# Path-integral derivation of gauge and gravitational chiral anomalies in theories with vector and axial-vector couplings in arbitrary even dimensions

John M. Gipson
*Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*
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The anomalies associated with general linear transformations of fermions coupled to external vector and axial-vector fields in curved spaces of arbitrary even dimensions are derived within the path-integral framework. The anomalies are due to the noninvariance of the fermionic functional measure under these transformations. The anomalies satisfy the Wess-Zumino consistency conditions. This implies that the choice of the fermionic functional measure is the correct one. Special cases of the general formula obtained agree with all previous results.

## I. INTRODUCTION

For some time it has been known that a naive application of symmetry arguments to models of fermions coupled to gauge bosons in even dimensions may lead to incorrect results.[1] A straightforward application of Noether's theorem leads one to believe that the vector (and axial-vector) currents generated by global vector (axial) rotations should be "covariantly" conserved. A careful evaluation of the "covariant" divergence of these currents leads to the surprising conclusion that at least some of these currents are not conserved. Such nonconserved currents are said to have an anomaly.

The modern rediscovery of the anomaly arose in the context of perturbation theory.[2] When one calculates the divergence perturbatively, some of the contributing diagrams are divergent, and must be regularized. It is impossible to impose a regularization scheme which simultaneously respects all symmetries. As demonstrated by Bardeen, it is always possible to choose a scheme such that the vector currents are conserved, but at least some of the axial-vector currents have anomalies. Alternatively, in a theory with pure left-handed coupling, one can always regularize so that the right-handed currents are anomaly free. The different results can be viewed as due to the addition of different counterterms to the Lagrangian.[2] The particular regularization scheme one employs is arbitrary. However, Wess and Zumino proved nonperturbatively that the anomalies must satisfy certain consistency conditions.[3]

An alternative, nonperturbative, derivation of the anomalies is due to Fujikawa.[4] He derived the axial-vector anomaly of Dirac fermions coupled to vector gauge bosons[4] using functional-integral techniques. The anomaly arises because the functional measure is not invariant under the axial transformation which generates the axial-vector current. Hence this symmetry is not a symmetry of the quantum theory.

The local form of the anomalies have also been known to mathematicians for some time, although under different names.[5] For the most part, their derivations of the anomalies have been indirect, utilizing abstract topological arguments. It has only been recently that direct derivations of the anomalies have appeared in the physics literature.

The absence or presence of anomalies has many important physical consequences. Anomalies in currents which couple to gauge bosons break the gauge invariance of the theory. Since gauge invariance is necessary for renormalizability, it is important that no such anomalies appear. This places constraints on the allowed particle content of these theories. Anomalies in currents which do not couple to gauge bosons are allowed, and at times welcome. For example, the U(1) isoscalar axial-vector anomaly cleared up the mystery of $\pi^0 \rightarrow 2\gamma$ (Ref. 2). Last, as 't Hooft has pointed out, any effective theory must reproduce the anomalies of the underlying fundamental theory.[6] This puts severe constraints on various composite models.

Recently there has been much renewed interest in anomalies in four and higher dimensions. The interest in four dimensions stems from some ambiguities in the path-integral derivation of anomalies in theories involving $\gamma_5$ couplings. It has been claimed that the anomalies derived using functional-integral techniques differ from those derived perturbatively.[7] This discrepancy can be attributed to the definition of the measure appearing in the functional integral. Since the anomaly is due to the noninvariance of the functional measure under vector and axial transformations, different definitions will give rise to different anomalies. Two essentially different measures have been proposed. The measure used in Ref. 8 leads to results which agree with those obtained in perturbation theory. One can verify after the fact that the anomalies satisfy the Wess-Zumino consistency conditions. In an alternative scheme, first proposed by Fujikawa,[4,7] one obtains a result in four dimensions which neither agrees with the perturbative calculation nor satisfies the consistency conditions.

The interest in higher-dimensional anomalies stems from the renewed activity in Kaluza-Klein theories. The simplest anomalies to calculate are the isoscalar and isovector axial-vector anomalies of a theory describing Dirac fermions coupled to vector gauge bosons. These were first derived perturbatively by Frampton and Kephart.[9] The isoscalar anomaly was derived using path integrals by Zumino, Wu, and Zee,[10] and by Matsuki.[11] Matsuki and Hill[12] derived the eight-dimensional chiral gravitational anomaly using path integrals. Alvarez-Gaumé and Getzler have derived the gravitational chiral anomaly using the path-integral formalism for a supersymmetric manifold.[13] In a previous paper[14] I used functional integrals to derive the general formula for the isoscalar and isovector anomalies of Dirac fermions coupled to vector gauge bosons in even-dimensional curved space-time. I made an error in that paper which was subsequently corrected by Endo and Takao.[15]

Many workers have examined the question of Weyl fermions coupled to gauge fields. The chiral isovector currents have anomalies which Zumino, Wu, and Zee[16] refer to as the "non-Abelian" anomalies. They, and others, have derived these anomalies by using the Wess-Zumino consistency conditions.[16] In this approach one constructs a solution to the consistency equations. Since a solution with given boundary conditions is unique, one has found the anomaly. Frampton and Kephart have obtained identical results perturbatively.[17] Alvarez-Gaumé,[18] and Alvarez-Gaumé and Witten,[19] using the measure suggested by Fujikawa,[4,7] derived a general form for the combined chiral gauge and gravitational anomalies using path-integral techniques. The formula they give is identical to the general formula I derived[14] for the axial-vector isovector anomalies of Dirac fermions coupled to vector fields. Their result, which transforms covariantly under gauge transformations, is not what is usually called the non-Abelian anomaly. In flat space their result does not obey the consistency conditions. Bardeen and Zumino[20] have examined this discrepancy, and refer to two kinds of anomalies which they call "consistent" or covariant. The consistent anomalies are associated with the currents generated by vector and axial transformations. The covariant anomalies are associated with currents which differ from these by the addition of polynomials in the gauge fields.

In this paper I derive the general form for the chiral anomalies of fermions coupled to external vector and axial-vector sources in arbitrary curved spaces using the path-integral formalism. Section II recapitulates the arguments relating the anomaly to the fermionic functional measure, and proves on general grounds that it must satisfy the Wess-Zumino consistency conditions. In Sec. III I define the functional measure I use, and verify formally that it leads to an anomaly which satisfies the Wess-Zumino consistency conditions. In Sec. IV I examine two regularization schemes one can impose to render the anomaly finite and well defined. I also briefly comment on the structure of the anomaly. In Sec. V I derive the explicit form of the anomaly in flat space. In Sec. VI I do the same in curved space. Section VII has some concluding remarks. A series of Appendixes collect various useful formulas.

