Evaluation of the chiral anomaly by the stochastic quantization method

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The method of stochastic quantization of Fermi fields is used to calculate the non-Abelian chiral anomaly. The calculation is based on the formalism for the stochastic quantization of Fermi fields. The regularization scheme followed is the one introduced by Breit, Gupta, and Zaks. The zero-mode ambiguity is avoided by introducing a small energy for the zero mode. The result obtained is the gauge-covariant anomaly.

I. INTRODUCTION

Recently the chiral anomalies were discussed extensively from various points of view—perturbative,[1] path integral,[2] and topological.[3] In this paper we develop a method for calculating the chiral anomaly in terms of stochastic quantization.[4]

We start from the (Euclidean) Dirac action

$$S=\int d^4x\,\bar{\psi}\not{D}\psi\ , \tag{1.1}$$

where

$$\not{D}=i\gamma^{\mu}[\partial_{\mu}-V_{\mu}(x)-i\gamma_5A_{\mu}(x)]$$

and $V_{\mu}(x),A_{\mu}(x)$ are $N\times N$ Hermitian matrices (vector and axial-vector fields).

Under the local chiral transformation

$$\begin{aligned}\psi(x)&\rightarrow e^{i\gamma_5\epsilon(x)}\psi(x)\ ,\\ \bar{\psi}(x)&\rightarrow\bar{\psi}(x)e^{i\gamma_5\epsilon(x)}\ ,\end{aligned} \tag{1.2}$$

the action transforms as

$$S\rightarrow S+\delta S\ , \tag{1.3}$$

$$\delta S=-\int d^4x\,\bar{\psi}(x)\{\epsilon(x)\gamma_5,\not{D}\}\psi(x)\ . \tag{1.4}$$

If $A_{\mu}(x)$ is changed simultaneously as

$$A_{\mu}(x)\rightarrow A_{\mu}(x)-\partial_{\mu}\epsilon(x)\ , \tag{1.5}$$

then δS is compensated by this change so that the action is invariant.

Let us consider a path integral over Fermi fields:

$$\int D\psi D\bar{\psi}\,e^{-S[\psi,\bar{\psi}]}=\int D\psi D\bar{\psi}\exp\left[-\int\bar{\psi}\not{D}\psi\,dx\right]=e^{W[V_{\mu},A_{\mu}]}=Z_0\ . \tag{1.6}$$

If the integration measure of (1.6) is invariant under (1.2), we should obtain

$$W[V_{\mu},A_{\mu}-\partial_{\mu}\epsilon]=W[V_{\mu},A_{\mu}]\ . \tag{1.7}$$

That is, W is chiral-gauge invariant. The existence of a chiral anomaly means that this relation is invalid and the functional $W[V_{\mu},A_{\mu}]$ is not chiral-gauge invariant. Therefore we intend to calculate a variation of W by infinitesimal chiral-gauge transformation:

$$\delta W=\frac{-\int D\psi D\bar{\psi}\int dx\,\epsilon(x)\partial_{\mu}(\bar{\psi}\gamma^{\mu}\gamma_5\psi)\exp\left[-\int\bar{\psi}\not{D}\psi\,dx\right]}{\int D\psi D\bar{\psi}\exp\left[-\int\bar{\psi}\not{D}\psi\,dx\right]}=\langle\delta S\rangle\ . \tag{1.8}$$

This relation is an ill-defined quantity if $\not{D}$ contains zero modes. In this case both the numerator and denominator become zero and the procedure for the calculation of the anomaly becomes ambiguous. It is known, however, that if we regularize the integral by assuming a small, finite, nonzero value for the zero-mode energy and by letting go to zero at the end of the calculations, we avoid the ambiguity. This is the infrared regularization. (This method of regularization is implicitly mentioned by Fujikawa,[5] but it was pointed out to me by Sakita.[6])

There are two distinct ultraviolet regularizations in the stochastic quantization procedure. The first (original) is to keep fictitious time τ finite. It is known that this is not sufficient to regularize all the perturbative terms. Thus, Breit, Gupta, and Zaks introduced an additional regularization on the random average procedure. We use this regularization scheme to compute (1.8).

II. STOCHASTIC QUANTIZATION OF FERMI FIELDS (REF. 7)

In this section, the stochastic quantization of Fermi fields is reviewed. The specific formalism we are going to use is due to Sakita. Since this work was published in an obscure publication[7] we shall outline only the significant points.

