Color radiation in the classical Yang-Mills theory

C. H. Oh and C. H. Lai
Department of Physics, National University of Singapore, Kent Ridge, Singapore 0511, Republic of Singapore

Rosy Teh
MARA Institute of Technology, Arau, Perlis, Malaysia
(Received 11 October 1984)

The total gauge-invariant color of a line source in Yang-Mills theory may or may not change as a result of color radiation, depending on the non-Abelian waves considered. We illustrate this by explicit examples.

I. INTRODUCTION

In classical electromagnetic theory, a point source, when accelerated, will radiate electromagnetic waves which carry energy and momentum, but the total charge of the source remains constant. In the non-Abelian Yang-Mills (YM) theory, the gluon field, apart from carrying energy and momentum, also possesses a non-Abelian charge (color) and furthermore, unlike the Abelian case, the total color of an external source need not necessarily be conserved. Thus it appears that the total color of an external source can be radiated away. The answer is, however, not so straightforward and it requires some careful analysis.[1–4] It has been shown by Trautman and others that for a single point source its color content will remain constant although gluon waves are radiated out, provided that the non-Abelian gauge group is compact and semisimple. In their analyses they have made use of the Abelian solution—the Lienard-Wiechert potential. For a system of point sources they[4] indicate that the total color may change as a result of the gluon radiation. Actually from the gauge-covariant conservation of the external source current j_μ^a, one anticipates that the integral of $j_0^a j_0^a$ can be time dependent as long as j_i^a is nonvanishing. That this integral may vary is either due to the flow of the current density j_i^a in the source region or due to the color exchange between the external source and the YM field.

In this paper, we consider, instead of a point source, a continuous line source lying along the x_3 axis. This line source must vanish sufficiently fast when $|x_3|$ tends to infinity so that the total source strength is finite. The non-Abelian wave solutions are then so constructed as to correspond to a line source. We find that, depending on the gluon field constructed, the total color of an external line source may or may not change as a result of color radiation. Although this is expected for an extended non-Abelian source, to our knowledge so far there are no explicit solutions illustrating this phenomenon.

In discussing the color radiation problem, we must first clearly define in a gauge-invariant manner the total color of the system (the external source together with the gluon field), that of the external source and that of the gluon field. These quantities could be susceptible to changes by arbitrary gauge transformations if they are not carefully defined.[4–6] The next stage is then to construct explicit non-Abelian wave solutions. In the following section we introduce our notations and describe the gauge-invariant color charge. In Sec. III we present non-Abelian wave solutions of a line source lying along the x_3 axis. We find that the gauge-invariant color charge of the external source and that of the YM field are nonzero and time dependent although their sum, which is conserved, vanishes. There is exchange of color between the external source and the YM field as time proceeds. Another type of non-Abelian wave solution in which the YM field carries no color, so that the gauge-invariant color of the external source is a constant, is illustrated in Sec. IV. For these types of solution there is no color transfer between the external source and the YM waves although the latter do transport energy and momentum. We end with some remarks in Sec. V.

II. GAUGE-INVARIANT COLOR CHARGES

The YM field equations in the presence of an external source are

$$D_\mu F^{\mu\nu} = j^\nu \ , \tag{1a}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu] \ , \tag{1b}$$

$$A_\mu = A_\mu^a (\sigma^a / 2i) \ , \tag{1c}$$

where σ^a are the Pauli matrices and our metric is $\mathrm{diag}(g_{\mu\nu}) = (-+++)$; the gauge-field coupling constant g is set equal to one. The external current j_μ transforms gauge covariantly and is gauge-covariantly conserved:

$$D_\mu j^\mu = 0 \ . \tag{2}$$

Hence even though j_i vanishes sufficiently fast at large distances, the quantity

$$\bar{Q}^a = \int d^3x \, j_0^a \tag{3}$$

is neither necessarily conserved nor gauge covariant. A gauge-invariant specification of the total external source is[6]

