Construction of nondegenerate non-Abelian solutions and Coulomb-type solution for the same charge distribution in SU(2) Yang-Mills theory

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We propose a new cylindrical ansatz for SU(2) Yang-Mills theory which lifts the degeneracy between the two non-Abelian solutions in the presence of a time-independent charge current. We then construct explicit solutions for which we find that (i) $Q=Q_1$ is the point of bifurcation between the two nondegenerate non-Abelian solutions and a Coulomb-type solution, (ii) for $Q_1<Q<Q_2$, $E_{\rm NA}^{\rm I}<E_C<E_{\rm NA}^{\rm II}$, (iii) for $Q=Q_2$, $E_{\rm NA}^{\rm I}<E_C=E_{\rm NA}^{\rm II}$, (iv) for $Q>Q_2$, $E_{\rm NA}^{\rm I}<E_{\rm NA}^{\rm II}<E_C$. Here Q is the gauge-invariant charge and E is the total energy. Finally, we show that our ansatz is related to the usual cylindrical ansatz by a nonsingular gauge transformation.

In the last few years the question of the stability of the classical solutions of the Yang-Mills equations in the presence of external sources has received a lot of attention.[1] One of the motivations for this study is the hope that even at the classical level there may be a definite indication of the phenomenon that leads to confinement in the quantum theory. One of the earliest works in this direction was by Mandula[2] who showed that for a spherically symmetric static source the Coulomb solution is unstable when the source strength exceeds some critical value. For a nonspherical static source a similar result does not exist. However, Sikivie and Weiss[3] have constructed a non-Abelian magnetic dipole solution which always has less energy[4] compared to the corresponding Coulomb solution. In the case of their ansatz it turns out that (see below for details) for a given charge distribution there are two degenerate non-Abelian solutions (i.e., they have the same energy).

It may be interesting to enquire if it is possible to lift this degeneracy between the two non-Abelian solutions. In this context it may be worthwhile to recall that such a degeneracy also exists in the case of the spherical ansatz. In that case it has been shown by Jacobs and Wudka[5] that such a degeneracy can be lifted by introduction of a current source. However, for the Sikivie-Weiss ansatz such a degeneracy is not lifted even in the presence of a current source.[6]

In this note we propose a new cylindrical ansatz and show that within this ansatz the degeneracy between the two non-Abelian solutions is lifted in the presence of a time-independent charge-current source. As an illustration we explicitly construct magnetic dipole and total screening solutions and show that in both cases we have three nondegenerate solutions for the same charge distribution. The comparison of their energies as a function of the gauge-invariant charge Q shows that (i) $Q=Q_1$ is a point of bifurcation between the three solutions, (ii) for $Q_1<Q<Q_2$, $E_{\rm NA}^{\rm I}<E_C<E_{\rm NA}^{\rm II}$, (iii) $Q=Q_2$ is a crossover point at which $E_{\rm NA}^{\rm I}<E_C=E_{\rm NA}^{\rm II}$, (iv) for $Q>Q_2$, $E_{\rm NA}^{\rm I}<E_{\rm NA}^{\rm II}<E_C$. Finally, we show that our cylindrical ansatz is related to the Sikivie-Weiss ansatz by a nonsingular gauge transformation.

The Yang-Mills equations in the presence of external source J_μ^a are[7]

$$D_\mu F^{a\mu\nu}=J^{a\nu} \quad , \tag{1}$$

where

$$F_{\mu\nu}^a=\partial_\mu A_\nu^a-\partial_\nu A_\mu^a+g\epsilon^{abc}A_\mu^b A_\nu^c \tag{2}$$

and

$$D_\mu\phi^a=\partial_\mu\phi^a+g\epsilon^{abc}A_\mu^b\phi^c \ . \tag{3}$$

The Sikivie-Weiss ansatz is[3]

$$A_0^a(x)=g^{-1}\delta^{a3}\phi(\rho,x_3) \ , \tag{4a}$$

$$A_i^a(x)=g^{-1}\delta^{a1}\epsilon_{i3j}\left(\frac{x_j}{\rho}\right)A(\rho,x_3) \ . \tag{4b}$$