## II. THE FUNCTIONAL MEASURE AND THE ANOMALY

The starting point of my derivation is the Euclidean action $S(\overline{\Psi},\Psi;V,A)$ describing $N$ fermions coupled to external vector and axial-vector sources in $2n$-dimensional curves space:

$$S(\overline{\Psi},\Psi;V,A)=\int d^{2n}x\, g^{1/2}(x)\overline{\Psi}i\nabla\!\!\!\!/\,\Psi\ , \tag{2.1}$$

where

$$\begin{aligned}\nabla\!\!\!\!/ &=\gamma^{\mu}\nabla_{\mu}\equiv\gamma^{\mu}(\partial_{\mu}+\omega_{\mu}+V_{\mu}+i\gamma_{2n+1}A_{\mu})\ ,\\ V_{\mu}&=iV_{\mu}^{j}T^{j},\quad A_{\mu}=iA_{\mu}^{j}T^{j}\ .\end{aligned} \tag{2.2}$$

The $\gamma^{\mu}$ are the space-time-dependent Hermitian Dirac matrices appropriate for curved space. $\gamma_{2n+1}$ is the generalization of $\gamma_5$, and is also Hermitian. $\omega_{\mu}$ is known as the spin connection and incorporates the effects of curvature. The $T^{j}$ are the Hermitian generators of U($N$) and the $V$'s and the $A$'s are Hermitian external fields. In Appendix A I summarize my notation and review some of the properties of the Dirac equation in curved space. For notational simplicity, it is convenient to introduce the abbreviation

$$\int dx\equiv\int d^{2n}x\, g^{1/2}(x)$$

and the standard fermion bilinears:

$$J^{k,\mu}=\overline{\Psi}T^{k}\gamma^{\mu}\Psi,\quad J_{2n+1}^{k,\mu}=i\overline{\Psi}T^{k}\gamma^{\mu}\gamma_{2n+1}\Psi\ .$$

With these conventions, the action becomes

$$S(\overline{\Psi},\overline{\Psi};V,A)=\int dx[\overline{\Psi}i\,D\!\!\!\!/\,\Psi+i\,\mathrm{Tr}(J^{\mu}V_{\mu}+J_{2n+1}^{\mu}A_{\mu})]\ ,$$

where $D\!\!\!\!/=\partial\!\!\!/+\omega\!\!\!/$.

There is some ambiguity in the Euclidean versions of field theories with axial-vector couplings. In continuing to Euclidean space I have let $A_{\mu}\rightarrow +iA_{\mu}$, which is necessary for $i\nabla\!\!\!\!/$ to be a Hermitian operator:

$$\int dx\,\psi_j^{\dagger}(x)i\nabla\!\!\!\!/\,\phi_k(x)=\left[\int dx\,\phi_k^{\dagger}(x)i\nabla\!\!\!\!/\,\psi_j(x)\right]^{*}\ .$$

The reason for this choice is that I want the eigenvalues of $i\nabla\!\!\!\!/$ to be real, which is desirable for the manipulations of the following section. Here $\psi_j$ and $\phi_k$ are arbitrary square-integrable spinors. The combined vector and axial-vector transformations generate the noncompact group GL($N,C$) (see below). This is in contrast with Minkowski space where they generate the compact group $U_L(N,C)\times U_R(N,C)$. At the end of this section I will comment on what happens if I let $A_{\mu}$ remain $A_{\mu}$.

The classical action is an extremum with respect to all variations (which vanish sufficiently fast at infinity) of the classical solutions $\overline{\Psi}$ and $\Psi$:

$$[\delta/\delta\alpha(x)]S[\overline{\Psi}(\alpha),\Psi(\alpha)]\,|_{\alpha=0}=0\ . \tag{2.3}$$

We are interested in the behavior of the action under vector and axial-vector transformations. Under the general vector transformation

$$\Psi(x)\rightarrow\Omega(\alpha)\Psi(x),\quad\overline{\Psi}(x)\rightarrow\overline{\Psi}(x)\Omega(-\alpha)\ , \tag{2.4}$$

where

$$\Omega(\alpha)=\exp[i\alpha(x)],\quad\alpha(x)=\alpha_j T^{j}$$

the action transforms into

$$S(\overline{\Psi},\Psi;V,A)\rightarrow S(\overline{\Psi},\Psi;V,A)+\int dx\,\mathrm{Tr}\{-(\partial_\mu\alpha)J^\mu+\alpha([V^\mu,J_\mu]+[A_\mu,J^\mu_{2n+1}])\}+O(\alpha^2)\,. \quad (2.5)$$

Similarly, under the general axial-vector transformation

$$\Psi(x)\rightarrow\Omega_{2n+1}(\beta)\Psi(x),\quad \overline{\Psi}(x)\rightarrow\overline{\Psi}(x)\Omega_{2n+1}(\beta)\,, \quad (2.6)$$

where

$$\Omega_{2n+1}(\beta)=\exp[\gamma_{2n+1}\beta(x)]\,,$$

the Lagrangian transforms into

$$S(\overline{\Psi},\Psi;V,A)\rightarrow S(\overline{\Psi},\Psi;V,A)+\int dx\,\mathrm{Tr}\{(\partial_\mu\beta)J^\mu_{2n+1}+\beta(-[V_\mu,J^\mu_{2n+1}]+[A^\mu,J_\mu])\}+O(\beta^2)\,. \quad (2.7)$$

Using Eq. (2.3), Eqs. (2.5) and (2.7) imply the identities

$$\partial_\mu[g^{1/2}(x)J^\mu]+g^{1/2}(x)([V^\mu,J_\mu]+[A_\mu,J^\mu_{2n+1}])=0\,, \quad (2.8)$$

$$\partial_\mu[g^{1/2}(x)J^\mu_{2n+1}]+g^{1/2}(x)([V_\mu,J^\mu_{2n+1}]-[A^\mu,J_\mu])=0\,. \quad (2.9)$$

Classically, the vector and the axial-vector currents are covariantly conserved.