The path-integral expression of Fermi fields to be considered is given by

$$\int D\psi D\bar{\psi}(\cdots)e^{-S[\psi,\bar{\psi}]}\Big/\int D\psi D\bar{\psi}\, e^{-S[\psi,\bar{\psi}]}, \quad (2.1)$$

where ψ and $\bar{\psi}$ are independent Grassmann variables. We restrict S to be a bilinear form

$$S=\int dx\, \bar{\psi}(x)G\psi(x)\ , \quad (2.2)$$

since we can always bilinearize S by introducing a set of auxiliary fields. G may contain not only derivative operators but also other (external) fields.

In general,

$$G^\dagger \neq G\ . \quad (2.3)$$

An appropriate form for the Fokker-Planck Hamiltonian is given by

$$\begin{aligned} H_{\rm FP}&=\int dx\left[\frac{\delta}{\delta\psi(x)}G^\dagger\left[\frac{\delta}{\delta\bar{\psi}(x)}+\frac{\delta S}{\delta\bar{\psi}(x)}\right]-\frac{\delta}{\delta\bar{\psi}(x)}G^{T\dagger}\left[\frac{\delta}{\delta\psi(x)}+\frac{\delta S}{\delta\psi(x)}\right]\right]\\ &=\int dx\left[\frac{\delta}{\delta\psi(x)}G^\dagger\left[\frac{\delta}{\delta\bar{\psi}(x)}+G\psi(x)\right]-\frac{\delta}{\delta\bar{\psi}(x)}G^{T\dagger}\left[\frac{\delta}{\delta\psi(x)}-G^T\bar{\psi}(x)\right]\right], \end{aligned} \quad (2.4)$$

which has been proved[7] that it has positive-definite eigenvalues.

The corresponding Langevin equations are

$$\begin{aligned} \frac{\partial}{\partial\tau}\psi(x,\tau)&=-G^\dagger G\psi(x,\tau)+\tfrac{1}{2}G^\dagger\eta_1+\eta_2\ ,\\ \frac{\partial}{\partial\tau}\bar{\psi}(x,\tau)&=-(GG^\dagger)^T\bar{\psi}(x,\tau)+\bar{\eta}_1+\tfrac{1}{2}(G^\dagger)^T\bar{\eta}_2\ , \end{aligned} \quad (2.5)$$

where

$$\langle \eta_\alpha(x,\tau)\bar{\eta}_\beta(x',\tau')\rangle_\eta=-\langle\bar{\eta}_\beta(x',\tau')\eta_\alpha(x,\tau)\rangle_\eta=2\delta_{\alpha\beta}\delta(x-x')a_\Lambda(\tau-\tau')\ . \quad (2.6)$$

The general expression for the η average is given by

$$\langle\cdots\rangle_\eta=\lim_{\Lambda\to\infty}\frac{\int D\bar{\eta}D\eta\cdots\exp\left[-\frac{1}{2}\int dx\, d\tau d\tau' a_\Lambda(\tau-\tau')\bar{\eta}(x,\tau)\eta(x,\tau')\right]}{\int D\bar{\eta}D\eta\exp\left[-\frac{1}{2}\int dx\, d\tau d\tau' a_\Lambda(\tau-\tau')\bar{\eta}(x,\tau)\eta(x,\tau')\right]}\ , \quad (2.7)$$

where $a_\Lambda(\tau-\tau')$ is a symmetric regulator function, which has the following properties:

$$\begin{aligned} &a_\Lambda(\tau)=a_\Lambda(-\tau)\ ,\\ &\int d\tau' a_\Lambda(\tau-\tau')=1\ ,\\ &\lim_{\Lambda\to\infty} a_\Lambda(\tau-\tau')=\delta(\tau-\tau')\ . \end{aligned} \quad (2.8)$$

III. CALCULATION OF $\langle\delta S\rangle$

Our action is in bilinear form

$$S=\int\bar{\psi}(x)\not{D}\psi(x)dx\ . \quad (3.1)$$

ψ and $\bar{\psi}$ are Fermi fields and $\not{D}$ contains derivatives as well as external fields.

The Langevin equations taken from (2.5) are

$$\begin{aligned} \frac{\partial}{\partial\tau}\psi(x,\tau)&=-\not{D}^\dagger\not{D}\psi(x,\tau)+\not{D}^\dagger\eta(x,\tau)\ ,\\ \frac{\partial}{\partial\tau}\bar{\psi}(x,\tau)&=-(\not{D}\not{D}^\dagger)^T\bar{\psi}(x,\tau)+\bar{\eta}(x,\tau)\ , \end{aligned} \quad (3.2)$$

where we assume that $\not{D}\not{D}^\dagger$ does not contain the zero-mode energy. According to our prescription for the correct definition of (1.8), we need a small energy for the zero mode. Nevertheless, allowing this to be zero at the end of the calculations we recover the right theory.

