Monte Carlo study of the four-dimensional anisotropic U(1) gauge theory (lattice dimensional reduction)

Neelima Gupte*
International Centre for Theoretical Physics, 34100 Trieste, Italy

E. Katznelson
Physikalisches Institut der Universität Bonn, Nussallee 12, D-5300 Bonn 1, Federal Republic of Germany

(Received 16 September 1985)

We study the four-dimensional U(1) gauge theory with anisotropic couplings (β_1 for the XY, XZ, XT plaquettes and β_2 for the YZ, YT, ZT plaquettes) by a Monte Carlo simulation. We obtain the phase diagram and find that the effects of the anisotropy are not washed out by the phase transition in the case of the plaquette expectation values. Our phase diagram confirms the phase diagram obtained by mean-field theory with one-loop corrections.

It has been seen recently that lattice gauge theories with anisotropic values of the coupling constant have a rich phase structure.[1-3] The study of such theories has also given interesting insight into questions of universality and Lorentz invariance (rotational invariance on the Euclidean lattice).[4]

Consider the action

$$S=\sum_{\mu>\nu} E_{\mu\nu}\,\mathrm{Tr}(UUUU)\ . \tag{1}$$

The U's are elements of the group U(1) and lie on the links of a four-dimensional hypercubic lattice. The product $UUUU$ is taken around the elementary plaquette. Each orientation of the elementary plaquette carries a different weight $E_{\mu\nu}$. Thus, in four dimensions there are six possible weights $E_{\mu\nu}$.

It has been shown that for the deconfining phase of the sixfold anisotropic Villain gauge theory, the plaquette-plaquette correlation function satisfies rotational invariance if and only if the $E_{\mu\nu}$ satisfy the factorizability condition $E_{\mu\nu}=E_\mu E_\nu$.[4]

However, it remains to be shown that the effects of anisotropy are not washed out at the critical point. The phase diagram of the theory is also of interest, both in the context of the work of Ref. 4 and the mean-field calculations of Nielsen and Fu[3] for the anisotropic U(1) gauge theory.

We use the method of Monte Carlo (MC) simulation to study the action

$$S=\beta_1\sum_x \cos(\theta_{p,x})+\beta_2\sum\cos(\theta_{p,\bar{x}})\ , \tag{2}$$

where $\theta_{p,x}$ is a plaquette with two x-directed links and $\theta_{p,\bar{x}}$ is a plaquette with no x-directed links,

$$\theta_p=\Delta_\mu\theta_\nu-\Delta_\nu\theta_\mu\ , \tag{3}$$

and θ_μ lies on a link in the μ direction. The plaquettes XY, XZ, XT are associated with a coupling β_1, and YZ, YT, ZT with a coupling β_2. This is a factorizable subcase of the general case defined by Eq. (1).

The action given by Eq. (2) is in fact the action for a three-dimensional layer in four-dimensional gauge theory. This theory has been studied by Nielsen and Fu.[3] They obtain the phase diagram of the theory using the mean-field theory with one-loop corrections; as expected, there is no stable layer phase in the theory. This is a consequence of the fact that U(1) in three dimensions does not undergo a phase transition.

The phase diagram has other interesting features. The line $\beta_1=0$ corresponds to a three-dimensional U(1) gauge theory. There will be no phase transition along this line. One can also check the theory on the $\beta_2=0$ line, along with $\theta_{i,x}=0$. In this limit all the plaquettes in the (y,z), (y,t), (z,t) planes do not contribute and one is left with a theory defined only on the (x,z), (x,y), (x,t) planes. These planes can interact only through $\theta_{i,x}$, which we have chosen to be zero and hence one has QED in two dimensions.

To see the feature of the theory in the limits of infinite β_1, β_2, we turn to the Villain approximation,[5]

$$e^{A_V}=\prod_{i,\mu\nu}\sum_{l_{i,\mu\nu}}\exp\left[il_{i,\mu\nu}\theta_{i,\mu\nu}-\frac{{l_{i,xn}}^2}{2\beta_1}-\frac{{l_{i,nk}}^2}{2\beta_2}\right]\ , \tag{4}$$

where $n,k=x,y,z,l_{i,\mu\nu}$ is a set of integer fields associated with the corresponding plaquettes.

Performing the integration one is left with the partition function

$$Z_V=\sum_l \exp\left[-\sum\left[\frac{l_{i,xn}}{2\beta_1}+\frac{l_{i,nk}}{2\beta_2}\right]\right]\prod\delta(\Delta_\mu l_{i,\mu\nu})\ . \tag{5}$$

The solution of the constraint is

$$l_{i,\mu\nu}=\epsilon_{\mu\nu\alpha\beta}\Delta_\beta\tilde{l}_{i,\alpha}\ , \tag{6}$$

where $\tilde{l}_{i,\beta}$ are integer fields associated with the (i,β) links of the dual lattice.