On substituting this ansatz in Eq. (1) we have the Yang-Mills equations

$$\nabla^2\phi-\phi A^2=-q \ , \tag{5a}$$

$$\nabla^2 A-\frac{A}{\rho^2}+A\phi^2=-m \ , \tag{5b}$$

where

$$J_0^a(x)=-g^{-1}\delta^{a3}q(\rho,x_3) \ , \tag{6a}$$

$$J_i^a(x)=-g^{-1}\delta^{a1}\epsilon_{i3j}\left(\frac{x_j}{\rho}\right)m(\rho,x_3) \ . \tag{6b}$$

With this ansatz the general expression for field energy given by[8]

$$E=\int d^3x\,[\tfrac{1}{2}(E_i^aE_i^a+B_i^aB_i^a)+A_i^aJ_i^a], \quad i=1,2,3 \ , \tag{7}$$

reduces to

$$E_{na}=\frac{1}{g^2}\int d^3x\,[\tfrac{1}{2}(\nabla\phi)^2+\phi^2A^2-\tfrac{1}{2}Am] \ . \tag{8}$$

Let us first look at Yang-Mills equations (5) in the case of $m\equiv0$. It is clear from Eqs. (5) and (8) that if (ϕ,A) is a solution to these equations for a given q, then $(\phi,-A)$ is also a solution for the same q possessing the same energy. Even if $m\equiv0$ it follows from Eqs. (5) and (8) that (ϕ,A) and $(\phi,-A)$ are still degenerate.

In order to lift the degeneracy between the two non-Abelian solutions we now propose the ansatz

$$A_0^a(x)=g^{-1}\epsilon^{a3l}\left(\frac{x_l}{\rho}\right)\phi(\rho,x_3) \ , \tag{9a}$$

$$A_i^a(x)=g^{-1}\delta^{a3}\epsilon_{i3j}\left(\frac{x_j}{\rho}\right)\left(A(\rho,x_3)-\frac{1}{\rho}\right) \ . \tag{9b}$$

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Substituting this ansatz in Eq. (1) yields the same Yang-Mills equations as given by Eqs. (5) with

$$J_0^a(x) = -g^{-1}\epsilon^{a3l}\left(\frac{x_l}{\rho}\right)q \ , \tag{10a}$$

$$J_i^a(x) = -g^{-1}\delta^{a3}\epsilon_{i3j}\left(\frac{x_j}{\rho}\right)m \ . \tag{10b}$$

However, the expression for field energy is now different from (8) and is given by

$$E_{\rm NA} = \frac{1}{g^2}\int d^3x\left(\tfrac{1}{2}(\nabla\phi)^2+\phi^2A^2-\tfrac{1}{2}Am+\frac{m}{\rho}\right) \ . \tag{11}$$

Only in the special case of $m\equiv 0$ [or $\int(m/\rho)\,d^3x=0$] are the two energy expressions (8) and (11) identical. Thus

$$\phi=\phi_{\rm SW}=\frac{\sqrt{18}}{a}\frac{y^2}{\cosh(y^3)}, \quad y=r/a \ , \tag{12a}$$

$$A=A_{\rm SW}=\frac{c}{a}\left(\frac{\sin\theta}{y^2}\right)\tanh(y^3) \ , \tag{12b}$$

are also solutions within the new ansatz (9) possessing the same energy and the same q in Ref. 3.

