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c-boundary of Taub's plane-symmetric static vacuum spacetime

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To describe singularities using the notion of c-boundary, it is necessary to divide the c-boundary into two portions, one of them represents the singularity while the other represents infinity. By investigating the global structure of Taub's plane-symmetric static spacetime in detail we prove that the singular portion of the c-boundary of this spacetime is a single point. This suggests that it might not be fruitful describing the structure of singularities using the c-boundary construction. The possible difficulty in singling out the "singular portion" of the c-boundary is also discussed.

I. INTRODUCTION

A lot of spacetimes are known to be singular within the framework of classical general relativity. To describe the singularity better, one wants to make precise the meaning of some relevant notions such as the location, shape, size, and the vicinity of a singularity. However, it has proved rather difficult to do so since the "singular points" do not belong to the spacetime manifold M. An intriguing idea is to attach some sort of boundary ∂ to M to obtain a topological space $\overline{M}$, the boundary points being interpreted as singular points of the singular spacetime. The construction of ∂ should be determined by the geometrical structure of the spacetime itself. At least three boundary constructions along this line (or with certain modifications) have been proposed: the g-boundary,[1] the b-boundary,[2] and the c-boundary.[3–6] It has been known since 1977 that the b-boundary construction gives an unacceptable topology even for the extended Schwarzschild solution.[7] Although the g-boundary does not suffer from a difficulty of this sort, it does yield an unphysical topology in an example pointed out by Geroch, Liang, and Wald.[8] The example is as follows in short. In two-dimensional Minkowski spacetime $(N=R^2,\eta_{ab})$ let $s\in N$ and let r lie on a future-directed null geodesic from s. Construct a four-dimensional spacetime (M,g_{ab}) by taking the cross product of $(N-s,\Omega^2\eta_{ab})$ and the flat spacelike plane, where Ω is a conformal factor. It was shown that by choosing Ω suitably one can make the singular boundary point s fail to be T_1 related to the regular point r, thus obtaining an unacceptable topology for $\overline{M}$. Moreover, this example is valid for any singular boundary construction, provided that it possesses the following two properties: (i) Every incomplete geodesic in the original spacetime terminates at a singular point of $\overline{M}$; (ii) the resulting space $\overline{M}$ is geodesically continuous, in a sense made precise in Ref. 8. The g- and b-boundary constructions both possess these two properties.

Assuming the reader is familiar with the indecomposable past (indecomposable future) [IP (IF)] approach, we here give a brief review of the c-boundary construction.[3,9] Let (M,g_{ab}) be a spacetime that is at least past and future distinguishing. Denote by $\hat{M}$ the collection of all IP's of M, and $\check{M}$ the collection of all IF's. Introduce an intermediate space $M^{\#}$ by taking the union $\hat{M}\cup\check{M}$ and identifying $I^+(p)\in\check{M}$ with $I^-(p)\in\hat{M}$ for each $p\in M$. The element of $M^{\#}$ corresponding to an element P of $\hat{M}$ or $\check{M}$ is written as P^*. Thus, if $p\in M$, then $I^+(p)^* =I^-(p)^*\in M^{\#}$. The open sets of $M^{\#}$ are defined to be the unions and finite intersections of subsets of the form A^{int}, A^{ext}, B^{int}, and B^{ext}, where

$$A^{\text{int}}=\{P^*\mid P\in\hat{M} \text{ and } P\cap A\neq\varnothing\}\ ,$$

$$A^{\text{ext}}=\{P^*\mid P\in\hat{M} \text{ and } P=I^-(S)\Longrightarrow I^+(S)\not\subset A \text{ for all } S\subset M\}\ ,$$

B^{int} and B^{ext} being defined similarly, with the roles of past and future interchanged. The topological space $M^{\#}$ so defined is not, in general, a Hausdorff space. To obtain a Hausdorff space $\overline{M}$, we might have to identify certain points of $M^{\#}$. More precisely, $\overline{M}$ is defined as the quotient space $M^{\#}/R_H$, where R_H is the intersection of all equivalence relations $R\subset M^{\#}\times M^{\#}$ such that $M^{\#}/R$ is a Hausdorff space. It can be shown that there exists a natural, dense topological embedding of M in $\overline{M}$; hence $\overline{M}=M\cup\partial$, where ∂ is the collection of all ideal points (after necessary identification) of (M,g_{ab}), and is referred to as the c-boundary of the original spacetime (M,g_{ab}).[9]