The stochastic quantization prescription gives the quantity to be computed as

$$\langle\delta S[\bar{\psi},\psi]\rangle=\lim_{\tau\to\infty}\langle\delta S[\bar{\psi}_\eta(x,\tau),\psi_\eta(x,\tau)]\rangle_\eta\ , \quad (3.3)$$

where $\psi_\eta(x,\tau)$ and $\bar{\psi}_\eta(x,\tau)$ are the solutions of Eqs. (3.2). These give the evolution of the Fermi fields $\bar{\psi}$ and ψ with respect to the fictitious time τ as

$$\begin{aligned} \bar{\psi}_\eta(x,\tau)&=\int_0^\tau\bar{\eta}(x,\tau_1)e^{-\not{D}\not{D}^\dagger(\tau-\tau_1)}d\tau_1\ ,\\ \psi_\eta(x,\tau)&=\int_0^\tau e^{-\not{D}^\dagger\not{D}(\tau-\tau_2)}\not{D}^\dagger\eta(x,\tau_2)d\tau_2\ . \end{aligned} \quad (3.4)$$

Then the η average of δS is specified by

$$\langle \delta S[\bar\psi_\eta(x,\tau),\psi_\eta(x,\tau)]\rangle_\eta=\left\langle -\int \bar\psi_\eta(x,\tau)\{\epsilon(x)\gamma_5,\not D\}\psi_\eta(x,\tau)d^4x\right\rangle_\eta$$
$$=\left\langle -\int d^4x\,d^4y\int_0^\tau d\tau_1\int_0^\tau d\tau_2\bar\eta_\alpha(x,\tau_1)e^{-\not D\not D^\dagger(\tau-\tau_1)}\{\epsilon(x)\gamma_5,\not D\}e^{-\not D^\dagger\not D(\tau-\tau_2)}\not D^\dagger\eta_\beta(y,\tau_2)\right\rangle_\eta . \quad (3.5)$$

Using the relation (2.6) and performing the η average of the noise function we have

$$\langle \delta S[\bar\psi_\eta(x,\tau),\psi_\eta(x,\tau)]\rangle_\eta=-\lim_{\Lambda\to\infty}\int d^4x\int_0^\tau d\tau_1\int_0^{\tau_2}2a_\Lambda(\tau_1-\tau_2)\mathrm{tr}[e^{-\not D\not D^\dagger(\tau-\tau_1)}\{\epsilon(x)\gamma_5,\not D\}e^{-\not D^\dagger\not D(\tau-\tau_2)}\not D^\dagger] . \quad (3.6)$$

We compute the last trace

$$\mathrm{tr}[e^{-\not D\not D^\dagger(\tau-\tau_1)}\{\epsilon(x)\gamma_5,\not D\}e^{-\not D^\dagger\not D(\tau-\tau_2)}\not D^\dagger]=\mathrm{tr}[\epsilon(x)\gamma_5\not D\not D^\dagger e^{-\not D\not D^\dagger(2\tau-\tau_1-\tau_2)}]+\mathrm{tr}[\epsilon(x)\gamma_5\not D^\dagger\not D e^{-\not D^\dagger\not D(2\tau-\tau_1-\tau_2)}] . \quad (3.7)$$

$\not D\not D^\dagger$ and $\not D^\dagger\not D$ are two Hermitian operators, which pick only even values of the gauge field since

$$\not D^\dagger(A_\mu)=\not D(-A_\mu) . \quad (3.8)$$

Then

$$\langle \delta S[\bar\psi,\psi]\rangle=\lim_{\tau\to\infty}\langle \delta S[\bar\psi_\eta(x,\tau),\psi_\eta(x,\tau)]\rangle_\eta$$
$$=-\lim_{\tau\to\infty}\lim_{\Lambda\to\infty}\int d^4x\int_0^\tau d\tau_1\int_0^\tau d\tau_2 2a_\Lambda(\tau_1-\tau_2)\{\mathrm{tr}[\epsilon(x)\gamma_5\not D\not D^\dagger e^{-\not D\not D^\dagger(2\tau-\tau_1-\tau_2)}]$$
$$+\mathrm{tr}[\epsilon(x)\gamma_5\not D^\dagger\not D e^{-\not D^\dagger\not D(2\tau-\tau_1-\tau_2)}]\} . \quad (3.9)$$