One might naively think that, unlike the Sikivie-Weiss ansatz, our A_i^a as given by Eq. (9b) is singular in case A is chosen to be regular [as is done above in Eq. (12b)]. However, this is not so. The apparent singular term in A_i^a,

$$g^{-1}\delta^{a3}\epsilon_{i3j}\frac{x_j}{\rho^2} \ , \tag{13}$$

does not contribute either to the electric or to the magnetic fields, which for our ansatz are given by

$$E_i^a=-g^{-1}\epsilon^{a3l}\left(\frac{x_l}{\rho}\right)\partial_i\phi+g^{-1}(x^a-\delta^{a3}x_3)\epsilon_{i3j}\left(\frac{x_j}{\rho^2}\right)\phi A \ , \tag{14a}$$

$$B_i^a=g^{-1}\delta^{a3}\epsilon_{ijk}\epsilon_{k3l}\partial_j(x_lA/\rho) \ . \tag{14b}$$

As a matter of fact our ansatz (9) can be obtained from the Sikivie-Weiss ansatz (4) by the time-independent nonsingular gauge transformation

$$A'_\mu=UA_\mu U^{-1}+g^{-1}U\partial_\mu U^{-1} \ , \tag{15a}$$

with

$$U=e^{-i\omega\sigma_3}e^{i\pi\sigma_2/2}e^{i\pi\sigma_1/2}, \quad \tan\omega=\frac{x_2}{x_1} \ . \tag{15b}$$

One might then wonder as to why the energy expression (11) differs from the Sikivie-Weiss expression (8). This is so because the term $\int d^3x\,(J_i^aA_i^a)$ in (7) is not gauge invariant. However, in the case of $m\equiv 0$ the two energy expressions are identical as they should be since $\int(E_i^aE_i^a+B_i^aB_i^a)\,d^3x$ is gauge invariant.

From Eqs. (5) and (11) it is clear that in the presence of m, if (ϕ,A) is a solution for a given charge distribution q then $(\phi,-A)$ is also a solution with same q but with different energy, i.e., within the new ansatz the two non-Abelian solutions are not degenerate. For the same q we also have a Coulomb-type solution $(\phi_c,0)$ which has different energy from either of the non-Abelian energies and is given by[6]

$$E_C=\frac{1}{g^2}\int d^3x\,[\tfrac{1}{2}(\nabla\phi)^2+\phi^2A^2+\tfrac{1}{2}(\nabla\chi)^2] \tag{16}$$

where

$$\nabla^2\chi=\phi A^2 \ . \tag{17}$$

We now present explicit solutions where all the features described above are present and compare the energies of the non-Abelian and Coulomb-type branches as a function of the gauge-invariant charge Q given by

$$Q=\int d^3x\,[J_0^a(x)J_0^a(x)]^{1/2}=\frac{1}{g}\int d^3x\,|q| \ , \tag{18}$$

in view of Eq. (10a).

As our first choice we choose

$$A=\frac{\lambda}{a}ye^{-y}\sin\theta \ , \tag{19}$$

where $y=r/a$, a being the scale factor with dimension of length and λ a dimensionless parameter which we always choose to be non-negative. In order to obtain a nonsingular and short-ranged ϕ we choose

$$\chi=\frac{\mu\lambda^2}{a}y^6e^{-3y}\sin^6\theta \ , \tag{20}$$

where $\mu=\mu(\lambda)$ is dimensionless. Using Eq. (17) we have

$$\phi=\frac{3\mu}{a}y^2e^{-y}\sin^2\theta[12+(3y^2-14y)\sin^2\theta] \ . \tag{21}$$

The corresponding electric and magnetic fields can be computed by using Eqs. (14) and it turns out that both of them are totally screened. Using Eq. (11) the energies corresponding to the solutions (ϕ,A) and $(\phi,-A)$, which we denote by $E_{\rm NA}^{\rm I}$ and $E_{\rm NA}^{\rm II}$, respectively, are given by

$$\frac{ag^2E_{\rm NA}^{\rm I}}{\pi}=2880\mu^2+\frac{55\,647}{1408}\mu^2\lambda^2-\lambda^2+8\lambda-\frac{237\,568}{2187}\lambda\mu^2 \ , \tag{22a}$$

$$\frac{ag^2}{\pi}E_{\rm NA}^{\rm II}=\frac{ag^2}{\pi}E_{\rm NA}^{\rm I}-16\lambda+\frac{475\,136}{2187}\lambda\mu^2 \ . \tag{22b}$$