An important difference between the c and g-boundary (or b-boundary) is that the c-boundary represents not only singular points (if any) but also "points at infinity." Therefore, in order to describe singularities it is essential to write $\partial=\partial_s\cup\partial_i$, where ∂_s and ∂_i represent, respectively, the singular portion and the nonsingular infinity portion of the c-boundary. Penrose suggested[4-6] a number of slightly different criteria to distinguish the singular ideal points from the infinity ones, the simplest is as follows. An ideal point [a terminal indecomposable past (TIP) or a TIF] is said to be at infinity (an ∞-TIP or ∞-TIF) if there exists a semi-infinite causal curve of infinite proper length which has that ideal point as its ideal end point; otherwise the ideal point is said to be a finite one. In the case that we have a spacetime which is maximally extended, all finite ideal points may be reasonably interpreted as singular points (singular TIP's or TIF's) of the spacetime. One might also consider some of the points at infinity to be singular. We shall get back to this issue later.

It is worth noting that the c-boundary construction does not admit property (ii) mentioned in Ref. 8, since, as pointed out by Geroch, c-boundary is purely conformal while geodesic continuity is very nonconformally invariant. Hence, the conclusion of the example in Ref. 8 does not necessarily hold. In this example there are simply two singular ideal points in $M^{\#}$, namely, s as a TIP and s as a TIF. Therefore, the only identification in passing from $M^{\#}$ to $\overline{M}$ is the identification of these two ideal points. The $\overline{M}$ thus obtained must be a Hausdorff space, as can be inferred from the following assertion on p. 564 of Ref. 3: "...the space $M^{\#}/R$ is Hausdorff, where R is the equivalence relation which simultaneously identifies all the elements of $M^{\#}$ not in $\phi(M)$."

Despite the fact that the c-boundary survives the example of Ref. 8, we verify in Sec. II that the singular portion ∂_s of the c-boundary of Taub's plane-symmetric static vacuum spacetime is, in fact, a single point of $\overline{M}$. This suggests that there might be some difficulty in describing the structure of singularities using the notion of c-boundary.

II. c-BOUNDARY OF TAUB'S STATIC SPACETIME

The line element of Taub's spacetime reads[10]

$$dS^2=z^{-1/2}(-dt^2+dz^2)+z(dx^2+dy^2),\quad z>0\ . \tag{1}$$

The spacetime is timelike but not null geodesically complete, and the singularity $z=0$ can be reached only by null geodesics confined in t-x planes. This is analogous to Reissner-Nordström spacetime. The singularity $z=0$ in Taub spacetime is also a scalar-polynomial curvature singularity,[9] since calculation shows that $R_{abcd}R^{abcd}\to\infty$ as $z\to 0$.

Taub's spacetime is obviously stably causal and hence the "TIP approach" applies. To figure out the c-boundary it is essential to categorize all TIP's and TIF's. It follows from (1) that $d(t+z)d(t-z)>0$ for any timelike curve; hence, $(t+z)$ and $(t-z)$ are increasing along its future direction. Therefore, a future inextendible timelike curve γ must fall into one of the following types: (i) $(t+z)$ and $(t-z)$ are both bounded along the future direction of γ; it then follows that γ approaches the singularity $z=0$; (ii) $(t+z)$ and $(t-z)$ are both unbounded along the future direction of γ; (iii) $(t-z)$ is bounded while $(t+z)$ is unbounded, it follows that $(t-z)$ approaches some limit while $(t+z)$ goes to infinity.

Lemma 1. Given two points $p_1=(t_1,z_1,x_1,y_1)$ and $p_2=(t_2,z_2,x_2,y_2)$ in Taub's spacetime manifold M, then $t_1-z_1\geq t_2+z_2$ implies $p_2\in I^-(p_1)$.

Proof. Consider two points $p_1'=(t_1',z',x_1,y_1)$ and $p_2'=(t_2',z',x_2,y_2)$ satisfying $t_1'-z'=t_1-z_1,\ t_2'+z'=t_2+z_2$ and $z'<z_1, z'<z_2$, then $p_1'\in J^-(p_1), p_2\in J^-(p_2')$. It follows from (1) that the coordinate straight line connecting p_1' to p_2' can be made timelike by choosing z to be sufficiently small. Thus $p_2'\in I^-(p_1')$ and $p_2\in I^-(p_1)$.