$$Q_S=\int d^3x\, j_a^0\eta^a\ , \tag{4}$$

where $\eta^a(x)$ is a unit vector in the internal group space and it transforms gauge covariantly. Note that if η^a is also gauge-covariantly constant then

$$\partial_\mu(j^{a\mu}\eta^a)=0\ , \tag{5}$$

but here we shall not impose such a condition on $\eta^a(x)$. For the total color of the system a perceptive expression is

$$I=I^a\frac{\sigma^a}{2i}=\int d^3x(j^0+[A_i,F^{i0}])\ . \tag{6}$$

If we restrict gauge transformations to those which are independent of space-time coordinates at large distances which in turn means we only consider YM connections that vanish faster than $1/r$ at large r, then I^a is a gauge-covariant and a gauge-invariant conserved total color charge is

$$Q=(I^aI^a)^{1/2}\ . \tag{7}$$

Expression (6) is however not suitable for the purpose of determining whether there is color exchange between the external source and the non-Abelian waves because neither (6) nor (7) can be written in a gauge-independent manner as a sum of the gauge-invariant color of the external source and that of the YM field. In particular the part

$$I_F=\int d^3x\,[A_i,F^{i0}] \tag{8}$$

in Eq. (6) is highly gauge dependent and cannot be used to indicate the color content of the YM field although unfortunately it has been employed in Ref. 4.

The gauge-invariant conserved color current density $\mathcal{J}^\nu$, introduced in Ref. 6, is appropriately useful for the purpose of this paper:

$$\mathcal{J}^\nu=\eta^aj^{a\nu}+(D_\mu\mu)^aF^{a\mu\nu}\equiv\mathcal{J}_S^\nu+\mathcal{J}_F^\nu\ . \tag{9}$$

Here the scalar field $\eta^a(x)$ transforms as the adjoint representation of the gauge group in question and $\eta^a(x)\eta^a(x)=1$. Evidently the decomposition of $\mathcal{J}^\nu$ into the gauge-invariant color current density $\mathcal{J}_S^\nu$ of the external source and that $\mathcal{J}_F^\nu$ of the YM field is gauge independent. The total gauge-invariant color of the whole system is defined by

$$Q_T=\int d^3x\,\mathcal{J}^0 \tag{10}$$

and that of the YM field is

$$Q_F=\int d^3x\,\mathcal{J}_F^0\ . \tag{11}$$

Thus

$$Q_T=Q_S+Q_F\ , \tag{12}$$

where Q_S is as given by expression (4).

III. YANG-MILLS WAVES WITH COLOR

We now proceed to next stage of work: construction of wave solutions to Eq. (1). As mentioned previously our wave solutions are associated with a continuous line source along the x_3 axis. Currently there exists no systematic method and after some effort we find the following gauge field potential will serve our purpose:

$$A_\mu^a(x)=\partial_\mu U\left[\phi f_1^a(U)+\left[\ln\tan\frac{\theta}{2}\right]f_2^a(U)+h^a(U)\right]\ , \tag{13a}$$

$$U=r\mp x^0\ , \tag{13b}$$

$$r=(x_1{}^2+x_2{}^2+x_3{}^2)^{1/2},\ \ x^0=t\ . \tag{13c}$$

Here $f_1^a(U)$, $f_2^a(U)$, and $h^a(U)$ are arbitrary functions of U, θ, and ϕ are spherical coordinates and we shall take $U=r-x^0$. The electric and magnetic field strengths are, respectively, given by

$$E_i^a=F_{i0}^a=-\frac{1}{\rho}(\hat{\boldsymbol{\phi}}_if_1^a+\hat{\boldsymbol{\theta}}_if_2^a)\ , \tag{14a}$$

$$B_i^a=\tfrac{1}{2}\epsilon_{ijk}F^{ajk}=\frac{1}{\rho}(\hat{\boldsymbol{\theta}}_if_1^a-\hat{\boldsymbol{\phi}}_if_2^a)\ , \tag{14b}$$

