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Path-integral bosonization for massive fermion models in two dimensions

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We study the bosonization of the massive Thirring model in the framework of path integrals.

The analysis of quantum field models in two space-time dimensions has proved to be a useful theoretical laboratory to understand phenomena such as dynamical mass generation, confinement, and topological excitations, all features expected to be present in the more realistics four-dimensional quantum theories.

Recently, a powerful nonperturbative technique has been used to analyze several two-dimensional nonmassive fermion models in the (Euclidean) path-integral approach. This technique is based on a suitable chiral change of variables.[1–4]

It is the purpose of this Brief Report to show how to deal with the case of massive fermion models in the framework of the above technique by studying the massive Abelian Thirring model.[5]

Let us start our analysis by considering the Euclidean Lagrangian of the model

$$\mathcal{L}_1(\psi,\bar{\psi})(x)=\left[-i\bar{\psi}\gamma_\mu\partial_\mu\psi+m\bar{\psi}\psi+\frac{g^2}{2}(\bar{\psi}\gamma_\mu\psi)^2\right](x)\ , \quad (1)$$

where $\psi=(\psi_1,\psi_2)$ denotes a two-dimensional massive fermion field of (bare) mass m and coupling constant g.

The Hermitian γ matrices we are using satisfy the (Euclidean) relations

$$\{\gamma_\mu,\gamma_\nu\}=2\gamma_{\mu\nu},\quad \gamma_\mu\gamma_5=i\epsilon_{\mu\nu}\gamma_\nu,\quad \gamma_5=i\gamma_0\gamma_1\ , \quad (2)$$
$$\epsilon_{01}=-\epsilon_{10}=1\ .$$

The Lagrangian (1) is invariant under the global Abelian group $\psi\rightarrow e^{i\alpha}\psi$ ($\alpha\in\mathbf{R}$) with the Noetherian conserved current

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi)(x)=0\ .$$

In order to construct an equivalent bosonic theory for the model (1), we consider the quantum partition functional

$$Z=\int D[\psi(x)]D[\bar{\psi}(x)]\exp\left[-\int d^2x\mathcal{L}_1(\psi,\bar{\psi})(x)\right]\ . \quad (3)$$

It will be useful for our purposes to write the interaction Lagrangian in (3) in a form closely parallel to the usual fermion-vector coupling in gauge theories by making use of the following identity:

$$\exp\left[-\frac{g^2}{2}\int d^2x(\bar{\psi}\gamma^\mu\psi)^2(x)\right]=\int D[A_\mu(x)]\exp\left[-\int d^2x\tfrac{1}{2}A_\mu{}^2(x)\right]\exp\left[+\int d^2x\, ig(\bar{\psi}\gamma^\mu\psi)(x)A_\mu(x)\right]\ , \quad (4)$$

where $A_\mu(x)$ is an auxiliary Abelian vector field.

After the use of (4), Z becomes

$$Z=\int D[A_\mu(x)]D[\psi(x)]D[\bar{\psi}(x)]\exp\left[-\tfrac{1}{2}\int d^2xA_\mu{}^2(x)\right]\exp\left[-\int d^2x\,[-i\bar{\psi}\gamma_\mu(\partial_\mu-gA_\mu)\psi+m\bar{\psi}\psi](x)\right]\ . \quad (5)$$

Now, we proceed as in the massless case by making the change of variables[3]

$$\psi(x)=\exp[ig\gamma_5\beta(x)+i\eta(x)]\chi(x)\ , \quad (6)$$

$$\bar{\psi}(x)=\bar{\chi}(x)\exp[ig\gamma_5\beta(x)-i\eta(x)]\ , \quad (7)$$

$$A_\mu(x)=\left[+\epsilon_{\mu\nu}\partial_\nu\beta-\frac{i}{g}\partial_\mu\eta\right](x)\ . \quad (8)$$

At this point, it becomes important to remark that the fermionic measure $D[\psi(x)]D[\bar{\psi}(x)]$ in (5) is defined in terms of the normalized eigenvectors of the Hermitian Dirac operator $-i\gamma_\mu(\partial_\mu-gA_\mu)$, since we are dealing with the massive Thirring model as a mass perturbation model around the massless case closely to the idea of the conventional bosonization scheme implemented by Coleman.[4,5]