Lemma 2. If γ is a future inextendible timelike curve with $z\to 0$ and t approaching a constant c along the future direction, then $I^-(\gamma)=\mathscr{D}_c$, where

$$\mathscr{D}_c=\{(t,z,x,y)\,|\,t+z<c\}\ .$$

Proof. Since $(t+z)$ is increasing along any future-directed timelike curve, it is clear that $I^-(\gamma)\subset\mathscr{D}_c$. To show the inverse, let $p\in\mathscr{D}_c$. Since $(t+z)_p<c$ and $(t-z)_\gamma\to c$, one can choose $q\in\gamma$ such that $(t-z)_q>(t+z)_p$, and hence $p\in I^-(q)\subset I^-(\gamma)$ by virtue of lemma 1.

Note that these two lemmas are also valid for Reissner-Nordström spacetime with the role of (t,z,x,y) replaced by (t,r,θ,ϕ).

All TIP's in Taub spacetime are ∞-TIP's since each of them [including those of type (i)] can be written as $I^-(\gamma)$ for some timelike curve γ of infinite proper time. To single out those TIP's which represent the singularity (those ∞-TIP's which are also singular), we make use of an alternative criterion given by Penrose:[4] A TIP is called null-finite if it is of the form $I^-(\mu)$, where μ is a null geodesic of finite affine length; otherwise it is called null-infinite. Each null-finite TIP may be thought of as defining a sort of singular ideal point. All TIP's of Taub spacetime generated by curves of type (i) represent singular ideal points for they are null-finite. All curves of type (ii) generate a single TIP—the whole spacetime manifold M. This is null-infinite yet is relevant to the singularity in some sense since it is generated by a timelike curve defined by $t=1/z$, x=const, y=const, and the curvature scalar $R_{abcd}R^{abcd}$ is unbounded along γ. Thus, if the criterion is widened a little such that a TIP is said to be singular if it is null-finite or if any curvature scalar is unbounded along some curve generating it (see p. 225 of Ref. 4), then all TIP's of types (i) and (ii) are singular. We shall adopt this criterion in this paper. On the other hand, TIP's of type (iii) are null-infinite, they are irrelevant to the singularity in any sense, and hence represent nonsingular, infinity ideal points.

Because of the static property, statements dual to lemmas 1 and 2 can be obtained by simply interchanging the roles of past and future, and TIF's can be categorized similarly. In the light of lemma 2 and its dual, the set consisting of all singular TIP's (TIF's) of type (i) is "one-dimensional" in the sense that each element of it is characterized by a real number c. Further, this set is timelike according to Penrose's definition,[4] since for any TIP of type (i), denoted W, there exists a PIP (proper IP), denoted V, contained in some TIP of type (i) such that $W\subset V$. It turns out that the set consisting of all nonsingular TIP's (TIF's) is a "three-dimensional" null set.

Introducing retarded Eddington-type coordinates

$u=t-z$ and $Z=1/\sqrt{z}$, one obtains a conformal metric

$$d\tilde{S}^2=\Omega^2 dS^2=-Z^3du^2+4\,du\,dZ+dx^2+dy^2$$

with $\Omega=Z^{-1}$. Define

$$\mathscr{I}^+=\{(u,Z,x,y)\,|\,Z=0,-\infty<u<\infty\ ,\ -\infty<x<\infty,-\infty<y<\infty\}$$

and $M'=M\cup\mathscr{I}^+$, since the conformal metric $\tilde{g}_{ab}$ is nonsingular at $\mathscr{I}^+$, one obtains an unphysical spacetime $(M',\tilde{g}_{ab})$. It is not difficult to show that any past-directed timelike curve having $p\in M$ as its future end-point never intersects $\mathscr{I}^+$; hence $I^-(p,M)=I^-(p,M')$ and $I^-(S,M)=I^-(S,M')$ for any $S\subset M$. We shall denote $I^-(S,M)$ simply by $I^-(S)$. If $\gamma\subset M$ is a curve of type (iii), i.e., if $v\,|_\gamma\to\infty$ and $u\,|_\gamma\to c$, then a simple calculation using the Schwartz inequality shows that $x\,|_\gamma$ and $y\,|_\gamma$ must approach some limits a and b, respectively; hence

$$I^-(\gamma)=I^-(e,M')=I^-(\eta_-^{c,a,b})\ , \tag{2}$$

where $e\in\mathscr{I}^+$ is the point with

$$(u,Z,x,y)_e=(c,0,a,b)\ ,$$

and $\eta_-^{c,a,b}$ denotes the null geodesic defined by $t-z=c$, $x=a$, $y=b$.