where $\hat{\boldsymbol{\phi}}_i$ and $\hat{\boldsymbol{\theta}}_i$ are unit vectors,

$$\hat{\boldsymbol{\phi}}_i=\epsilon_{i3j}x_i/\rho,\ \ \rho^2=x_1{}^2+x_2{}^2\ , \tag{15a}$$

$$\hat{\boldsymbol{\theta}}=(\delta_i^Ax_3x_A/\rho-\delta_i^3\rho)/r\ , \tag{15b}$$

and the index A takes values 1 and 2 only. The energy density T_{00} and momentum density T_{0i} can be easily evaluated

$$T_{0i}=n_iT_{00}=n_i(f_1{}^2+f_2{}^2)/\rho^2\ , \tag{16a}$$

$$n_i=x_i/r,\ \ f_A{}^2=f_A^af_A^a,\ \ A=1,2\ . \tag{16b}$$

This expression (16) indicates that the wave propagates radially outward, but the amplitude varies across the spherical wave front. For this reason we say that solution (13) corresponds to non-Abelian spherical fronted wave. Note that

$$E_i^aB_i^a=T_{0i}B_i^a=T_{0i}E_i^a=0\ .$$

Furthermore, as the energy and momentum densities are equal in magnitude, the wave field, when quantized, describes a massless particle. Expressions (14), (15), and (16) suggest that there may be a line source along the x_3 axis. We find indeed from solution (13)

$$j_i^a=-n_ij_0^a=2\pi\delta(x_1)\delta(x_2)n_in_3f_2^a(U)\ . \tag{17}$$

It is straightforward to verify that expression (17) does satisfy the constraint $D_\mu j^\mu=0$.

The total color of the external source is given by

$$\begin{aligned}Q_s&=\int d^3x\,\eta^aj^{a0}\\&=2\pi\int_{-\infty}^{\infty}dx_3\frac{x_3}{|x_3|}f_2^a(|x_3|-t)\eta^a\ .\end{aligned} \tag{18}$$

For the color content of the YM field we find

$$Q_F=\int d^3x\,\partial_i\eta^aF_a^{i0}=-\int d^3x\,\eta^aj^{a0}\ , \tag{19}$$

where the surface integral of $(\eta^aF_a^{i0}n_i)$ vanishes identically by using Eq. (14a). Thus at any instant and for any choice of η^a, the YM field always screens the external source so that the total color of the system Q_T vanishes. To be more explicit we choose

$$\eta^a = j^{a0}/|j^{a0}| = \frac{x_3}{|x_3|} f_2^a (f_2{}^2)^{-1/2} \tag{20}$$

and set

$$f_2{}^2(U) = \mathrm{sech}^4(r-t) , \tag{21}$$

whence we obtain

$$Q_S = 4\pi(1+\tanh t) , \tag{22a}$$

$$Q_F = -4(1+\tanh t) . \tag{22b}$$

In this case the source density $\eta^a j^{a0}$ has a maximum at the origin at time $t=0$. As t increases, the maximum splits into two maxima located at $x_3 = \pm t$ on the x_3 axis, which move apart along the x_3 axis at the speed of light. The total source (gauge field) color assumes the value 4π (-4π) at $t=0$ and when $t\to\infty$ it increases (decreases) to a maximum (minimum) of 8π (-8π). This means that there is an inflow of color from the gauge field to the external source. Had we chosen $U=r+t$ instead of $U=r-t$, then

$$Q_S = -Q_F = 4\pi(1-\tanh t) .$$

Hence in the process of color radiation, the flow of color is from the external source to the gauge field since initially the source color $Q_s = 4\pi$ and finally when $t=\infty$, $Q_S = 0$. Thus the non-Abelian wave as given by expression (13), apart from carrying energy and momentum, also transports color.