The breakdown of the classical identities (2.8) and (2.9) at the quantum level is due to the noninvariance of the functional measure under the transformations associated with these currents. This was first demonstrated by Fujikawa[4] in the evaluation of the isoscalar axial-vector anomaly of Dirac fermions coupled to vector gauge bosons. The generating functional $Z(i\nabla\!\!\!\!/\,)$ in Euclidean space is

$$Z(i\nabla\!\!\!\!/\,)=\int d\mu(\overline{\Psi},\Psi)\exp\left[-\int dx\,\overline{\Psi}i\nabla\!\!\!\!/\,\Psi\right]\,. \quad (2.10)$$

Here $d\mu(\overline{\Psi},\Psi)$ is the fermionic functional measure. A straightforward evaluation of

$$-[\delta/\delta\alpha(x)]Z[\Omega(-\alpha)i\nabla\!\!\!\!/\,\Omega(\alpha)]\,|_{\alpha=0}$$

using Eq. (2.5) generates the vacuum expectation value of the left-hand side (LHS) of Eq. (2.8). On the other hand,

$$\begin{aligned}Z[\Omega(-\alpha)i\nabla\!\!\!\!/\,\Omega(\alpha)]&=\int d\mu(\overline{\Psi},\Psi)\exp\left[-\int dx\,\overline{\Psi}[\Omega(-\alpha)i\nabla\!\!\!\!/\,\Omega(\alpha)]\Psi\right]\\&=\int d\mu(\overline{\Psi}\Omega(\alpha),\Omega(-\alpha)\Psi)\exp\left[-\int dx\,\overline{\Psi}i\nabla\Psi\right]\\&=\int d\mu(\overline{\Psi},\Psi)J(-\alpha)\exp\left[-\int dx\,\overline{\Psi}i\nabla\Psi\right]\,.\end{aligned}$$

$J(-\alpha)$ is the Jacobian of the inverse of the transformation given in Eq. (2.4). Equation (2.8) is replaced by

$$\begin{aligned}&\langle\partial_\mu[g^{1/2}(x)J^\mu]+g^{1/2}(x)([V^\mu,J_\mu]+[A_\mu,J^\mu_{2n+1}])\rangle\\&\quad=-[\delta/\delta\alpha(x)]J(-\alpha)\,|_{\alpha=0}\equiv G(V(x),A(x))\,. \quad (2.11)\end{aligned}$$

$G(V(x),A(x))$ is the vector current anomaly. Similarly, evaluating

$$[\delta/\delta\beta(x)]Z[\Omega_{2n+1}(\beta)i\nabla\!\!\!\!/\,\Omega_{2n+1}(\beta)]\,|_{\beta=0}$$

in two different ways leads to the quantum version of Eq. (2.9):

$$\begin{aligned}&\langle\partial_\mu[g^{1/2}(x)J^\mu_{2n+1}]+g^{1/2}(x)([V_\mu,J^\mu_{2n+1}]-[A^\mu,J_\mu])\rangle\\&\quad=[\delta/\delta\beta(x)]J_{2n+1}(-\beta)\,|_{\beta=0}\\&\quad\equiv G_{2n+1}(V(x),A(x))\,, \quad (2.12)\end{aligned}$$

where

$$d\mu[\overline{\Psi}\Omega_{2n+1}(-\beta),\Omega_{2n+1}(-\beta)\Psi]=d\mu(\overline{\Psi},\Psi)J_{2n+1}(-\beta)\,.$$

[The relative minus sign between the vector and axial-vector anomalies can be traced to Eqs. (2.5) and (2.7).] If either $G\neq0$ or $G_{2n+1}\neq0$ then some of the classical identities are violated at the quantum level. The corresponding Ward-Takahashi identities are said to be anomalous.

Before proceeding to derive a more explicit form of $G$ and $G_{2n+1}$, it is worth considering what we can say about them on general grounds. Following Wess and Zumino,[3] I introduce the generators of infinitesimal gauge transformations:

$$\begin{aligned}\alpha\cdot M(x)&=\{i[\partial_\mu\alpha(x)]+i[V_\mu(x),\alpha(x)]\}\delta/\delta V_\mu(x)\\&\quad+i[A_\mu(x),\alpha(x)]\delta/\delta A_\mu(x)\,,\\\beta\cdot N(x)&=\{-i[\partial_\mu\beta(x)]-i[V_\mu(x),\beta(x)]\}\delta/\delta A_\mu(x)\\&\quad+i[A_\mu(x),\beta(x)]\delta/\delta V_\mu(x)\,.\end{aligned} \quad (2.13)$$

These obey the following commutation relations:

$$\begin{aligned}&[\alpha\cdot M(x),\beta\cdot M(y)]=i[\alpha(x),\beta(y)]\cdot M(x)\delta(x-y)\,,\\&[\alpha\cdot M(x),\beta\cdot N(y)]=i[\alpha(x),\beta(y)]\cdot N(x)\delta(x-y)\,,\\&[\alpha\cdot N(x),\beta\cdot N(y)]=-i[\alpha(x),\beta(y)]\cdot M(x)\delta(x-y)\,.\end{aligned} \quad (2.14)$$

Acting on the generating functional, these operators generate the anomalies:

$$\begin{aligned}\int dx\,\alpha\cdot M(x)Z(i\nabla\!\!\!\!/\,)&=-\int d\mu(\overline{\Psi},\Psi)\exp\left[-\int dx\,\overline{\Psi}i\nabla\!\!\!\!/\,\Psi\right]\int dx\,\mathrm{Tr}[-(\partial_\mu\alpha)J^\mu+\alpha([V^\mu,J_\mu]+[A_\mu,J^\mu_{2n+1}])]\\&=-\int dx\,\mathrm{Tr}[\alpha(x)G(x)]\,,\\\int dx\,\beta\cdot N(x)Z(i\nabla\!\!\!\!/\,)&=\int dx\,\mathrm{Tr}[\beta(x)G_{2n+1}(x)]\,.\end{aligned} \quad (2.15)$$

These equations, combined with Eq. (2.14), imply that the anomalies must satisfy the consistency conditions

$$\int dx\, dy\{\alpha\cdot M(x)\mathrm{Tr}[\beta G(y)]-\beta\cdot N(x)\mathrm{Tr}[\alpha G(y)]\} = \int dx\,\mathrm{Tr}(i[\alpha,\beta]G(x))\ , \quad (2.16)$$

$$\int dx\, dy\{\alpha\cdot M(x)\mathrm{Tr}[\beta G_{2n+1}(y)]+\beta\cdot N(x)\mathrm{Tr}[\alpha G(y)]\} = \int dx\,\mathrm{Tr}(i[\alpha(x),\beta(x)]G_{2n+1}(x))\ , \quad (2.17)$$

$$\int dx\, dy\{\beta\cdot N(x)\mathrm{Tr}[\alpha G_{2n+1}(y)]-\alpha\cdot N(x)\mathrm{Tr}[\beta G_{2n+1}(y)]\} = +\int dx\,\mathrm{Tr}(i[\alpha(x),\beta(x)]G(x))\ . \quad (2.18)$$

[The unconventional choice of signs in Eq. (2.17) can be traced to Eqs. (2.11) and (2.12).] Since the anomaly is related to the functional measure, one can interpret these relations as restricting the allowed definition of the functional measure.