As has been shown by Fujikawa[1] the transformations (6) and (7) are not free of cost due to the noninvariance of the functional fermionic measure under chiral change of variables. The resulting Jacobian is given by[3]

$$D[\psi(x)]D[\bar{\psi}(x)]=\exp\left\{-\int d^2x\left[\frac{1}{2}\left(1+\frac{g^2}{\pi}\right)(\partial_\mu\beta)^2-\frac{1}{2}\frac{(1+g^2/\pi)}{g^2}(\partial_\mu\eta)^2\right]\right\}D[\chi(x)]D[\bar{\chi}(x)]\ . \quad (9)$$

Concerning the transformation (8), we have the result[3]

$$D[A_\mu(X)]=\mathrm{Det}\left[\frac{i}{g}(\partial_0{}^2-\partial_1{}^2)\right]D[\beta(x)]D[\eta(x)]\ . \quad (10)$$

Substituting Eqs. (9) and (10) in Eq. (5), we obtain the expression

$$Z=\int D[\beta(x)]D[\eta(x)]\mathrm{Det}\left[\frac{i}{g}(\partial_0{}^2-\partial_1{}^2)\right]\exp\left\{-\int d^2x\left[\frac{1}{2}\left[1+\frac{g^2}{\pi}\right](\partial_\mu\beta)^2-\frac{1}{2}\frac{(1+g^2/\pi)}{g^2}(\partial_\mu\eta)^2\right]\right\}$$
$$\times\int D[\chi(x)]D[\bar\chi(x)]\exp\left[-\int d^2x\,(-\bar\chi i\gamma_\mu\partial_\mu\chi+m\bar\chi e^{2ig\gamma_5\beta}\chi)(x)\right] . \tag{11}$$

Now we note that the (unphysical) $\eta(x)$ field is decoupled in the effective partition functional given by Eq. (11); since it is related to the spurious longitudinal part of the conserved U(1) current of the model [note that at the classical level the field $A_\mu(x)$ coincides with $(\bar\psi\gamma^\mu\psi)(x)$]. As a consequence solely the transversal part of $A_\mu(x)$ effectively contributes to the partition functional (11). Thus, we get the effective result

$$Z=\int D[\beta(x)]\exp\left[-\int d^2x\,\frac{1}{2}\left[1+\frac{g^2}{\pi}\right](\partial_\mu\beta)^2(x)\right]\int D[\chi(x)]D[\bar\chi(x)]\exp\left[-\int d^2x\,(-\bar\chi i\gamma_\mu\partial_\mu\chi+m\bar\chi e^{2ig\gamma_5\beta}\chi)(x)\right] . \tag{12}$$

Now we note that, opposite to the massless case, we have not decoupled completely the massive fermions from the field $A_\mu(x)$, since there remains in Eq. (12) the mass coupling term

$$m\left[\bar\chi e^{2ig\gamma_5\beta}\chi\right](x)=m\left[\bar\chi\left[\frac{1+\gamma_5}{2}\right]\chi e^{2ig\beta}+\bar\chi\left[\frac{1-\gamma_5}{2}\right]\chi e^{-2ig\beta}\right](x) . \tag{13}$$

We, then, face the problem to evaluate the fermionic functional integral

$$I[\beta(x)]=\int D[\chi(x)]D[\bar\chi(x)]\exp\left[-\int d^2x\{[-\bar\chi(i\gamma_\mu\partial_\mu)\chi](x)+m(\sigma_+e^{2ig\beta}+\sigma_-e^{-2ig\beta})(x)\}\right] , \tag{14}$$

where we have introduced the objects

$$\sigma_\pm(x)=\left[\bar\chi\left[\frac{1\pm\gamma_5}{2}\right]\chi\right](x) . \tag{15}$$

In order to evaluate (14), we make a series expansion of the term

$$\exp\left[-m\int d^2x(\sigma_+e^{2ig\beta}+\sigma_-e^{-2ig\beta})(x)\right]$$

in powers of the (bare) fermion mass m.