First we perform the integration over time τ. We change the variables

$$\tau_1-\tau_2=t,\quad \frac{\tau_1+\tau_2}{2}=T . \quad (3.10)$$

Then the first term of (3.9) becomes

$$-\lim_{\tau\to\infty}\lim_{\Lambda\to\infty}\int d^4x\left[\int_0^{\tau/2}dT\int_{-2T}^{2T}dt\,2a_\Lambda(t)+\int_{\tau/2}^{\tau}dT\int_{-2(\tau-T)}^{2(\tau-T)}dt\,2a_\Lambda(t)\right]\mathrm{tr}[\epsilon(x)\gamma_5\not D\not D^\dagger e^{-2\not D\not D^\dagger(\tau-T)}] . \quad (3.11)$$

We keep Λ finite during the integration over τ. Because of the relations (2.7) the regulator function gives the following:

$$\int_{-2T}^{2T}dt\,a_\Lambda(t)=1 \text{ only if } 2T>\frac{1}{\Lambda} \text{ or } T>\frac{1}{2\Lambda} ,$$
$$\int_{-2(\tau-T)}^{2(\tau-T)}dt\,a_\Lambda(t)=1 \text{ only if } 2(\tau-T)>\frac{1}{\Lambda} \text{ or } T<\tau-\frac{1}{2\Lambda} , \quad (3.12)$$

and (3.11) takes the form

$$-\lim_{\tau\to\infty}\lim_{\Lambda\to\infty}\int d^4x\left[\left[\int_{1/2\Lambda}^{\tau/2}dT\,2+\int_{\tau/2}^{\tau-1/2\Lambda}dT\,2\right]\mathrm{tr}[\epsilon(x)\gamma_5\not D\not D^\dagger e^{-2\not D\not D^\dagger(\tau-T)}]\right]$$
$$=-\lim_{\tau\to\infty}\lim_{\Lambda\to\infty}\int d^4x\,\mathrm{tr}\left[\epsilon(x)\gamma_5\int_{1/2\Lambda}^{\tau-1/2\Lambda}dT\,2\not D\not D^\dagger e^{-2\not D\not D^\dagger(\tau-T)}\right]$$
$$=-\lim_{\tau\to\infty}\lim_{\Lambda\to\infty}\int d^4x\,\mathrm{tr}[\epsilon(x)(\gamma_5 e^{-2\not D\not D^\dagger/2\Lambda}-\gamma_5 e^{-2\not D\not D^\dagger(\tau-1/2\Lambda)})] . \quad (3.13)$$

As we already discussed, $\not D\not D^\dagger$ has a small finite value for the zero mode. Thus we can take the limit $\tau\to\infty$, while $\not D\not D^\dagger$ is finite. Then (3.13) gives

$$-\lim_{\Lambda\to\infty}\int d^4x\,\mathrm{tr}[\epsilon(x)\gamma_5 e^{-\not D\not D^\dagger/\Lambda}] . \quad (3.14)$$

The trace in the last expression can be evaluated by using the plane-wave basis.[3] Taking both terms of (3.9) into account we have

$$\langle \delta S[\bar{\psi},\psi]\rangle = -\lim_{\Lambda\to\infty}\lim_{x\to y}\int d^4x \left[\mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 e^{-\not{D}\not{D}^\dagger/\Lambda} e^{ik(x-y)} + \mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 e^{-\not{D}^\dagger\not{D}/\Lambda} e^{ik(x-y)} \right]$$

$$= -\lim_{\Lambda\to\infty}\lim_{x\to y}\int d^4x \left[\mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 e^{-(D^\mu D_\mu+\sigma^{\nu\mu}F_{\nu\mu}/2)/\Lambda} e^{ik(x-y)} \right.$$

$$\left. + \mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 e^{-(\tilde{D}^\mu \tilde{D}_\mu+\sigma^{\nu\mu}\tilde{F}_{\nu\mu}/2)/\Lambda} e^{ik(x-y)} \right]$$

$$= -\lim_{\Lambda\to\infty}\int d^4x \left[\mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 \exp\left[-[(D^\mu+ik^\mu)(D_\mu+ik_\mu)+\tfrac{1}{2}\sigma^{\nu\mu}F_{\nu\mu}]\frac{1}{\Lambda}\right] \right.$$

$$\left. + \mathrm{tr}\epsilon(x)\int \frac{d^4k}{(2\pi)^4}\gamma_5 \exp\left[-[(\tilde{D}^\mu+ik^\mu)(\tilde{D}_\mu+ik_\mu)+\tfrac{1}{2}\sigma^{\nu\mu}\tilde{F}_{\nu\mu}]\frac{1}{\Lambda}\right] \right] , \quad (3.15)$$