As mentioned at the beginning of this section, there is some ambiguity in continuing a theory with $\gamma_{2n+1}$ couplings to Euclidean space. Fujikawa and others[4–7] let $A_\mu$ remain $A_\mu$. This has the advantage that the combined axial-vector and vector transformations form the compact group $U_L(N,C)\times U_R(N,C)$, instead of the noncompact group $GL(N,C)$, as is the case with us. This has the disadvantage that the $i\nabla$ is not Hermitian. All of the results of this section can be transcribed to this case by replacing the axial-vector source $A_\mu$ by $-iA_\mu$, and the axial-vector gauge parameter $\beta$ by $i\beta$. In particular, the anomalies continue to be related to the Jacobians of vector and axial-vector transformations. They are also related to the generators of gauge transformations. Since these transformations generate an algebra, the anomalies must satisfy certain consistency equations.

## III. CONSTRUCTING THE FUNCTIONAL MEASURE AND THE ANOMALY

A standard means of defining the functional integral involves expanding $\Psi(x)$ and $\bar{\Psi}(x)$ in terms of a complete set of eigenfunctions of some Hermitian operator $\Delta_0$ which acts on spinors. Let $\{\psi_k(x)\}$ be one such complete set:

$$\begin{aligned}&\Delta_0\psi_k(x)=\lambda_{0_k}\psi_k(x)\ ,\\ &\sum_k \psi_k(x)\psi_k^\dagger(y)=\delta^{2n}(x-y)g^{-1/4}(x)g^{-1/4}(y)\ ,\\ &\int dx\,\psi_k^\dagger(x)\psi_m(x)=\delta_{k,m}\ .\end{aligned} \quad (3.1)$$

*A priori*, since $\Psi(x)$ and $\bar{\Psi}(x)$ are independent, it is possible to expand them in terms of different complete sets. For the sake of generality, I will introduce a second basis $\{\phi_k(x)\}$ which satisfies the above equations with $\Delta_0$ and $\lambda_{0_k}$ replaced by $\Delta_1$ and $\lambda_{1_k}$. $\Psi(k)$ and $\bar{\Psi}(x)$ can then be expanded as

$$\Psi(x)=\sum_k a_k\psi_k(x),\quad \bar{\Psi}(x)=\sum_k \phi_k^\dagger(x)\bar{b}_k\ . \quad (3.2)$$

$a_k$ and $\bar{b}_k$ are the generators of the infinite-dimensional Grassman algebra. The fermion functional measure is then defined as

$$d\mu(\bar{\Psi},\Psi)=\prod_k d\bar{b}_k\prod_k da_k\ . \quad (3.3)$$

The functional measure implicitly depends on the basis one chooses, and hence on the Hermitian operator $\Delta_0$.

Following Fujikawa, we compute the Jacobian of the vector and axial-vector transformations. Under the inverse of the vector transformation of Eq. (2.4) $\Psi(x)$ transforms as

$$\Psi'(x)=\exp[-i\alpha(x)]\Psi(x)=\sum_k a_k'\psi_k(x)\ . \quad (3.4)$$

The $a_k'$ are related to the $a_k$ by a unitary transformation

$$\begin{aligned}&a_k'=\sum_k U_{k,m}a_m\ ,\\ &U_{k,m}=\int dx\,\psi_k^\dagger(x)\exp[-i\alpha(x)]\psi_m(x)\ .\end{aligned} \quad (3.5)$$

The change in $\Psi$ contributes the following factor to the Jacobian:

$$\prod_k da_k'=\det U^{-1}\prod_k da_k\ .$$

From Eq. (3.1) it is easy to show that

$$(\ln U)_{k,m}=-i\int dx\,\psi_k^\dagger(x)\alpha(x)\psi_m(x)$$

which holds for arbitrary $\alpha(x)$. It follows from this that

$$\det U^{-1}=\exp(-\mathrm{Tr}\ln U) =\exp\left[i\int dx\sum_k\psi_k^\dagger(x)\alpha(x)\psi_k(x)\right]\ .$$

$\bar{\Psi}$ contributes a similar factor, but with $\psi$ replaced by $\phi$ and an opposite sign for $\alpha$. The total Jacobian of the inverse vector transformation is

$$J(-\alpha)=\exp\left[i\int dx\sum_k\psi_k^\dagger(x)\alpha(x)\psi_k(x) -\phi_k^\dagger(x)\alpha(x)\phi_k(x)\right]\ . \quad (3.6)$$

Therefore, from Eq. (2.11), the vector anomaly is

$$\mathrm{Tr}[\alpha(x)G(x)] =-i\sum_k[\psi_k^\dagger(x)\alpha(x)\psi_k(x)-\phi_k^\dagger(x)\alpha(x)\phi_k(x)]\ . \quad (3.7)$$

Similarly, the Jacobian of the inverse axial-vector transformations is

$$J_{2n+1}(-\beta)=\exp\left[\int dx\sum_k\phi_k^\dagger(x)\beta(x)\gamma_{2n+1}\phi_k(x) +\psi_k^\dagger(x)\beta(x)\gamma_{2n+1}\psi_k(x)\right] \quad (3.8)$$

which, via Eq. (2.12), leads to the axial-vector anomaly

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=\sum_k[\phi_k^\dagger(x)\beta(x)\gamma_{2n+1}\phi_k(x) +\psi_k^\dagger(x)\beta(x)\gamma_{2n+1}\psi_k(x)]\ . \quad (3.9)$$

The computation of the anomalies redues to the evaluation of these infinite sums.

A naive application of the completeness relations, Eq. (3.1), would lead to the conclusion that the vector current had no anomaly. However, the sums appearing in Eqs. (3.8) and (3.9) are divergent (or at best, conditionally convergent) and must be regulated to be rendered meaningful. I will return to this momentarily. By choosing $\{\phi_j\}=\{\psi_j\}$ we can ensure that the vector current has no anomaly. The axial-vector anomaly is then twice the first sum appearing on the RHS of Eq. (3.9).