IV. COLORLESS YANG-MILLS WAVES

We now investigate another kind of wave solution in which the total source color remains constant in the process of emitting YM waves. For this purpose we write

$$A_\mu^a = \partial_\mu U (x_1 g_1^a + x_2 g_2^a + \ln\rho g_3^a + \phi g_4^a) , \tag{23a}$$

$$U = x_3 \pm t , \tag{23b}$$

where the $g_{\bar\lambda}^a$, $\bar\lambda = 1,2,3,4$ are arbitrary functions of U and we choose $U=(x_3-t)$. The field strengths are

$$E_i^a = -\delta_i^1 g_1^a - \delta_i^2 g_2^a - (\hat\rho_i g_3^a + \hat\phi_i g_4^a)/\rho , \tag{24a}$$

$$B_i^a = \delta_i^1 g_2^a - \delta_i^2 g_1^a + (\hat\rho_i g_4^a - \hat\phi_i g_3^a)/\rho , \tag{24b}$$

where

$$\hat\rho_i = \delta_i^A x_a/\rho, \quad A=1,2 . \tag{24c}$$

The momentum and energy densities can be computed and we arrive at

$$\begin{aligned} T_{0i} = \delta_i^3 T_{00} = \delta_i^3 \rho^{-2} [& \rho^2 (g_1{}^2 + g_2{}^2) + g_3{}^2 + g_4{}^2 \\ & + x_1 (g_1^a g_3^a + g_2^a g_4^a) \\ & + x_2 (g_2^a g_3^a - g_1^a g_4^a)] . \end{aligned} \tag{25}$$

Both quantities are bounded whenever g_1^a and g_2^a are bounded and $g_3^a = g_4^a = 0$. The energy flows along the x_3 direction and the amplitude of the wave varies over the wave front, this indicates that solution (23) describes a plane-fronted wave.[7] The source density for solution (23) is found to be

$$j_i^a = -\delta_i^3 j_0^a = \delta_i^3 (2\pi) \delta(x_1) \delta(x_2) g_3^a(U) . \tag{26}$$

It is easy to check that $D_\mu j^\mu = 0$ as expected.

The total color charge of the external source is found to be

$$Q_S = 2\pi \int_{-\infty}^{\infty} dx_3 g_3^a(U) \eta^a , \tag{27}$$

while that of the gauge field is

$$Q_F = \int d^3x\, \partial_i \eta^a F_a^{i0} = \int dS\, n_i F_a^{i0} \eta^a - \int d^3x\, \eta^a j_a^0 . \tag{28}$$

Using expression (24a) for F_a^{i0} and choosing the surface of a cylinder, we find

$$\int dS\, n_i F_a^{i0} \eta^a = 2\pi \int_{-\infty}^{\infty} dx_3 g_3^a(U) \eta^a .$$

This means that the gauge field conveys no color and the total color of the system is contributed by the external source only,

$$Q_T = Q_S \tag{29}$$

which is a constant. As an illustration of the above, we put

$$\eta^a = j^{a0}/|j^{a0}| = g_3^a(U)(g_3{}^2)^{-1/2} \tag{30}$$

and

$$g_3{}^2 = g_3^a g_3^a = \mathrm{sech}^4(x_3 - t) . \tag{31}$$

We find

$$Q_T = Q_S = -4\pi . \tag{32}$$

In contrast with expression (22a), Q_S here is a constant. Thus although the gauge field (23) carries energy and momentum, it has no color and consequently there is no flow of color between the external source and the gauge field.

From the above we learn that for a non-Abelian line source emitting non-Abelian waves, the total source color may or may not alter depending on the non-Abelian wave solutions considered and the expression for the source function.

V. COMMENTS

We make some remarks.

