume of the unit N-sphere, G_D is the $D=m+N$ dimensional Newton constant, and F_1 and F_2 are constants that are to be obtained from the zeta function. The first term is the classical contribution while the second is the one-loop correction. Furthermore, $F_2=-\frac{1}{2}\zeta(0)$, where $\zeta(s)$ is the sum of the zeta functions for the graviton and ghost determinants in Eq. (6) (with appropriate factors of $+\frac{1}{2}$ and -1, respectively) and F_1 is obtained from the derivative of $\zeta(s)$. The constants F_1 and F_2 are expected to be real, because without a higher-dimensional cosmological constant there are no "tachyonic" negative eigenvalues in the zeta function. (Negative eigenvalues were found to cause the effective potential calculated in an odd number of dimensions to have an imaginary part.[7])

Using Eq. (7) and imposing condition (5b) gives

$$\frac{{r_0}^{N+m-2}}{G_D}=\frac{16\pi}{(2\pi)^m\Omega_N}\frac{F_1+F_2\ln(2\pi{\rho_0}^2)}{N(N-1)}\,, \tag{8}$$

where ${\rho_0}^2=\mu^2{r_0}^2$. Condition (5a) becomes

$$0=(N+m-2)[F_1+F_2\ln(2\pi{\rho_0}^2)-2F_2]\,. \tag{9}$$

This equation determines ${\rho_0}^2$ in terms of the constants F_1 and F_2. Using this in Eq. (8) then gives

$$\frac{{r_0}^{N+m-2}}{G_D}=\frac{16\pi}{(2\pi)^m\Omega_N}\frac{2F_2}{N(N-1)(N+m-2)}\,. \tag{10}$$

Thus r_0 is determined entirely by the constant $F_2=-\frac{1}{2}\zeta(0)$. This expression only gives a positive r_0 if F_2 is positive. Assuming $F_2>0$ the asymptotic behavior of the effective potential is then known: as r gets large the quantum term vanishes as $(\ln r)/r^m$ and the classical term dominates, making the potential negative; while when r becomes small the classical term vanishes as r^{N-2} and the quantum term becomes negative because $\ln\rho<0$. In short,

$$V_{\text{eff}}(r)\to-\infty \quad \text{as } r\to\infty \quad \text{(classical term)}\,, \tag{11a}$$

$$V_{\text{eff}}(r)\to-\infty \quad \text{as } r\to 0 \quad \text{(quantum term)}\,. \tag{11b}$$

Thus we have a real potential with a single, uniquely determined stationary point, which unfortunately is a maximum instead of a minimum.

Even though a maximum of the potential is expected it is still of interest to evaluate the constant F_2, both to determine when $F_2>0$ and to see how strong the Casimir "force" is. The zeta function for both of the functional determinants in Eq. (6) can be written as the sums of several zeta functions of the form[12]

$$\zeta_i(s)=(-1)^s\kappa_i\int\frac{d^mk}{(2\pi)^m}\sum_{l=l_0}^{\infty}d_i(l)\left[\frac{l^2+l(N-1)+c_i}{r^2}+k^2\right]^{-s} . \tag{12}$$

The sum on l is a sum over the orders of the spherical harmonics on S^N, with $d_i(l)$ an appropriate degeneracy factor, while the integral over k^2 is the sum over the continuous eigenvalue spectrum of the Laplacian on the external Minkowski space; κ_i is an overall degeneracy factor, while c_i is some given constant for a given set of modes (scalars, vectors, or tensors) on S^N. This zeta function may be analytically continued to the contour integral

$$\zeta_i(s)=\frac{\kappa_i}{(2\pi)^m}\frac{(-r^2)^s}{\Gamma(s)}\frac{1}{1+e^{\pm i\pi p}}\int_{-\infty+i\Delta}^{+\infty+i\Delta}\frac{\sinh(\alpha_i z)}{[2\sinh(z/2)]^{N+1}}\left[\frac{z}{2\beta}\right]^{\nu}I_\nu(\beta z)dz , \tag{13}$$

where $p=2s-(m+N+1)$ and $\nu=2s-(m+1)/2$, and α_i is either 1 or 2 depending upon the mode under consideration. The difference between even and odd dimensions is contained in the factor $(1+e^{\pm i\pi p})^{-1}$. If $N+m$ is odd then $1+e^{\pm i\pi p}=2$ and the zeta function can be written as

$$\zeta_i(s)=\frac{\tilde{\zeta}_i(s)}{\Gamma(s)} , \tag{14}$$

where $\tilde{\zeta}_i(s)$ is a well-behaved function of s. Then not only is $\zeta_i(0)=0$, but the derivative at $s=0$ is $\tilde{\zeta}_i(0)$. For $N+m$ even $(1+e^{\pm i\pi p})\approx\mp s$ as $s\to0$, while $\Gamma(s)\approx 1/s$ in this limit, so $\zeta_i(0)$ does not necessarily vanish.