Following Fujikawa,[4,7] I regularize the infinite sums by inserting a function $f(\lambda^2 t)$ and employ point splitting:

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)] = 2\lim_{\substack{y\to x\\ t\to 0}}\mathrm{Tr}\left[\beta(x)\sum_k \gamma_{2n+1}f(t\lambda_k^2)\psi_k(x)\psi_k^\dagger(y)\right], \quad (3.10)$$

here $t$ has the appropriate dimensions to make $t\lambda^2$ dimensionless. The function $f$ obeys the following conditions:

$$f(0)=1,\ \ f(\infty)=f'(\infty)=f''(\infty)=\cdots=0\ .$$

The effect of including $f$ is to cut off the contribution to the sum from the large eigenvalues. Replacing $\lambda_k$ by $\Delta_0$, and using the completeness relations, Eq. (3.1), the RHS of Eq. (3.10) becomes

$$2\lim_{\substack{y\to x\\ t\to 0}}\mathrm{Tr}\{\beta(x)\gamma_{2n+1}f(t\Delta_0^2)\delta^{2n}(x-y)[g(x)g(y)]^{-1/4}\}\ . \quad (3.11)$$

The usual choice for the regulating function $f(\lambda^2 t)$ is $\exp(-t\lambda^2)$.[4,6,7] [If the eigenvalues of $\Delta_0$ are positive, one can take $\exp(-t\lambda)$ as the regulating function.] I find it more convenient to use $(1+t\lambda^2)^{-1}$. Formally, both functions should yield the same results, since

$$\lim_{t\to 0}\exp(-t\Delta_0^2)=\int_0^\infty ds\, e^{-s}\lim_{t\to 0}\exp(-st\Delta_0^2) \quad (3.12)$$

interchanging the integral and the limit, and doing the integration, one obtains

$$\lim_{t\to 0}(1+t\Delta_0^2)^{-1}\ . \quad (3.13)$$

Strictly speaking, to ensure that all steps in the above argument are well defined, one would have to impose additional assumptions to ensure convergence at every step along the way. In any case, the $t$-independent terms obtained using Fujikawa's and my regulating function should yield the same answer.

Customarily one chooses

$$\Delta_0=i\nabla\!\!\!\!/\ . \quad (3.14)$$

It is for this reason that we demanded that the Euclidean dirac operator be Hermitian. This ensures convergence of Eq. (3.10) with either my choice of regulating function or $\exp(-t\lambda^2)$. In the case of a non-Hermitian dirac operator, some workers[7] have used $\exp(-t\lambda\lambda^*)$, which leads to an anomaly which is gauge covariant, but does not satisfy the Wess-Zumino conditions. [One can prove Eq. (3.10) with my regulating function would converge even if $\lambda$ were not real; as the magnitude of $\lambda$ becomes large, the imaginary part of $\lambda$ becomes small with respect to the real part.]

With my choice of regulating function, the anomaly is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=2\lim_{\substack{y\to x\\ t\to 0}}\mathrm{Tr}[\beta(x)\gamma_{2n+1}(1-t\nabla\!\!\!\!/^2)^{-1}] \times\delta^{2n}(x-y)[g(x)g(y)]^{-1/4}\ . \quad (3.15)$$

This can be further simplified. Since the trace of an odd number of $\gamma$ matrices vanishes, one has

$$\begin{aligned}\mathrm{Tr}[\beta(x)\gamma_{2n+1}(1-t\nabla\!\!\!\!/^2)^{-1}] &=\mathrm{Tr}[\beta(x)\gamma_{2n+1}(1-\sqrt{t}\,\nabla\!\!\!\!/)(1-t\nabla\!\!\!\!/^2)^{-1}\}\\ &=\mathrm{Tr}[\beta(x)\gamma_{2n+1}(1+\sqrt{t}\,\nabla\!\!\!\!/)^{-1}]\ . \end{aligned}\quad (3.16)$$

Using the plane-wave representation of the $\delta$ function:

$$\delta^{2n}(x-y)=(2\pi)^{-2n}\int d^{2n}k\,\exp[ik^\mu(x-y)_\mu]\ , \quad (3.17)$$

commuting $\exp[ik\cdot(x-y)]$ through the differential operators, and taking the limit $y\to x$, the LHS of Eq. (3.15) becomes

$$2\lim_{t\to 0}\mathrm{Tr}\left[\beta(x)\gamma_{2n+1}[g(x)]^{-1/4}\left[(2\pi)^{-2n}\int d^{2n}k\frac{1}{[1+\sqrt{t}(\nabla\!\!\!\!/+ik\!\!\!/)]}\right]g(x)^{-1/4}\right]\ . \quad (3.18)$$

Although the term appearing in large parentheses looks like a differential operator, it is in fact not, since

$$\begin{aligned}\left[\int d^{2n}kf(\nabla\!\!\!\!/+ik\!\!\!/)\right]\exp(ip\cdot x) &=\exp(ip\cdot x)\left[\int d^{2n}k\,f(\nabla\!\!\!\!/+ik\!\!\!/+ip\!\!\!/)\right].\\ &=\exp(ip\cdot x)\left[\int d^{2n}k\,f(\nabla\!\!\!\!/+ik\!\!\!/)\right]. \end{aligned}\quad (3.19)$$

After shifting the integration variable, one finds that $\exp(ip\cdot x)$ commutes with the integral. Since $\exp(ip\cdot x)$ form a complete set locally, it follows that the integral is a $c$-number valued matrix. Therefore, the total anomaly is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)] =2(2\pi)^{-2n}[g(x)]^{-1/2} \times\int d^{2n}k\lim_{t\to 0}\mathrm{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!\!/+ik\!\!\!/)}\right]\ . \quad (3.20)$$

In previous derivations of the anomaly within the path-integral framework one had to wait until the very end of the calculations to verify that the final results satisfied the Wess-Zumino consistency conditions. An advantage of our choice of regulating function is that one can verify at the very beginning that the anomalies satisfy the Wess-Zumino conditions. One can then proceed with confidence to the arduous task of actually calculating them. Since the vector anomalies are zero, Eq. (2.16) is trivially satisfied. Under a vector gauge transformation

$$\int dx\, \alpha\cdot M(x) \nabla\!\!\!/(y) = [\nabla\!\!\!/(y), i\alpha(y)]\ ; \tag{3.21}$$

hence,

$$\begin{aligned}
\int dx\, \alpha\cdot M(x) \int d^{2n}k \operatorname{Tr}\left[\beta(x)\gamma_{2n+1}\frac{1}{\{1+\sqrt{t}\,[\nabla\!\!\!/(x)+ik\!\!\!/]\}}\right]
&= \int d^{2n}k \operatorname{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}[i\alpha,\sqrt{t}\,\nabla\!\!\!/]\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\\
&= \int d^{2n}k \operatorname{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}i\alpha - i\alpha\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\\
&= \int d^{2n}k \operatorname{Tr}\left[i[\alpha,\beta]\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right].
\end{aligned}$$