Explicitly,

$$I[\beta(x)]=\sum_{n=0}^{\infty}\frac{(-m)^n}{n!}\int d^2x_1\cdots d^2x_n\int D[\chi(x)]D[\bar\chi(x)]\exp\left[-\int d^2x[\bar\chi(-i\gamma_\mu\partial_\mu)\chi](x)\right]$$
$$\times(\sigma_+e^{2ig\beta}+\sigma_-e^{-2ig\beta})(x_1)\cdots(\sigma_+e^{2ig\beta}+\sigma_-e^{-2ig\beta})(x_n) . \tag{16}$$

Now it is a well-known result that the only nonzero terms in (16) are those with equal number of σ_+'s and σ_-'s:[5,6] i.e.,

$$\int D[\chi(x)]D[\bar\chi(x)]\exp\left[-\int d^2x[\bar\chi(-i\gamma_\mu\partial_\mu)\chi](x)\right]\prod_{i=1}^{k}\sigma_+(x_i)\prod_{i=1}^{k}\sigma_-(y_i)=\left[\frac{1}{2\pi}\right]^{2k}\frac{\prod_{i>j}^{k}(x_i-x_j)^2(y_i-y_j)^2}{\prod_{(i,j)}^{k}(x_i-y_j)^2} , \tag{17}$$

with the massless fermion propagator given by

$$(i\gamma_\mu\partial_\mu)^{-1}(x,y)=+\frac{1}{2\pi}\gamma_\mu\frac{(x_\mu-y_\mu)}{(x-y)^2} . \tag{18}$$

By following Coleman,[5] we introduce a massless scalar field $\phi(x)$ with the (infrared regularized) Green's function given by $\Delta(x)=-(1/4\pi)\ln(x^2/\epsilon^2)$ (ϵ is a infrared cutoff with mass dimension) and rewrite Eq. (17) in the form

$$\left[\frac{1}{2\pi}\right]^{2k}e^{-2k\Delta(0)}\int D[\phi(x)]\exp\left[-\int d^2x\,\tfrac{1}{2}(\partial_\mu\phi)^2(x)\right]\exp\left[\sqrt{4\pi}\,i\left[\sum_{i=1}^{k}\phi(x_i)-\phi(y_i)\right]\right] . \tag{19}$$

By noting that the averages

$$\int D[\phi(x)]\exp\left[-\int d^2x\,\tfrac{1}{2}(\partial_\mu\phi)^2(x)\right]\exp\left[\sqrt{4\pi}i\left[\sum_{i=1}^{k}\phi(x_i)-\sum_{i=1}^{l}\phi(y_i)\right]\right]$$

are zero for $k\neq l$ due to the infrared divergences of the massless scalar field $\phi(x)$ (Ref. 5) we can write $I[\beta(x)]$ [see Eq. (14)] in the form

$$I[\beta(x)]=\int D[\phi(x)]\exp\left[-\int d^2x\,\tfrac{1}{2}(\partial_\mu\phi)^2(x)\right]\exp\left[-\int d^2x[me^{-\Delta(0)}]\frac{1}{4\pi}\cos(2g\beta+\sqrt{4\pi}\phi)(x)\right] , \tag{20}$$

where we see that the (bare) mass parameter gets a multiplicative (ultraviolet) renormalization $m_R=me^{-\Delta(0)}$.[5]

Finally, substituting Eq. (20) in Eq. (12), we get the effective bosonic action for the fermionic massive Thirring model [see Eq. (3)]:

$$Z=\int D[\beta(x)]D[\phi(x)]\exp\left\{-\int d^2x\,\tfrac{1}{2}\left[\left[1+\frac{g^2}{\pi}\right](\partial_\mu\beta)^2+(\partial_\mu\phi)^2\right](x)\right\}\exp\left[-\frac{m_R}{4\pi}\int d^2x\cos(2g\beta+\sqrt{4\pi}\phi)(x)\right] . \tag{21}$$

In order to analyze the physical spectrum associated to the effective bosonic action Eq. (21) we introduce the new fields

$$2g\tilde{\beta}(x)=(2g\beta+\sqrt{4\pi}\phi)(x) ,$$
$$\tilde{\phi}(x)=(c_0\beta+c_1\phi)(x) , \tag{22}$$

where the arbitrary constants c_0, c_1 satisfy the relation

$$\frac{c_1}{c_0}=-\frac{1}{\sqrt{\pi}(1+g^2/\pi)} ,$$
$$\frac{c_1}{c_0}\neq\frac{\sqrt{\pi}}{g} . \tag{23}$$