Using (6) in (5) one arrives at (with $\tilde{l}_{x,k}=0$)

$$Z_V=\sum \exp\left[-\left[\frac{1}{2\beta_1}(\Delta_n \tilde{l}_{i,k}-\Delta_k \tilde{l}_{i,n})^2 - \frac{1}{2\beta_2}\sum(\Delta_x \tilde{l}_{i,k})^2\right]\right] . \quad (7)$$

The first term in Eq. (7) is just the Villain form of the XY model in three dimensions and the second is the Villain form of two-dimensional periodic QED.[5] Bhanot[2] showed that the couplings β_1 and β_2 are dual under the transformation that transforms a plaquette with x-directed links into one with no x-directed links. The duality of the line $\beta_1=0$, $\beta_2\to\infty$ and $\beta_2=0$, $\beta_1\to\infty$ we found is a consequence of this relation. The resulting phase diagram is presented in Fig. 1.

The region A in Fig. 1 is the Coulombic phase of four-dimensional QED. The region B is the electric confinement phase. The line of critical points between the above regions ends at the XY phase transition point.

To complete the phase diagram we make use of the fact that a two-dimensional U(1) theory is a one-plaquette theory for all couplings and that the region $\beta_1\approx\beta_2>1$ is Coulomb type. Thus one expects a line of phase transitions between the above region and the region $\beta_1>\beta_2$ and $\beta_1\gg1$. As on the line $\beta_1=\infty$ there is no phase transition for finite β_2, the line of phase transitions is expected to meet the line $\beta_1=\infty$ at the point $\beta_2\to\infty$ from below.

In the two limits $\beta_1\to\infty$ and $\beta_2\to0$ the theory becomes two-dimensional QED. But these regions are in a different phase. The region near the line $\beta_2=0$ is a disordered phase or a magnetic ordered phase. The phase C (i.e., the limit $\beta_1\to\infty$) is in the two-dimensional U(1) theory in the dual variable and therefore it is the magnetic disordered phase or the electric ordered phase. The same structure of phases is found by Bhanot[2] for the $Z(2)$ theory.

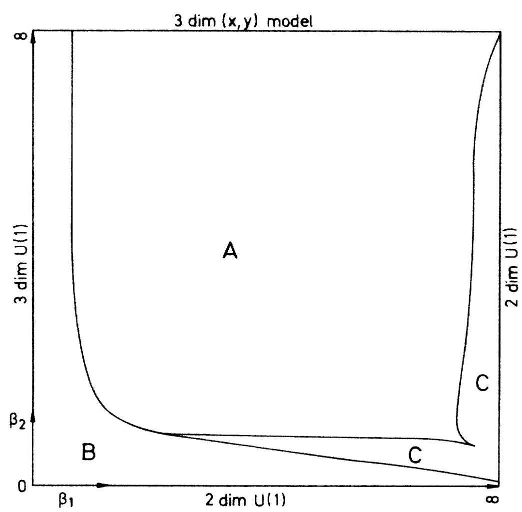

FIG. 1. The phase diagram expected from the duality of the line $\beta_1=0,\beta_2\to\infty$ and $\beta_1\to\infty,\beta_2=0$. Region A is the Coulombic phase, B is the electric confinement phase, and C is the ordered phase.

MONTE CARLO STUDY

The theory is studied on a 5^4 lattice. We use the approximation $Z(128)=\mathrm{U}(1)$. Our simulation reproduces the known behavior of the isotropic theory correctly.

The vertical line of phase transitions between phase C and A (Fig. 1) is expected for $\beta_1\gg1$, in this region the correlation length is very large, so we cannot expect to see it on a 5^4 lattice. The only signal for the phase C is the bifurcation of the horizontal line. To see the behavior of the line of phase transitions we define the average plaquette energy for plaquettes with x-directed links:

$$\langle S_x\rangle=\frac{1}{N_x}\left\langle\sum_x \cos\theta_{p,x}\right\rangle , \quad (8)$$

where N_x is the total number of plaquettes with x-directed links; and

$$\langle S_{\bar{x}}\rangle=\frac{1}{N_{\bar{x}}}\left\langle\sum_{\bar{x}} \cos\theta_{p,\bar{x}}\right\rangle , \quad (9)$$

where $N_{\bar{x}}$ is the total number of plaquettes with no x-directed links; and the quantity

$$\langle S_{\mathrm{an}}\rangle=\langle S_{\bar{x}}\rangle-\langle S_x\rangle , \quad (10)$$

which measures the anisotropy of the system. Figure 2 is a plot of $\langle S_{\mathrm{an}}\rangle$ against β_1 for $\beta_2=\beta_1/10$. We see that the anisotropy increases as β_1 increases and reaches its maximum at the phase-transition point (i.e., $\beta_1\approx2$). Then $\langle S_{\mathrm{an}}\rangle$ decreases rapidly up to the second phase transition at $\beta_2\approx0.4$.