Analogously, in advanced Eddington-type coordinates $v=t+z$ one obtains an unphysical spacetime $(M'',\tilde{g}_{ab})$ where $M''=M\cup\mathscr{I}^-$, with

$$\mathscr{I}^-=\{(v,Z,x,y)\,|\,Z=0,-\infty<v<\infty\ ,\ -\infty<x<\infty,-\infty<y<\infty\}\ .$$

Since $I^+(S,M)=I^+(S,M'')$ for any $S\subset M$, we shall denote $I^+(S,M)$ by $I^+(S)$. For a past inextendible timelike curve $\gamma\subset M$ of type (iii)' [the dual to type (iii)], i.e., if $u\,|_\gamma\to-\infty, v\,|_\gamma\to c$, one has

$$I^+(\gamma)=I^+(f,M'')=I^+(\eta_+^{c,a,b})\ , \tag{3}$$

where $f\in\mathscr{I}^-$ is the point with $v=c, Z=0, x=a\equiv\lim_{v\to c}x\,|_\gamma, y=b\equiv\lim_{v\to c}y\,|_\gamma$, and $\eta_+^{c,a,b}$ denotes the null geodesic defined by $t+z=c, x=a, y=b$. Further, lemma 2 implies that $I^-(\eta_+^{c,a,b})$ and $I^+(\eta_-^{c,a,b})$ are independent of a and b; hence, we shall denote them simply by $I^-(\eta_+^c)$ and $I^+(\eta_-^c)$.

Expressions (2) and (3) imply that TIP's (TIF's) of type (iii) are in one-to-one correspondence with points at $\mathscr{I}^+(\mathscr{I}^-)$. The set $\mathscr{I}^+(\mathscr{I}^-)$, therefore, is a representation of the set of TIP's (TIF's) of type (iii), and it is "three-dimensional" in the sense that each element is characterized by a triple of real numbers. Furthermore, it is easily checked that this set is a null set according to Penrose's definition.[4]

To sum up, the set $\hat{M}$ of Taub spacetime can be divided into four disjoint subsets:

$$\hat{M}=\hat{M}_1\cup\hat{M}_3\cup\hat{W}\cup\hat{M}_r\ ,$$

where $\hat{M}_1$ represents the set of TIP's of type (i), $\hat{M}_3$ the set of TIP's of type (iii), $\hat{W}$ the set consisting of a single TIP—the TIP generated by any curve of type (ii)—which is equal to the whole spacetime manifold M, and $\hat{M}_r$ the set of all PIP's. Similarly, $\check{M}=\check{M}_1\cup\check{M}_3\cup\check{W}\cup\check{M}_r$, where each subset has a dual meaning to its counterpart.

The intermediate space $M^\#$ of Taub spacetime with topology defined by Ref. 3 is non-Hausdorff. We next verify that in order to get a Hausdorff space $\overline{M}$ all singular ideal points (points of $\hat{M}_1$, $\check{M}_1$, $\hat{W}$, and $\check{W}$) have to be identified.

Lemma 3. Any open set of $M^\#$ containing $I^+(\eta_-^c)^*\in M^\#$ contains

$$\mathscr{O}_1=I^-(\eta_+^{c+\alpha})^{\text{int}}\cap I^-(\eta_-^{c-\alpha,a_1,b_1})^{\text{ext}}\cap\cdots\cap I^-(\eta_-^{c-\alpha,a_k,b_k})^{\text{ext}}\cap I^-(\eta_+^c)^{\text{ext}}\ , \tag{4}$$

while any open set containing $I^-(\eta_+^{\tilde{c}})^*\in M^\#$ contains

$$\mathscr{O}_2=I^+(\eta_-^{\tilde{c}-\tilde{\alpha}})^{\text{int}}\cap I^+(\eta_+^{\tilde{c}+\tilde{\alpha},\tilde{a}_1,\tilde{b}_1})^{\text{ext}}\cap\cdots\cap I^+(\eta_+^{\tilde{c}+\tilde{\alpha},\tilde{a}_l,\tilde{b}_l})^{\text{ext}}\cap I^+(\eta_-^{\tilde{c}})^{\text{ext}}\ , \tag{5}$$

where $\alpha>0$; $a_1,\ldots,a_k;b_1,\ldots,b_k$; $\tilde{\alpha}>0$; $\tilde{a}_1,\ldots,\tilde{a}_l$; $\tilde{b}_1,\ldots,\tilde{b}_l$ are some real numbers.