In terms of these new fields, the effective Lagrangian in Eq. (21) takes, then, the more transparent form

$$\mathscr{L}_2(\tilde{\beta},\tilde{\phi})(x)=\tfrac{1}{2}(\partial_\mu\tilde{\beta})^2\left[\frac{(1+g^2/\pi)4c_1{}^2g^2+4g^2c_0{}^2}{(2gc_1-\sqrt{4\pi}c_0)^2}\right]+\tfrac{1}{2}(\partial_\mu\tilde{\phi})^2\left[\frac{(1+g^2/\pi)4\pi+4g^2}{(2gc_1-\sqrt{4\pi}c_0)^2}\right]+\frac{m_R}{4\pi}\cos[2g\tilde{\beta}(x)] . \tag{24}$$

There is thus, in the spectrum of the model, a massless scalar field $\tilde{\phi}(x)$ and a sine-Gordon field $\tilde{\beta}(x)$.

It is instructive to point out that the massless scalar field $\tilde{\phi}(x)$ is a remnant of "almost long-range order" of the Kosterlitz-Thouless type which occurs in the infrared region of the massless Thirring model ($m_R=0$).[7]

Finally, we note that a similar analysis can be straightforwardly implemented for the non-Abelian version of the model,[3] or for the massive (Abelian and non-Abelian) fermion gauge theories.[8–10]

As we have shown, chiral changes in path integrals even for massive fermion models provide a quick, mathematically, and conceptually simple way to analyze two-dimensional fermion models.

Note added. We would like to make some clarifying remarks on the analysis implemented in this Brief Report. First, the expansion (16) is possible because the fermionic functional measure $D[\chi(x)]D[\bar{\chi}(x)]$ in Eq. (16) is now defined in terms of the eigenvectors of the free Dirac operator $i\chi_\mu\partial_\mu$ [see Eq. (9)]. Second, we observe that the average before Eq. (20) is zero for $k\neq l$ [see Eqs. (4.3)–(4.6) and Eq. (4.11) of Ref. 5] by its direct evaluation. The infrared-regularized result reads

$$\left[\prod_{j<j'}^{k}(|x_j-x_{j'}|/\epsilon)^2\prod_{j<j'}^{l}(|y_j-y_{j'}|/\epsilon)^2\right]\left[\prod_{j}^{k}\prod_{j'}^{l}(|x_j-y_{j'}|/\epsilon)^2\right]^{-1} ,$$

where we have ignored the field's contractions at the same point and the functional ϕ-independent determinant $\mathrm{Det}(\partial_0\partial_0+\partial_1\partial_1)$. Taking now the limit $\epsilon\to 0^+$ we see easily that the only nonzero terms are those with $k=l$.

Third, after this work was sent for publication, we became aware of the paper of Ref. 10, where some comments related to the non-Abelian case are made, and of other papers,[11] where the problem analyzed in this paper is studied by using an *inappropriate* fermion mass-dependent Jacobian [see Eq. (9) of this paper and final comments of Ref. 4] and leading to a bosonic sine *hyperbolic* theory in the Abelian case, a result opposite to the usual operator analysis (see Refs. 5, 10, and 12).

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APPENDIX: THE NON-ABELIAN CASE

We now briefly discuss the non-Abelian SU(N) Thirring model.

Proceeding as in the Abelian case by introducing an SU(N) auxiliary gauge field $A_\mu(x)=A_\mu^a(x)\lambda_a$, with λ_a being the generators of the associated SU(N) Lie algebra, we write the quantum partition functional of the model in the following form:

$$Z=\int D[A_\mu(x)]\exp\left[-\tfrac{1}{2}\int d^2x\,\mathrm{Tr}A_\mu\cdot A_\mu\right]Z_F[A_\mu] , \tag{A1}$$

where

$$Z_F[A_\mu]=\int D[\psi(x)]D[\bar{\psi}(x)]\times\exp\left[-\int d^2x[-i\bar{\psi}\gamma_\mu(\partial_\mu-gA_\mu)\psi+m\bar{\psi}\psi](x)\right] \tag{A2}$$

is the vacuum energy density of a theory of SU(N) massive fermions interacting with the external gauge field $A_\mu(x)$.