(a) We note that solution (13) is a special case of the non-Abelian spherical plane-fronted waves discussed in Ref. 8. For both solutions (13) and (23), the external source densities and the field strengths are null since

$$j_\mu^a j^{a\mu} = 0 , \tag{33a}$$

$$\epsilon_{\mu\nu\alpha\beta} F_a^{\mu\nu} F_a^{\alpha\beta} = 0, \quad F_{\mu\nu}^a F^{b\mu\nu} = 0 . \tag{33b}$$

(b) Although the wave solutions (13) and (23) satisfy

$$[A_\mu, A_\nu] = 0 , \tag{34a}$$

$$[A_\mu, F^{\mu\nu}]=0 \tag{34b}$$

they are essentially non-Abelian since

$$[A_\mu, F^{\alpha\beta}]\neq 0 \ . \tag{35}$$

This is also the sense in which the Coleman plane wave is non-Abelian.[9]

(c) Solution (23) can be regarded as a linear superposition of the Coleman plane wave[9] and a plane-fronted wave solution since we can decompose Eq. (23a) as

$$A_\mu^a = b_\mu^a + W_\mu^a \ , \tag{36a}$$

where

$$b_\mu^a = (\delta_\mu{}^3 - \delta_\mu^0)[x_1 g_1^a(U) + x_2 g_2^a(U)] \tag{36b}$$

is the Coleman potential and

$$W_\mu^a = (\delta_\mu{}^3 - \delta_\mu^0)[\ln\rho g_3^a(U) + \phi g_4^a(U)] \tag{36c}$$

is the plane-fronted wave potential.

(d) From remark (c) it follows that solution (23) is the sourceless Coleman plane-wave solution when $g_3^a(U)=g_4^a(U)=0$. Thus although the Coleman plane wave transports energy and momentum it carries no color since $Q_F=0$. The authors of Ref. 4 computed the quantity I_F as defined in Eq. (8) here for the Coleman plane wave and found that it is zero in one gauge choice and nonzero in the transverse gauge. As I_F is highly gauge dependent, one should use the gauge-invariant quantity Q_F and not I_F, to determine the color content of the gluon wave. Using Q_F, we conclude that the Coleman non-Abelian wave is not a truly colored wave.

(e) For solutions (13) and (23), the gauge-covariant conservation of the external source current, Eq. (2), becomes

$$\partial_\mu j^\mu = 0 \ . \tag{37}$$

However this conservation of j^μ is due to our particular gauge choice. In a different gauge choice, Eq. (2) is always valid but not Eq. (37).

(f) The actual value of the gauge-invariant total color Q_T as well as those of the external source color Q_S and gauge field color Q_F depend on the scalar function $\eta^a(x)$. However, our conclusion, that for solution (13) the non-Abelian wave does carry color opposite to that of the external source and for solution (23) the non-Abelian wave has no color, is valid for any choice of η^a.

(g) Solution (13) carries no color if the function $f_2^a(U)$ vanishes. In order for the total color of the external source to have a finite value, $f_2^a(U)$ must vanish sufficiently fast at large distances. Thus the part of the non-Abelian wave which has color, namely, the second term in Eq. (13a), resides mainly at the vicinity of the external source. In other words the colored wave is more or less confined.

ACKNOWLEDGMENT

One of us (C.H.O.) wishes to thank Professor N. P. Chang for very helpful discussions.

[1] A Trautman, Phys. Rev. Lett. **46**, 875 (1981).
[2] W. Drechsler, Phys. Lett. **90B**, 258 (1980).
[3] W. Drechsler and A. Rosenblum, Phys. Lett. **106B**, 81 (1981).
[4] J. Tafel and A. Trautman, J. Math. Phys. **24**, 1087 (1983).
[5] S. Schlieder, Nuovo Cimento **63A**, 137 (1981).
[6] C. H. Lai and C. H. Oh, Phys. Rev. D **29**, 1805 (1984).
[7] W. B. Bonner, Nature (London) **225**, 932 (1970).
[8] W. B. Campbell and T. A. Morgan, Phys. Lett. **84B**, 87 (1979).
[9] S. Coleman, Phys. Lett. **70B**, 59 (1977).