The evaluation of $\zeta_i(0)$ for even N follows closely the evaluation of $\tilde{\zeta}_i(0)$ for N odd described in Ref. 7. Taking $s\to0$ in Eq. (13) gives, in the notation of Ref. 7,

$$\zeta_i(0)=\frac{\kappa_i}{(2\pi r)^m}\frac{-1}{2\pi i}\frac{1}{4}\{3[C_5(\beta_i+\alpha_i)-C_5(\beta_i-\alpha_i)]-3\beta_i[C_4(\beta_i+\alpha_i)-C_4(\beta_i-\alpha_i)]+\beta_i{}^2[C_3(\beta_i+\alpha_i)-C_3(\beta_i-\alpha_i)]\} . \tag{15}$$

For N even or odd the functions $C_p(\gamma)$ are given by

$$C_p(\gamma)=-2\pi i\sum_{j=p}^{(N+p)}\sum_{k=0}^{(N+p-j)}a_{j,k}(-\gamma)^k\left[[1-(-1)^N]\mathrm{Le}_j(\gamma)+\frac{1}{j!}B_j(\gamma)\right] . \tag{16}$$

Note that for N even this expression is a polynomial in γ, independent of the polylogarithmic functions $\mathrm{Le}_j(\gamma)$.

Using Eq. (16) in Eq. (15) the values of $F_2=-\frac{1}{2}\zeta(0)$ were evaluated numerically for $N=2,4,\ldots,20$ and the results are listed in Table I. The condition $F_2>0$ is satisfied only for $N \bmod 4=2$. Similar behavior was observed[7] in the odd-dimensional calculation of the corresponding constant $F(0)$, and I conjecture that the sign of F_2 is $-(-1)^{N/2}$ for all even N. In all cases F_2 is found to be real, as expected. For $F_2>0$ the values of the distance around the internal manifold, $L_{KK}=2\pi r_0$, are also tabulated in Table I. In all cases L_{KK} is above the Planck length, so the one-loop approximation can be trusted. For the purpose of comparison with previous calculations the values of the gauge coupling constants corresponding to these values of r_0 are also listed in the table.

The example above shows that it is indeed possible to find a unique stationary point of the gravitational effective action in an even number of dimensions, although in this particular case the solution is unstable. The behavior of the solution for large r described in Eqs. (11) also demonstrates that it should in fact be quite difficult to get a globally stable quantum effective potential for the graviton on any background manifold $M^m \otimes B^N$ without a higher-dimensional cosmological constant. In general one

TABLE I. Values of the constant $F_2=-\frac{1}{2}\zeta(0)$ for various values of N. When F_2 is positive the distance around the internal manifold and the resulting inverse gauge coupling constant are also given.

N	F_2	$L_{KK}=2\pi r_0$	α^{-1}
2	$+15.94$	1.998	0.002 14
4	-2.156×10^3		
6	$+6.242\times10^4$	5.772	1.20
8	-1.074×10^6		
10	$+1.453\times10^7$	8.577	39.4
12	-1.690×10^8		
14	$+1.775\times10^9$	10.91	1.31×10^3
16	-1.732×10^{10}		
18	$+1.600\times10^{11}$	12.95	4.44×10^4
20	-1.414×10^{12}		

expects (as is demonstrated here) that the classical part of the potential will dominate for large r, with the quantum fluctuations becoming significant only when r approaches the Planck length. (Here r represents some characteristic length scale of the compact internal manifold B^N.) The classical dimensionally reduced action goes like $S_E \approx -r^N R/G_D$ with $R \approx 1/r^2$, so the effective potential behaves like $V_{\rm eff}(r) \approx -r^{N-2}/G_D$ for large r and is unbounded below. This defect may be cured by adding a cosmological constant Λ to the action, so that R is replaced by $R-2\Lambda$. Then for $\Lambda > 0$ the leading behavior of the potential for large r is $V_{\rm eff}(r) \approx r^N \Lambda/G_D$, which favors r becoming small. But this takes the problem back to the uncomfortable situation of requiring that the four-dimensional cosmological constant vanish while at the same time requiring that the higher-dimensional cosmological constant *not* vanish. Furthermore, in an even number of dimensions there do not appear to be other conditions one could impose to uniquely determine both Λ and μ^2 along with r_0, although perhaps some renormalization condition on a higher derivative of the effective potential could be found, as is done for scalar fields.

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