In going from the third to the fourth line I have used the reasoning following Eq. (3.19) to justify neglecting terms which vanish on integration over $k$. Comparison with Eq. (2.17) shows that the second consistency equation is satisfied. Under an axial-vector transformation,

$$\int dx\, \alpha\cdot N(x)\nabla\!\!\!/(y) = \{\nabla\!\!\!/(y), i\alpha(y)\gamma_{2n+1}\}\ . \tag{3.22}$$

Now,

$$\begin{aligned}
\int dx\, \alpha\cdot N(x) \int d^{2n}k \operatorname{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]
&= -\int d^{2n}k \operatorname{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\{\sqrt{t}\,\nabla\!\!\!/(y), i\alpha(y)\gamma_{2n+1}\}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\\
&= -\int d^{2n}k \operatorname{Tr}\left[i\beta\gamma_{2n+1}\left[\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\alpha\gamma_{2n+1}+\alpha\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\right]\\
&\quad -\int d^{2n}k \operatorname{Tr}\left[i\beta\gamma_{2n+1}\left[\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}2\alpha\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\right]\\
&= -\int d^{2n}k \operatorname{Tr}\left[(\alpha\gamma_{2n+1},\beta\gamma_{2n+1})\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\\
&\quad +\int d^{2n}k \operatorname{Tr}\left[\beta\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}(2\alpha\gamma_{2n+1})\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right].
\end{aligned}$$

I have used Eq. (3.19) to justify going from the second to third line. After simplification, the consistency equation (2.18) becomes

$$2\int dx \int d^{2n}k \left[\operatorname{Tr}\left[\beta\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}(\alpha)\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]-\operatorname{Tr}\left[\alpha\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}(\beta)\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!/+ik\!\!\!/)}\right]\right]=0\ . \tag{3.23}$$

Using the formal identity

$$\int dx \int d^{2n}k \operatorname{Tr}[\alpha(\nabla\!\!\!/+ik\!\!\!/)^j\beta(\nabla\!\!\!/+ik\!\!\!/)^m] = \int dx \int d^{2n}k \operatorname{Tr}[\beta(\nabla\!\!\!/+ik\!\!\!/)^m\alpha(\nabla\!\!\!/+ik\!\!\!/)^j]$$

which can be proved by repeated integration by parts, the first term in Eq. (3.23) cancels the second.

## IV. REGULARIZING THE ANOMALY

In the previous section I obtained a closed form for the anomaly as a limit of function [see Eq. (3.20)]:

$$\operatorname{Tr}[\beta(x)G_{2n+1}(x)] = \lim_{t\to 0}\operatorname{Tr}[\beta(x)G_{2n+1}(x,t)]\ . \tag{4.1}$$

The RHS of this satisfies the Wess-Zumino consistency conditions before taking the limit $t \to 0$. Unfortunately, the RHS diverges in this limit. To render the LHS meaningful, the RHS must be further regularized. In this section I will address this problem and propose several solutions. I will also make some brief comments on the structure of the anomaly in curves spaces.

The function appearing on the RHS of Eq. (4.1) is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x,t)]=2(2\pi)^{-2n}[g(x)]^{-1/2}\int d^{2n}k\,\mathrm{Tr}\left[\beta(x)\gamma_{2n+1}\frac{1}{1+\sqrt{t}\,(\not\nabla+i\not k)}\right]. \tag{4.2}$$

There is an alternative form of this which will prove more useful in calculations. Namely,

$$\begin{aligned}\mathrm{Tr}\{\beta\gamma_{2n+1}[1+\sqrt{t}\,(\not\nabla+i\not k)]^{-1}\}&=\mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}\,(\bar{\not\nabla}+i\not k)]\frac{1}{[1+\sqrt{t}\,(\not\nabla+i\not k)]\times[1-\sqrt{t}\,(\bar{\not\nabla}+i\not k)]}\right]\\&=\mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}\,(\bar{\not\nabla}+i\not k)]\frac{1}{1-t(\not\nabla_{+}+i\not k)^2P_{+}-t(\not\nabla_{-}+i\not k)^2P_{-}-2\sqrt{t}\;i\not A\gamma_{2n+1}}\right],\end{aligned} \tag{4.3}$$

where I have introduced the notation

$$\begin{aligned}&\bar{\not\nabla}=\not D+\not V-i\gamma_{2n+1}\not A=\not\nabla_{+}P_{+}+\not\nabla_{-}P_{-}\;,\\&\not\nabla_{\pm}=\not D+\not A_{\pm},\quad \not A_{\pm}=\not V\pm i\not A,\quad P_{\pm}=\tfrac{1}{2}(1\pm\gamma_{2n+1})\;.\end{aligned} \tag{4.4}$$

To simplify the denominator of the Eq. (4.3) expression, define

$$X_{\pm}=1-t(\not\nabla_{\pm}+i\not k)^2=1-t[(\nabla_{+}+ik)_{\mu}(\nabla_{+}+ik)^{\mu}+F_{\pm}\cdot\sigma+r/4]\;. \tag{4.5}$$

Here $r$ is the scalar curvature and $F_{\pm}$ is the gauge field associated with $A_{\pm}$ in the representation of the fermions. (See Appendix A for more details.) Note that the $X$'s involve an even number of $\gamma$ matrices. Equation (4.3) becomes

$$\mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}\,(\not\nabla_{+}P_{+}+\not\nabla_{-}P_{-}+i\not k)]\frac{1}{X_{+}P_{+}+X_{-}P_{-}+2\sqrt{t}\,i\gamma_{2n+1}\not A}\right]. \tag{4.6}$$

Using Eq. (B5), one has

$$\begin{aligned}[X_{+}P_{+}+X_{-}P_{-}+2\sqrt{t}\,i\gamma_{2n+1}\not A]^{-1}=&\left[1+\frac{1}{X_{+}}\sqrt{t}\,i2\not A\right]P_{-}\frac{1}{X_{-}-t2\not A(X_{+})^{-1}2\not A}\\&+\left[1-\frac{1}{X_{-}}\sqrt{t}\,i2\not A\right]P_{+}\frac{1}{X_{+}-t2\not A(X_{-})^{-1}2\not A}\;.\end{aligned} \tag{4.7}$$

Using this, the LHS of Eq. (4.2) becomes

$$\begin{aligned}2(2\pi)^{-2n}[g(x)]^{-1/2}\int d^{2n}k\,\mathrm{Tr}\Bigg\{\beta\gamma_{2n+1}&\left[\left[1-it(\not\nabla_{+}+i\not k)\frac{1}{X_{+}}2\not A\right]P_{-}\frac{1}{X_{-}-t2\not A(X_{+})^{-1}2\not A}\right]\\&+\left[\left[1+it(\not\nabla_{-}+i\not k)\frac{1}{X_{-}}2\not A\right]P_{+}\frac{1}{X_{+}-t2\not A(X_{-})^{-1}2\not A}\right]\Bigg\},\end{aligned} \tag{4.8}$$

where I have discarded terms involving an odd number of $\gamma$ matrices.