Proof. Any open set of $M^\#$ containing $I^+(\eta_-^c)^*$ contains

$$\mathscr{O}_1=I^-(\gamma_1)^{\text{int}}\cap\cdots\cap I^-(\gamma_m)^{\text{int}}\cap I^-(\gamma_{m+1})^{\text{ext}}\cap\cdots\cap I^-(\gamma_{m+n})^{\text{ext}}\ ,$$

where γ_i $(i=1,\ldots,m)$ and γ_j $(j=m+1,\ldots,m+n)$ are future-directed timelike curves satisfying $I^+(\eta_-^c)^*\in I^-(\gamma_i)^{\text{int}}$ and $I^+(\eta_-^c)^*\in I^-(\gamma_j)^{\text{ext}}$. A straightforward argument using lemmas 1 and 2 then shows that $\mathscr{O}_1$ can be sharpened by writing $I^-(\eta_+^{c'})^{\text{int}}$ instead of

$$I^-(\gamma_1)^{\text{int}}\cap\cdots\cap I^-(\gamma_m)^{\text{int}}\ ,$$

where $c'>c$ is some real number; and writing $I^-(\eta_-^{c_j,a_j,b_j})^{\text{ext}}$ instead of $I^-(\gamma_j)^{\text{ext}}$ if γ_j is of type (iii); and writing $I^-(\eta_+^c)^{\text{ext}}$ instead of $I^-(\gamma_j)^{\text{ext}}$ if γ_j is of type (i). Setting c'' to be

$$\max\{c_{m+1},\ldots,c_{m+n}\}$$

and α to be

$$\min\{(c'-c),(c-c'')\}$$

one finally obtains (4). Expression (5) can be shown similarly.

Lemma 4. Let (M,g_{ab}) be a past and future distinguishing spacetime, $p\in M, B\subset\hat{M}$ and $A\subset\check{M}$, then $I^+(p)^*\in B^{\text{ext}}$ if and only if $p\notin\overline{B}$, $I^-(p)^*\in A^{\text{ext}}$ if and only if $p\notin\overline{A}$.

Proof. Suppose $I^+(p)^*\notin B^{\text{ext}}$; then there exists $S\subset M$ with $I^+(S)=I^+(p)$ and $I^-(S)\subset B$. The first together with the future distinguishing condition[9] require $p\in\overline{S}$; hence $p\in\overline{I^-(p)}\subset\overline{I^-(S)}\subset\overline{B}$. To show the inverse, it suffices to note that $p\in\overline{B}$ implies $I^-(p)\subset B$, contradicting $I^+(p)^*\in B^{\text{ext}}$. The dual statment can be similarly verified.

Lemma 5. $I^+(\eta_-^c)^*\in M^\#$ and $I^-(\eta_+^{\tilde{c}})^*\in M^\#$ are non-T_2 related in $M^\#$ provided that $\tilde{c}<c$.

Proof. It suffices to show that any open sets $\mathscr{O}_1$ and $\mathscr{O}_2$

of the form (4) and (5) with $\tilde{c} < c$ must intersect each other. Without loss of generality, we set $\tilde{\alpha}=\alpha$ in (4) and (5). Define

$$\mathscr{R}=\{(t,z,x,y)\,|\,\tilde{c}-\alpha<t-z<\tilde{c}<c<t+z<c+\alpha\}\ .$$

Consider a point $e\in\mathscr{I}^+$ with $u_e=c-\alpha$ and a point $p\in\mathscr{R}\cap I^-(e,M)$, let γ be a timelike curve connecting p to e, define

$$r_p^{(i)}=[(x_p-a_i)^2+(y_p-b_i)^2]^{1/2}\ ,$$

where a_i,b_i $(i=1,\ldots,k)$ are those in (4). Then a simple calculation using the Schwartz inequality shows

$$r_p^{(i)}<\left[(u_p-c+\alpha)\left[\int_{c-\alpha}^{u_p}Z^3du-4Z_p\right]\right]^{1/2}\ .$$