As in the Abelian case there is a chiral change which decouples the fermion fields $\psi(x)$ $\bar{\psi}(x)$ from the SU(N) gauge field $A_\mu(x)$. This change is implemented by using the Roskie decoupling gauge[13,14]

$$(\gamma_\mu A_\mu)(x)=-\frac{1}{g}e^{-i\gamma_5\beta(x)}(\gamma_\mu\partial_\mu)e^{i\gamma_5\beta(x)} , \tag{A3}$$

where $\beta(x)=(\beta^a\lambda_a)(x)$ takes value in the Lie algebra of SU(N).

So, we make the variable change in Eq. (A2) $[U_5(x)\equiv e^{-i\gamma_5\beta(x)}]$

$$\psi(x)=e^{-i\gamma_5\beta(x)}\chi(x)=U_5(x)\chi(x)\ , \tag{A4}$$

$$\bar\psi(x)=\bar\chi(x)e^{-i\gamma_5\beta(x)}=\bar\chi(x)U_5(x)\ , \tag{A5}$$

which yields the result $[U(x)\equiv e^{-i\beta^a(x)\lambda_a}]$

$$Z_F[A_\mu(x)]=\int D[\chi(x)]D[\bar\chi(x)]J[A_\mu]\exp\left[-\left(\int d^2x[\bar\chi(i\gamma_\mu\partial_\mu)\chi]+m(\sigma_+(U^2)+\sigma_-(U^2)^\dagger)(x)\right)\right]\ , \tag{A6}$$

where the quantum aspect of (A4) and (A5) is taken into account by the Jacobian [see Eq. (7) of Ref. (13)]

$$J[G_\mu]=\frac{\mathrm{Det}[\not{D}(A_\mu)]}{\mathrm{Det}(i\gamma_\mu\partial_\mu)}\ , \tag{A7}$$

which can be explicitly evaluated [see Eqs. (9)–(15) of Ref. 13]

$$\ln J[G_\mu]=-\frac{g^2}{2\pi}\int d^2x\,\mathrm{Tr}(\tfrac{1}{2}A_\mu\cdot A_\mu)(x)+\frac{i}{2\pi}\int_0^1 dt\int d^2x\,\mathrm{Tr}[(\partial_t U)U^{-1}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}\epsilon_{\mu\nu}](x,t)\ , \tag{A8}$$

with

$$U(t,x)=e^{-it\beta(x)}\quad(0\le t\le 1)\ . \tag{A9}$$

The first term in Eq. (A8) is the non-Abelian Schwinger-mechanism mass term. The second term can be written as $4\pi i\Gamma_{\mathrm{WZ}}[U(t,x)]$, where $\Gamma_{\mathrm{WZ}}[U(t,x)]$ is the well-known two-dimensional Wess-Zumino functional[12,13,15]

$$\Gamma_{\mathrm{WZ}}[U(t,x)]=\frac{1}{24\pi^2}\int_B d^3x\,\epsilon^{ijk}\mathrm{Tr}[(\partial_i UU^{-1})(\partial_j UU^{-1})(\partial_k UU^{-1})](x)\ , \tag{A10}$$

where B is the upper hemisphere on S^3, whose boundary S^2 can be considered as the compactified space-time R^2.

The $(\chi(x),\bar\chi(x))$ fermionic functional integrations are evaluated as in the Abelian case by considering the power series in the (bare) fermion mass m [see Eq. (16)] with the result

$$I[U(x)]=\int D[\phi(x)]\exp\left(-\tfrac{1}{2}\int d^2x(\partial_\mu\phi)^2(x)\right)\exp\left(-\frac{1}{4\pi}\int d^2x\,me^{-\Delta(0)}\mathrm{Tr}[e^{i\sqrt{4\pi}\phi}U^2+e^{-i\sqrt{4\pi}\phi}(U^2)^\dagger](x)\right)\ , \tag{A11}$$

which is the SU(N) version of the sine-Gordon term Eq. (20) in the text.

Finally, we remark that one can implement the variable change $A_\mu^a(x)\to\beta^a(x)$ in Eq. (A1) leading to an effective theory in terms of the fields $\beta^a(x)$ [see Eqs. (18) and (19) of Refs. 13 and 15].

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