One can evaluate Eq. (4.8) by scaling the momentum variable, $k=q/\sqrt{t}$, expanding the trace in powers of $\sqrt{t}$ and doing the $q$ integration. Upon using symmetric integration all half-integral powers of $t$ vanish. All integer powers of $t\geq -n$ contribute to the anomaly

$$\begin{aligned}&2(2\pi)^{-2n}\int d^{2n}k\,\mathrm{Tr}\{\beta\gamma_{2n+1}[1+\sqrt{t}\,(\not\nabla+i\not k)]^{-1}\}\\&\quad=t^{-n}A_{-n}+t^{1-n}A_{1-n}+\cdots A_{0}+B_{0}+O(t)\;.\end{aligned} \tag{4.9}$$

Here the $A$'s which do not contain the totally antisymmetric $\epsilon$ symbol, are the so-called "normal parity naive anomalies" [Hu, Young, and McKay (Ref. 8)]. These same terms appear when one calculates the anomalies perturbatively. In perturbative calculations they can be removed by the addition of counterterms of the Lagrangian. These counterterms can be chosen in such a way that the vector Ward identities are preserved. $B_0$, which depends on the $\epsilon$ symbol, cannot be removed by the addition of local counterterms which preserve the vector Ward identities. Because of this, $B_0$ is called the "minimal" anomaly.

Because of the presence of $A_{-n}\cdots A_{-1}$, Eq. (4.9) diverges upon taking the $t\to 0$ limit. Therefore, these terms must be regulated away in some manner. One possibility is to use a Pauli-Villars regularization, in which the single variable $t$ and its associated limit is replaced by the set of $n$ variables $\{C_a\}$ and $\{t_a\}$ [Hu, Young, and McKay (Ref. 8)]. In Eq. (3.10) we make the replacement

$$\lim_{t\to 0} f(t\Delta_0^2) \to \lim_{t_a\to 0} \sum_a C_a f(t_a \Delta_0^2) \; . \tag{4.10}$$

The $C$'s and $t$'s satisfy the subsidiary conditions

$$\sum_a C_a = 1 \; ,$$
$$\sum_a C_a (t_a)^{-1} = \sum_a C_a (t_a)^{-2} = \ldots = \sum_a C_a (t_a)^{-n} = 0 \; . \tag{4.11}$$

By construction, this procedure removes the first $n$ terms on the RHS of Eq. (4.9). The anomaly then becomes the sum of $A_0$ and $B_0$, which is the result one obtains using $\zeta$-function regularization. The normal parity anomaly $A_0$ can be removed by adding a counterterm to the Lagrangian, as was done by Balachandran *et al.* in four dimensions.[8]

I prefer to use a different method to regulate the anomaly. Namely, taking a hint from the procedure used by Hu, Young, and McKay in four dimensions,[8] I make the replacement

$$\mathrm{Tr}\{\beta\gamma_{2n+1}[1+\sqrt{t}(\nabla\!\!\!/ + i k\!\!\!/)]^{-1}\}$$
$$\to \tfrac{1}{2}\,\mathrm{Tr}(\beta\gamma_{2n+1}\{[1+\sqrt{t}(\nabla\!\!\!/ + i k\!\!\!/)]^{-1} + [1+\sqrt{t}(\bar{\nabla\!\!\!/} + i k\!\!\!/)]^{-1}\}) \; . \tag{4.12}$$

Note that with this replacement the "new" axial-vector anomaly will still satisfy the consistency conditions. Equation (4.8) is replaced by

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x,t)] = (2\pi)^{-2n}[g(x)]^{-1/2}\int d^{2n}k\,\mathrm{Tr}\left\{\beta\gamma_{2n+1}\left[\left[1 - it(\nabla\!\!\!/_+ + i k\!\!\!/)\frac{1}{X_+}2A\!\!\!/\right]\frac{1}{X_- - t2A\!\!\!/(X_+)^{-1}2A\!\!\!/}\right.\right.$$
$$\left.\left. + \left[1 + it(\nabla\!\!\!/_- + i k\!\!\!/)\frac{1}{X_-}2A\!\!\!/\right]\frac{1}{X_+ - t2A\!\!\!/(X_-)^{-1}2A\!\!\!/}\right]\right\} \; . \tag{4.13}$$

In expanding, this one obtains only "abnormal parity" contributions, i.e., terms involving the $\epsilon$ symbol. I will now prove directly that this is finite in the limit $t\to 0$.

Equation (4.13) can be rewritten in the schematic form

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x,t)] = \mathrm{Tr}[\beta(x)\gamma_{2n+1}B(\nabla_{+\mu},\nabla_{-\nu},F_+\cdot\sigma,F_-\cdot\sigma,A,r;t)]$$
$$+ \mathrm{Tr}[\beta(x)\gamma_{2n+1}\gamma^\alpha B_\alpha(\nabla_{+\mu},\nabla_{-\nu},F_+\cdot\sigma,F_-\cdot\sigma,A,r;t)] \; . \tag{4.14}$$

The functions $B$ and $B_\alpha$ are polynomials in their arguments. Now, as emphasized following Eq. (3.19), the RHS of the above equation is not a differential operator. This restricts the possible polynomials that can appear. In particular, the $\nabla$'s must appear in the form of commutators among themselves and with the $A$'s, $r$'s, and $F\cdot\sigma$'s. Let

$$W_{\mu\nu} \equiv [\nabla_{+\mu},\nabla_{+\nu}] = \tfrac{1}{2}R_{\mu\nu} + F_{+\mu\nu} \; ,$$
$$W_{\mu'\nu'} \equiv [\nabla_{-\mu},\nabla_{-\nu}] = \tfrac{1}{2}R_{\mu\nu} + F_{-\mu\nu} \; , \tag{4.15}$$
$$W_{\mu\nu'} \equiv [\nabla_{+\mu},\nabla_{-\nu}] = \tfrac{1}{2}R_{\mu\nu} + [\partial_\mu + A_{+\mu}, \partial_\nu + A_{-\nu}] \; .$$