A possible explanation for this behavior is the following: in the region $0<\beta_1\lesssim2$ $(0<\beta_2\lesssim0.2)$ the system is in the disordered phase and $\langle S_{\mathrm{an}}\rangle$ increases because $\beta_2=0.1\beta_1$. In the region $0.2\lesssim\beta_2\lesssim0.4$ the $\bar{x}$ plaquettes are in the disordered phase. On the other hand, the x plaquettes are ordered. Consequently, $\langle S_{\mathrm{an}}\rangle$ changes faster than $\langle S_x\rangle$ and hence $\langle S_{\mathrm{an}}\rangle$ decreases. In the Coulombic phase $\langle S_{\mathrm{an}}\rangle$ decreases slowly. From this explanation it seems that after the bifurcation of the phase-transition

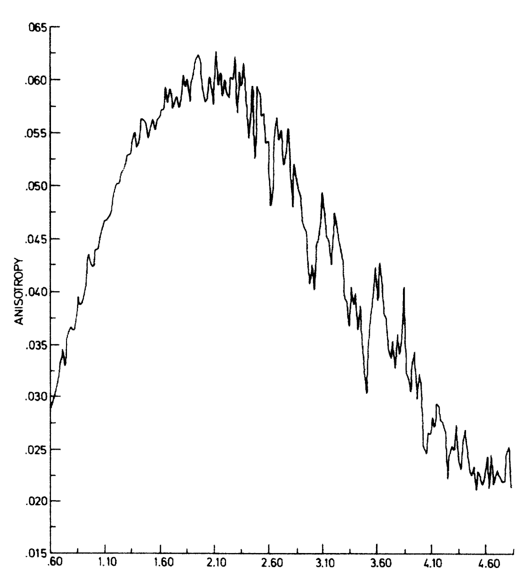

FIG. 2. $\langle S_{\mathrm{an}}\rangle$ vs β_1 for the anisotropic line $\beta_2=\beta_1/10$.

line the phase between the two branches is a partially ordered phase that changes to an ordered phase as β_1 increases. [On some lines (but not on the line $\beta_2=\beta_1/10$) we measure the anisotropy of the 2×2 Wilson loop. It shows the same behavior as the anisotropy of the plaquette.]

It is interesting that the anisotropy has a nonzero value even at the critical point. Since the case under consideration belongs to the factorizable category, the anisotropy can be scaled away by rescaling of the lattice spacing and rotational invariance can be restored. However, for the general nonfactorizable case, the anisotropy will be nonzero and cannot be eliminated by rescaling. Recently Landau and Swendsen[6] studied the anisotropic two-dimensional Ising model using Monte Carlo renormalization-group methods. They found that the spatial anisotropy is a marginal operator and that the critical behavior of the general anisotropic model is governed by a line of fixed points. As in our model the anisotropy has a nonzero value for a local operator on the critical line.

Figure 3 is the phase diagram found by Monte Carlo calculation. The bars are the locations of the hysteresis loops. The bifurcation of the horizontal line is seen by having two nonoverlapping hysteresis loops in $\langle S_{\bar{x}} \rangle$ and $\langle S_x \rangle$.

After the bifurcation the hysteresis of the lower line is closed when the number of sweeps increases. Because of the size of our lattice one cannot rule out the possibility that this is a line of fast crossover and not a critical line.

Our result is in agreement with the result of Nielsen and Fu.[3]

DISCUSSION

The numerical Monte Carlo study of the anisotropic U(1) theory shows a rich phase diagram. As expected there is no region of three-dimensional layers enclosed by a line of phase transitions. However, it is possible that one of the second-order phase-transition points (e.g., the three-dimensional X-Y transition) can be an infrared attractive fixed point. In this case, for some neighborhood of this point in coupling space the theory has the long-distance behavior of a three-dimensional theory (i.e., an effective layers phase).

To get a three-dimensional theory out of four dimensions the coupling (β_1) of all plaquettes with two x-directed links is taken to be smaller than the coupling β_2 of the plaquettes with no x-directed links. Let us rewrite Eq. (2) in the form

$$S=\sum_{x}\beta_1(x)\cos\theta_{p,x}+\beta_2\sum_{\bar{x}}\cos\theta_{p,\bar{x}}\ , \qquad (11)$$

with $\beta_1(x)\neq\beta_2$ for some fraction (P) of the points.[7] The interesting questions are as follows.

(a) What is the critical fraction P_c for which one gets a phase diagram similar to Fig. 3?

(b) Does P_c depend on the way the isotropic plaquettes are distributed?