Since $Z<\sqrt{2/\alpha}$ along γ, there exists a real number ρ depending only upon c, $\tilde{c}$, and α such that $r_p^{(i)}\le\rho$ for any $p\in\mathscr{R}$. Similarly, there exists $\tilde{\rho}$ depending only upon c, $\tilde{c}$, and α such that

$$\tilde{r}_p^{(j)}\equiv[(x_p-\tilde{a}_j)^2+(y_p-\tilde{b}_j)^2]^{1/2}\le\tilde{\rho}\ ,$$

where $\tilde{a}_j$ and $\tilde{b}_j$ $(j=1,\ldots,l)$ are those in (5). Choose $q\in\mathscr{R}$ with $r_q^{(i)}>\rho$ $(i=1,\ldots,k)$ and $\tilde{r}_q^{\,j}>\tilde{\rho}$ $(j=1,\ldots,l)$, then

$$q\notin\overline{I^-(\eta_-^{c-\alpha,a_i,b_i})}\,(i=1,\ldots,k)$$

and

$$q\notin\overline{I^+(\eta_+^{\tilde{c}+\alpha,\tilde{a}_j,\tilde{b}_j})}\,(j=1,\ldots,l)\ .$$

This, on account of lemma 4, establishes $I^+(q)^*=I^-(q)^*\in\mathscr{O}_1\cap\mathscr{O}_2$.

Divide $M^\#$ into three disjoint subsets: $M^\#=M_S^\#\cap M_i^\#\cup M_r^\#$, where $M_S^\#=\hat{M}_1\cup\check{M}_1\cup\hat{W}\cup\check{W}$ represents the set of singular ideal points, while $M_i^\#=\hat{M}_3\cup\check{M}_3$ represents the set of nonsingular infinity points, and $M_r^\#$ represents the image of $\hat{M}_r\cup\check{M}_r$ under the identification mapping $I^+(p)^*=I^-(p)^*$ for any $p\in M$, then an argument using lemma 5 shows that the subset $\psi(M_S^\#)\subset\overline{M}$ is a single point of $\overline{M}$ (where ψ denotes the projection from $M^\#$ to $\overline{M}$). We next prove that the only identification one has to make in passing from $M^\#$ to $\overline{M}$ is to identify all elements of $M_S^\#$. On account of the statement of Ref. 3 quoted near the end of Sec. I, the nontrivial task is to show the following lemma.

Lemma 6. For any $P_1^*=I^-(\eta_-^{c_1,a_1,b_1})^*\in\check{M}_3$ and $P_2^*\in M_S^\#\cup\check{M}_3\cup\hat{M}_3$, there exist open sets $\mathscr{O}_1,\mathscr{O}_2\subset M^\#$ such that (i) $P_1^*\in\mathscr{O}_1$, $P_2^*\in\mathscr{O}_2$, $\mathscr{O}_1\cap\mathscr{O}_2=\phi$, (ii) $\mathscr{O}_1\cap M_S^\#=\phi$; $\mathscr{O}_2\cap M_S^\#=\phi$ or $M_S^\#\subset\mathscr{O}_2$.

Proof. Choose a point $p_1=(t_1,z_1,a_1,b_1)\in M$ and a number h_1 with $t_1-z_1<c_1<c_1+h_1<t_1+z_1$, define $\mathscr{O}_1=I^+(p_1)^{\rm int}\cap I^+(\eta_-^{c_1+h_1})^{\rm ext}$, then $I^-(\eta_-^{c_1,a_1,b_1})^*\in\mathscr{O}_1$ and $\mathscr{O}_1\cap M_S^\#=\phi$. The choice of $\mathscr{O}_2$, however, depends on the P_2^* given.

Case A. $P_2^*\in M_S^\#$. Define

$$\mathscr{O}_2=I^+(\eta_+^{t_1+z_1,a_1,b_1})^{\rm ext}\cup I^+(\eta_-^{c_1+h_1})^{\rm int}$$
$$\cup I^-(\eta_+^{t_1+z_1})^{\rm int}\bigcup_{\tilde{c}\ge t_1+z_1}\mathscr{O}_{2,\tilde{c}}\ ,$$

with

$$\mathscr{O}_{2,\tilde{c}}=I^-(\eta_+^{\tilde{c}+\alpha})^{\rm int}\cap I^-(\eta_-^{c_1+h_1,a_1',b_1'})^{\rm ext}$$
$$\cap\cdots\cap I^-(\eta_-^{c_1+h_1,a_k',b_k'})^{\rm ext}\ ,$$

where α is any positive number, while a_i', b_i' $(i=1,\ldots,k)$ are so chosen that

$$I^+(p_1)\cap\mathscr{E}\cap\mathscr{D}\subset\bigcup_{1\le i\le k}I^-(e_i,M')\ ,$$

where

$$\mathscr{D}=\{(t,z,x,y)\in M\,|\,t-z<c_1+h_1\}\ ,$$
$$\mathscr{E}=\{(t,z,x,y)\in M\,|\,z<(\tilde{c}+\alpha-t_1+z_1)/2\}\ ,$$

and $e_i\in\mathscr{I}^+$ with $u_i=c_1+h_1$, $x_i=a_i'$, $y_i=b_i'$ (this can always be done). It is easily seen that $M_S^\#\subset\mathscr{O}_2$. A longer argument also shows $\mathscr{O}_1\cap\mathscr{O}_2=\phi$.