The Coulomb-type energy E_C corresponding to the same charge distribution is obtained from (16), (19), (20), and (21) as

$$\frac{ag^2}{\pi}E_C=2880\mu^2+\frac{18\,549}{704}\mu^2\lambda^2+\frac{3200}{19\,683}\mu^2\lambda^4 \ . \tag{22c}$$

The corresponding gauge-invariant charge Q can be calculated numerically by using Eqs. (5a), (18), (19), and (21). A plot of $E_{\rm NA}^{\rm I}$, $E_{\rm NA}^{\rm II}$, and E_C (in units of $25\pi/ag^2$) as a function of Q (in units of $8\pi/g$) is shown in Fig. 1, where we have used the parametrization

$$\mu(\lambda)=\mu_0+0.002\lambda^2, \quad \mu_0=\left(\frac{2187}{29\,696}\right)^{1/2} \ . \tag{23}$$

As we shall show below, for this parametrization $Q_1=Q(\lambda=0)$ is a bifurcation point while $Q_2=Q(\lambda=\lambda_{\rm crit})$ is a crossover point. For $Q_1<Q\leq Q_2$ we find that $E_{\rm NA}^{\rm I}<E_C\leq E_{\rm NA}^{\rm II}$, while for $Q>Q_2$ we have $E_{\rm NA}^{\rm I}<E_{\rm NA}^{\rm II}<E_C$.

Let us now show that $\lambda=0$ is a bifurcation point. From Eqs. (5a), (18), (19), (21), and (23) it is clear that Q in-

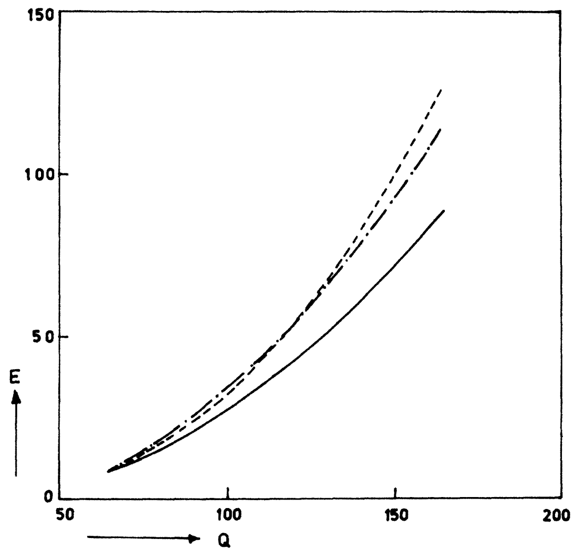

FIG. 1. Plot of E (in units of $25\pi/ag^2$) vs Q (in units of $8\pi/g$), where the solid line denotes $E^{\mathrm{I}}_{\mathrm{NA}}$ vs Q, the dot-dashed line denotes $E^{\mathrm{II}}_{\mathrm{NA}}$ vs Q, and the dashed line denotes E_C vs Q.

creases with λ. In such a case bifurcation will occur at $\lambda=\lambda_0$ if E and Q have simultaneous extrema at λ_0 and λ_0 is not a point of inflection of Q^1. Note that here E stands for any one of $E^{\mathrm{I}}_{\mathrm{NA}}$, $E^{\mathrm{II}}_{\mathrm{NA}}$, and E_C. Therefore $\mu=\mu(\lambda)$ is obtained from

$$\frac{\partial E}{\partial\lambda}+\frac{\partial E}{\partial\mu}\frac{d\mu}{d\lambda}=0\ , \tag{24a}$$

$$\frac{\partial Q}{\partial\lambda}+\frac{\partial Q}{\partial\mu}\frac{d\mu}{d\lambda}=0\ . \tag{24b}$$