These questions are under study.

We propose the above model as a toy model to study the large-distance effect of local anisotropy in space (such anisotropy can be due to gravitational fluctuations). In reality the anisotropy is distributed in all directions, but

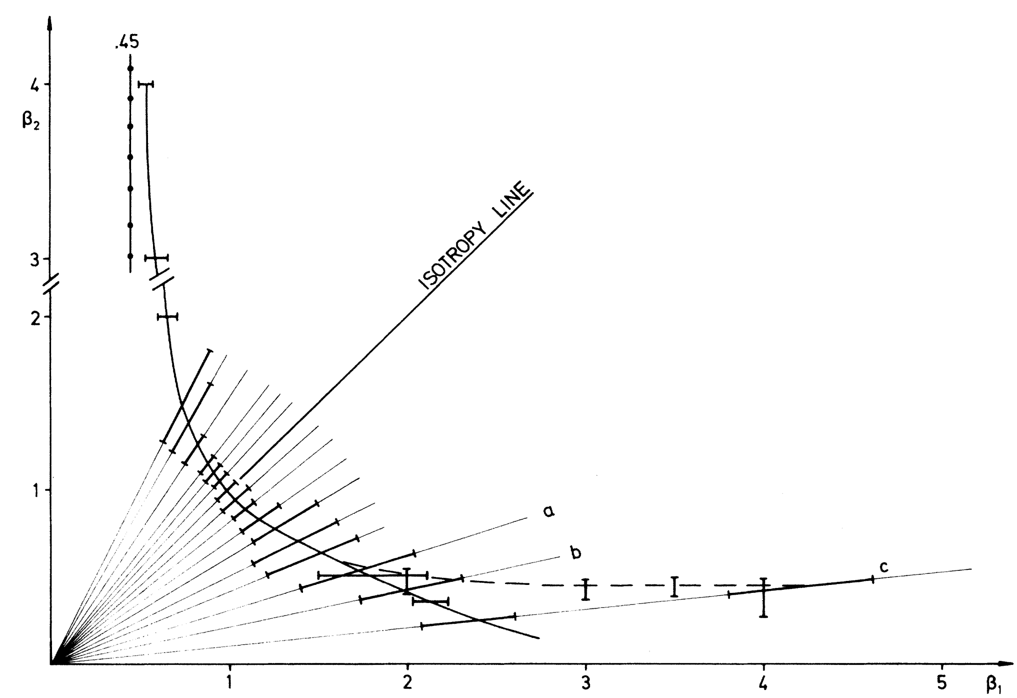

FIG. 3. The phase diagram found in the MC study. The bars correspond to the thermal cycles. On line a and b we found two completely overlapping hystereses for $\langle S_x \rangle$ and $\langle S_{\bar{x}} \rangle$. On line c the two hystereses have no overlap. On all other diagonal lines the hystereses of $\langle S_x \rangle$ and $\langle S_{\bar{x}} \rangle$ are completely overlapping. The horizontal and vertical bars correspond to thermal cycles of the variables $\langle S_x \rangle$ and $\langle S_{\bar{x}} \rangle$, respectively. The dotted line at $\beta_1=0.45$ is the one to which the phase-transition line approaches asymptotically as $\beta_2\rightarrow\infty$.

then the lower-dimensional layers are not flat and it will be very difficult to see them in a numerical study.

ACKNOWLEDGMENTS

We wish to thank Dr. Baskaran for discussions. One of us (N.G.) wishes to thank Professor M. Tossi for hospitality at the International Centre for Theoretical Physics (ICTP). E.K. also wishes to thank Professor H. Nielsen for discussion. Both of us would like to thank Professor Abdus Salam for hospitality at the ICTP where this work was done. One of us (E.K.) would like to thank the Alexander von Humboldt Foundation for financial support.

*Present address: Department of Physics, University of Poona, Pune 411007, India.

[1]T. T. Wu and B. M. McCoy, Nucl. Phys. **B190** [FS3], 519 (1981).

[2]G. Bhanot, Phys. Lett. **121B**, 401 (1983).

[3]H. B. Nielsen and Y. K. Fu, Nucl. Phys. **B236**, 167 (1984).

[4]N. Gupte, Phys. Lett. **103B**, 297 (1983).

[5]T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. **B129**, 493 (1977).

[6]D. P. Landau and R. H. Swendsen, Phys. Rev. B **30**, 2787 (1984).

[7]H. B. Nielsen and N. Brane, in *Proceedings of the XVIII International Symposium on the Theory of Elementary Particles, Ahrenshoop Institut für Hochenergiesphysik, 1984* (Akademie der Wissenschaften der DDR Berlin, Zeuthen, 1985); H. B. Nielsen *et al.*, in Proceedings of the Niels Bohr Centennial Conference, 1983 (unpublished).