Case B. $P_2^*=I^+(\eta_+^{c_2,a_2,b_2})^*\in\check{M}_3$. Choose a point

$$p_2=(t_2,z_2,a_2,b_2)\in M$$

and a number h_2 with

$$t_2-z_2<c_2-h_2<c_2<t_2+z_2<c_1\ ,$$

then $\mathscr{O}_2=I^-(p_2)^{\rm int}\cap I^-(\eta_+^{c_2-h_2})^{\rm ext}$ is the desired open set.

Case C. $P_2^*=I^-(\eta_-^{c_2,a_2,b_2})^*\in\hat{M}_3$. Choose a point $p_2=(t_2,z_2,a_2,b_2)\in M$ and a number h_2 with

$$t_2-z_2<c_2<c_2+h_2<t_2+z_2\ .$$

Define

$$\mathscr{O}_2=I^+(p_2)^{\rm int}\cap I^+(\eta_-^{c_2+h_2})^{\rm ext}\ ,$$

then $I^-(\eta_-^{c_2,a_2,b_2})^*\in\mathscr{O}_2$ and $\mathscr{O}_2\cap M_S^\#=\phi$. If $c_1\ne c_2$ ($c_2>c_1$, say), it is easy to check that $\mathscr{O}_1\cap\mathscr{O}_2=\phi$ provided that p_2 is so chosen that $c_1+h_1<t_2-z_2$. For the case $c_1=c_2$, a longer argument proves that $\mathscr{O}_1\cap\mathscr{O}_2=\phi$ provided that z_1 and z_2 are sufficiently small.

Combining the previous results with lemma 6 and its dual statement we finally obtain the following conclusion.

Conclusion. The c-boundary of Taub's plane-symmetric static spacetime consists of a single point $\psi(M_S^\#)$ that represents the singularity, and two "three-dimensional" portions $\psi(\hat{M}_3)$ and $\psi(\check{M}_3)$ that represent infinity.

It is worth noting that the c-boundary of the plane-symmetric solution to Einstein-Maxwell equations (the Kar[11] or McVittie[12] solution) bears a similar property as Taub's. The line element of Kar's solution is

$$dS^2=-\left[\frac{m}{z}+\frac{e^2}{z^2}\right]dt^2+\left[\frac{m}{z}+\frac{e^2}{z^2}\right]^{-1}dz^2$$
$$+z^2(dx^2+dy^2)\ ,$$

where m and e are two parameters. If $m>0$ and $z>0$, there is a singularity at $z=0$ which, by an argument

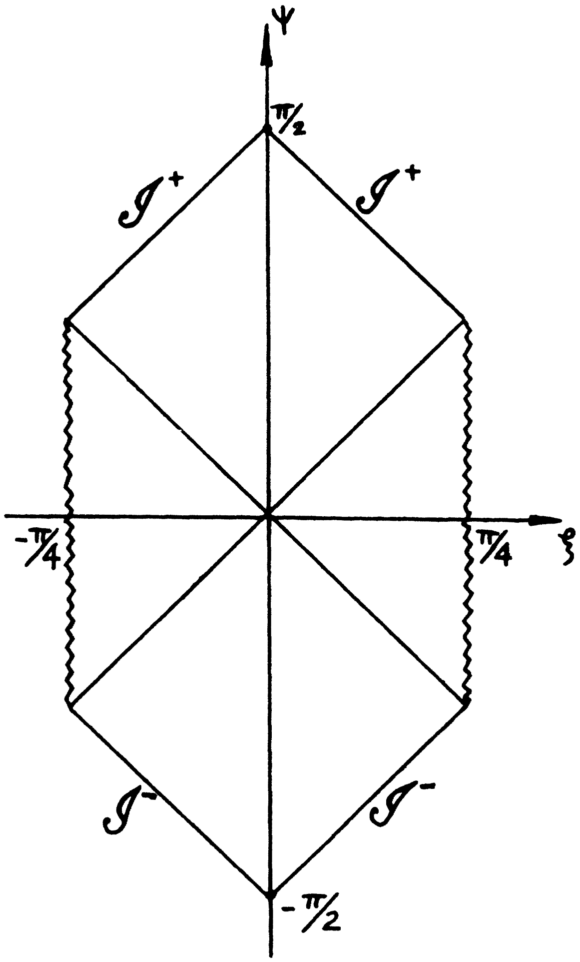

FIG. 1. Penrose diagram of Kar's spacetime.