where

$$D_\mu=\partial_\mu - iV_\mu + i\gamma_5 A_\mu,\quad \tilde{D}_\mu=\partial_\mu - iV_\mu - i\gamma_5 A_\mu,\quad F_{\nu\mu}=F'_{\nu\mu}+\gamma_5 f_{\nu\mu},\quad \tilde{F}_{\nu\mu}=F'_{\nu\mu}-\gamma_5 f_{\nu\mu}\ , \quad (3.16)$$

and

$$F'_{\nu\mu} = -(\partial_\nu V_\mu - \partial_\mu V_\nu) + i[V_\nu,V_\mu] + i[A_\nu,A_\mu],\quad f_{\nu\mu} = (\partial_\nu A_\mu - \partial_\mu A_\nu) - i[V_\nu,A_\mu] + i[V_\mu,A_\nu]\ . \quad (3.17)$$

After the expansion of (3.15) and rescaling

$$k^\mu \to \sqrt{\Lambda}k^\mu\ , \quad (3.18)$$

we have

$$\langle \delta S[\bar{\psi},\psi]\rangle = -\lim_{\Lambda\to\infty}\int d^4x\, \mathrm{tr}\epsilon(x)\gamma_5 \left[\left[\frac{\sigma^{\nu\mu}F_{\nu\mu}}{2}\right]^2 + \left[\frac{\sigma^{\nu\mu}\tilde{F}_{\nu\mu}}{2}\right]^2 \right] \frac{\Lambda^2}{2!\Lambda^2}\int \frac{d^4k}{(2\pi)^4} g''(k^2)\ , \quad (3.19)$$

where we replaced $e^{-k^2/\Lambda}$ by $g(k^2/\Lambda)$.

The result of (3.19) is independent of the specific form of $g(z)$, if we require that $g(z)$ is any smooth function which approaches zero as $z=\infty$,

$$g(\infty)=g'(\infty)=g''(\infty)=\cdots=0$$

and

$$g(0)=1\ . \quad (3.20)$$

In obtaining (3.19) all other terms vanish either due to the trace operation over the γ matrices or because they contain the factor $1/\Lambda^l$, $l>0$. Finally, using

$$\int \frac{d^4k}{(2\pi)^4} g''(k^2) = \frac{1}{16\pi^2} \quad (3.21)$$

we obtain

$$\langle \delta S\rangle = -\frac{1}{16\pi^2}\frac{1}{8}\mathrm{tr}\int d^4x\, \epsilon(x)\gamma_5[(\sigma^{\nu\mu}F_{\nu\mu})^2 + (\sigma^{\nu\mu}\tilde{F}_{\nu\mu})^2]$$

$$= -\frac{1}{16\pi^2}\mathrm{tr}\int d^4x\, \epsilon(x)\epsilon^{\nu\mu\rho\sigma}(F'_{\nu\mu}F'_{\rho\sigma} + f_{\nu\mu}f_{\rho\sigma})\ , \quad (3.22)$$

where $F, F', \tilde{F}, f$ are given at (3.16) and (3.17).

IV. CONCLUSION

We have shown by explicit calculation that the stochastic quantization method can be applied to the evaluation of the chiral anomaly. Notice, however, that it would be impossible to achieve the result without the specific regularization scheme which we used. This method of regularization—stochastic regularization—due to Breit, Gupta, and Zaks respects the chiral-gauge symmetry of the system and recovers the unregularized theory at the limit $\Lambda\to\infty$. The crucial point in our calculation was the regularization procedure in which Λ was kept finite while the limit $\tau\to\infty$ was taken. Then the limit $\Lambda\to\infty$ is taken. Also the infrared regulator was kept finite but nonzero during all calculations. Letting it go to zero at the end of the calculations the right theory can be recovered. We also emphasize the important role played in this work by the computation of the Langevin equations for anticommuting Fermi fields.[7]

Our result for the chiral anomaly is in covariant form. According to Bardeen and Zumino,[8] the chiral anomaly can assume covariant form when the definition of the composite current is covariant. In that sense the "consistent" anomaly, which can be obtained by varying the functional with respect to gauge potential and the "covariant" anomaly, are allowable.[9–11]

When we had completed our calculation we received a paper prior to publication from Alfaro and Gavela.[12] Our prescription for infrared regularization which we used to avoid ambiguities at the zero mode (assuming $\not{D}\not{D}^\dagger$ different than zero), is equivalent to that used by Alfaro and Gavela (assuming a finite mass in the Langevin equations).

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