These equations will admit a nontrivial solution if[5]

$$\frac{\partial E}{\partial\lambda}\frac{\partial Q}{\partial\mu}-\frac{\partial E}{\partial\mu}\frac{\partial Q}{\partial\lambda}=0\ . \tag{25}$$

Hence any $\mu(\lambda)$ satisfying (25) with derivative $d\mu/d\lambda$ satisfying (24) at $\lambda=\lambda_0$ will give a bifurcation point at λ_0 provided $d^2Q/d\lambda^2|_{\lambda=\lambda_0}\neq 0$. The value of μ_0 in our parametrization (23) is obtained from Eq. (25) by using either $E^{\mathrm{I}}_{\mathrm{NA}}$ or $E^{\mathrm{II}}_{\mathrm{NA}}$. Note that the parametrization (23) is also consistent with Eqs. (24). Further, for the same parametrization (23) and for the same values of $\mu(0)$ and $\mu'(0)$, E_C also satisfies Eqs. (24), (25), and $d^2Q/d\lambda^2|_{\lambda=0}\neq 0$. Hence $\lambda=0$ is a point of bifurcation between the two nondegenerate non-Abelian solutions and Coulomb-type solution. It is also clear from the above analysis that the point $\lambda=\lambda_{\mathrm{crit}}$ is merely a crossover point.

Finally, we show that with our new ansatz the degeneracy between the two Sikivie-Weiss magnetic dipole solutions[3] $(\phi_{\mathrm{SW}},A_{\mathrm{SW}})$ and $(\phi_{\mathrm{SW}},-A_{\mathrm{SW}})$ is lifted in the presence of nonzero m. We choose

$$\phi=\nu\phi_{\mathrm{SW}},\quad A=A_{\mathrm{SW}},\quad \nu>0\ , \tag{26}$$

as the solution of Eqs. (5) in our ansatz. Using Eq. (11) it is then easy to show that

$$E^{\mathrm{I}}_{\mathrm{NA}}-E^{\mathrm{II}}_{\mathrm{NA}}=-144\pi\nu^2(\nu^2-1)\frac{c}{a}\int_0^\infty dy\,\frac{y^3\sinh y^3}{\cosh^2 y^3}\ , \tag{27}$$

so that $E^{\mathrm{I}}_{\mathrm{NA}}>E^{\mathrm{II}}_{\mathrm{NA}}$ for $0<\nu^2<1$, while $E^{\mathrm{II}}_{\mathrm{NA}}>E^{\mathrm{I}}_{\mathrm{NA}}$ for $\nu^2>1$. The corresponding Coulomb-type energy can also be computed, and one can show that for $\nu^2>1$ (i) $Q=Q_1[=Q(C=0)]$ is a point of bifurcation between the two non-Abelian and the Coulomb-type branches, (ii) for $Q_1<Q<Q_2$, $E^{\mathrm{I}}_{\mathrm{NA}}<E_C<E^{\mathrm{II}}_{\mathrm{NA}}$, (iii) $Q=Q_2[Q(C=C_{\mathrm{crit}})]$ is merely a crossover point at which $E^{\mathrm{I}}_{\mathrm{NA}}<E_C=E^{\mathrm{II}}_{\mathrm{NA}}$, (iv) for $Q>Q_2$, $E^{\mathrm{I}}_{\mathrm{NA}}<E^{\mathrm{II}}_{\mathrm{NA}}<E_C$. The degeneracy of the magnetic multipole solutions will also be similarly lifted within this ansatz.

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[7]We follow the metric $(-+++)$ in this note and $a,b,c=1,2,3$ for the SU(2) gauge group.

[8]Notice that $E^a_i=F^a_{0i}$, $B^a_i=\frac{1}{2}\epsilon_{ijk}F^{ajk}$.