analogous to that for Taub's, shrinks to a single point in passing from $M^{\#}$ to $\overline{M}$. In the case $m<0$, $z>0$, there is also a coordinate singularity at $z_1=-e^2/m$. This can be eliminated and the spacetime extended by introducing new coordinates. The Penrose diagram of the maximal extension is shown in Fig. 1. The two timelike singularities shown will also shrink into two disjoint points in the passage from $M^{\#}$ to $\overline{M}$.

On the other hand, despite the close analogy between Taub spacetime and Reissener-Nordström spacetime with $e^2>m^2$ (or Schwarzschild spacetime with $m<0$), the singular portion of the latter is not a point but rather a "one-dimensional" set, the essential reason responsible is that lemma 5 fails for spacetimes with spherical symmetry due to the compactness of S^2. Similarly, the singular portion of the c-boundary of Reissner-Nordström spacetime with $e^2<m^2$ is not two points but two disjoint "one-dimensional" sets.

III. DISCUSSION

In order to describe singularities using the notion of c-boundary, it is essential that the c-boundary, $\partial\subset\overline{M}$, of any singular spacetime be divided into two disjoint subsets—the singular portion ∂_s and the nonsingular infinity portion ∂_i. This division should be compatible with the criterion distinguishing singular ideal points from nonsingular infinity ideal points. According to the latter, $M^{\#}$ can be written as $M^{\#}=M_r^{\#}\cup M_S^{\#}\cup M_i^{\#}$, where $M_S^{\#}$ is the subset consisting of all singular TIP's and TIF's, and $M_i^{\#}$ all nonsingular infinity ones. A necessary premise for giving any reasonable definition of ∂_s and ∂_i is that $\psi(M_S^{\#})\cap\psi(M_i^{\#})=\varnothing$, for if there is an element $Q\in\psi(M_S^{\#})\cap\psi(M_i^{\#})$, then there exist $P_1^*\in M_S^{\#}$ and $P_2^*\in M_i^{\#}$ such that $Q=\psi(P_1^*)=\psi(P_2^*)$, and hence it is impossible to tell whether Q should be in ∂_s or ∂_i. However, the answer to the question whether $\psi(M_S^{\#})\cap\psi(M_i^{\#})$ is empty depends heavily on the criterion used to distinguish singular ideal points from nonsingular ones. Take Taub spacetime, for example. If we stick to the criterion that an ideal point is singular if and only if it is null-finite, then

$$\psi(M_S^{\#})\cap\psi(M_i^{\#})=\psi(\check{W}^*)=\psi(\hat{W}^*)\neq\varnothing\ ,$$

since $\hat{W}$ and $\check{W}$ are both null-infinite. Nonetheless, if we choose the widened criterion that an ideal point is singular if it is null-finite or it can be generated by a curve along which some curvature scalar is unbounded, then $\hat{W}^*,\check{W}^*\in M_S^{\#}$ and hence $\psi(M_S^{\#})\cap\psi(M_i^{\#})=\varnothing$, as has been adopted in Sec. II. We can prove no assertion claiming that there exists a reasonable criterion to divide $M_S^{\#}$ and $M_i^{\#}$ such that $\psi(M_S^{\#})\cap\psi(M_i^{\#})=\varnothing$ for all singular spacetimes. Suppose no such criterion exists, then there will be no reasonable general definition for ∂_S and ∂_i; i.e., "the singular portion" makes no sense at all. On the other hand, if such a criterion does exist, then it is natural to define ∂_S and ∂_i to be $\psi(M_S^{\#})$ and $\psi(M_i^{\#})$ and hence "the singular portion ∂_S of the c-boundary" does make good sense. However, it has turned out that ∂_S of Taub spacetime (and also Kar's) is a single point of $\overline{M}$, this suggests that it might not be fruitful to describe the structure of singularities using the c-boundary construction.

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