Furthermore, for any matrix or tensor $E$, define

$$[\nabla_{+\mu},E_{\ldots}] = E_{\ldots\mu}\,, \quad [\nabla_{-\mu},E_{\ldots}] = E_{\ldots\mu'} \; , \tag{4.16}$$

where ... denotes indices. Of particular interest are

$$[\nabla_{+\mu},r] = r_\mu\,, \quad [\nabla_{-\mu},r] = r_{\mu'},\ldots \; ,$$
$$[\nabla_{+\mu},A] = A_\mu\,, \quad [\nabla_{-\mu},A] = A_{\mu'},\ldots \; ,$$
$$[\nabla_{+\mu},F_+\sigma] = (F_+\cdot\sigma)_\mu \; , \tag{4.17}$$
$$[\nabla_{-\nu},(F_+\cdot\sigma)_\mu] = (F_+\cdot\sigma)_{\mu\nu'} \; ,$$

etc., as well as terms with $F_-$. The RHS of Eq. (4.14) can be expressed as a polynomial in terms of the $W$'s, $F\cdot\sigma$'s, $A$'s, and $r$'s defined above. $G_{2n+1}(x,t)$ has dimension $M^{2n}$, while $t$ has dimension $M^{-2}$. By dimensional analysis, the terms of dimension $2j$ in the fields must carry a power of $t^{2j-2n}$. To survive the trace over $\gamma$ matrices a term must involve at least $2n$ $\gamma$ matrices. All such terms have at least dimension $2n$ in the fields, and are therefore finite in the limit $t\to 0$. Therefore, the LHS of Eq. (4.14) is well defined in this limit.

Actually, we can say a bit more about the structure of the anomaly. After taking the limit $t\to 0$, the only terms which contribute to Eq. (4.14) have exactly $2n$ $\gamma$ matrices and dimension $2n$ in the fields. This implies that $B(t=0)$ is a polynomial in $W$'s with two indices and $A$'s and $F\cdot\sigma$ with none. Similarly, $B_\alpha(t=0)$ is a polynomial in $W$'s with two indices, and $A$'s and $F$'s with none with the exception of a single three-index $W$ or one-index $A$ or $F$. Note that in both cases one can make the substitutions

$$W_{\mu\nu} = W_{\mu\nu'} = W_{\mu'\nu'} = \tfrac{1}{2}R_{\mu\nu} \tag{4.18}$$

since the gauge field terms vanish upon taking the trace. Many of the terms allowed on dimensional grounds turn out to vanish when one uses the symmetry properties of the Riemann curvature tensor $R_{\mu\nu\alpha\beta}$:

$$R_{\mu[\nu\alpha\beta]} = 0 \to \mathrm{Tr}(\gamma_{2n+1}\gamma^\alpha\cdots W_{\mu\alpha}\cdots) = 0 \; ,$$
$$R_{\mu[\nu\alpha\beta];\omega} = 0 \to \mathrm{Tr}(\gamma_{2n+1}\gamma^\alpha\cdots W_{\mu\alpha\omega}\cdots) = 0 \; , \tag{4.19}$$
$$R_{\mu\omega[\nu\beta;\alpha]} = 0 \to \mathrm{Tr}(\gamma_{2n+1}\gamma^\alpha\cdots W_{\mu\omega\alpha}\cdots) = 0 \; .$$

The first of these implies that in any term with a single index $A$ or $F$, this index must contract with the external $\gamma$ matrix. If it does not, the external $\gamma$ matrix contracts with a two-index $W$, and the trace vanishes. The next two equations imply that any term involving a three-index $W$ vanishes upon taking the trace.

Effectively, we can summarize these considerations with the following set of rules:

$$W_{\mu\nu}=W_{\mu\nu'}=W_{\mu'\nu'}=\tfrac{1}{2}R_{\mu\nu}\ ,$$
$$\gamma^{\mu}W_{\mu\nu}=0\cdots\ , \tag{4.20}$$
$$r=0\ .$$

All two and higher index $F\cdot\sigma$'s and $A$'s vanish.

All three and higher index $W$'s vanish.

Any term whose dimension is higher than the number of $\gamma$'s it contains vanishes.

No direct coupling between gauge and curvature terms.

One consequence of the above rules is that the anomaly can be written as a sum

$$\sum_k \mathrm{Tr}[\beta\gamma_{2n+1}P_k(F_+\cdot\sigma,F_-\cdot\sigma,A)Q_k(R_{\mu\nu})]\ ,$$

where $P_k$ and $Q_k$ are polynomials in their arguments. After taking the trace, the only coupling between gauge and gravitational effects comes about through the $\epsilon$ symbol. This "factorizability" becomes important in computing the anomaly in curved space.

## V. EVALUATING THE ANOMALY IN FLAT SPACE

In this section I will derive the general formula for the axial-vector anomaly in flat space and look at several special cases. From Eqs. (4.2) and (4.12), the regularized anomaly is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=\lim_{t\to 0}(2\pi)^{-2n}\int d^{2n}k\,\mathrm{Tr}\left[\beta\gamma_{2n+1}\left[\frac{1}{1+\sqrt{t}\,(\nabla\!\!\!\!/+ik\!\!\!/)}+\frac{1}{1+\sqrt{t}\,(\overline{\nabla\!\!\!\!/}+ik\!\!\!/)}\right]\right]. \tag{5.1}$$

I will concentrate on evaluating the first term in the integral. An alternative form of this term is given in Eq. (4.3):

$$\int d^{2n}k\,\mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}\,(\overline{\nabla\!\!\!\!/}+ik\!\!\!/)]\frac{1}{1-t(\nabla\!\!\!\!/_{+}+ik\!\!\!/)^2P_+-t(\nabla\!\!\!\!/_{-}+ik\!\!\!/)^2P_--2\sqrt{t}\,iA\!\!\!/\gamma_{2n+1}}\right]. \tag{5.2}$$

Now, the only terms which will contribute to the trace after taking the $t\to 0$ limit involve $2n$ $\gamma$ matrices and have dimension $2n$ in the gauge fields and their derivatives. Bearing this in mind, Eq. (5.2) becomes

$$\int d^{2n}k\,\mathrm{Tr}\left[\beta\gamma_{2n+1}\left[(1-\sqrt{t}\,\overline{\nabla\!\!\!\!/})\frac{1}{1+tk^2-X}+\sqrt{t}\,k\!\!\!/\frac{1}{1+tk^2-X}t\,2k\cdot\overline{\nabla\!\!\!\!/}\frac{1}{1+tk^2-X}\right]\right], \tag{5.3}$$

where

$$X=t(F_+\cdot\sigma P_++F_-\cdot\sigma P_-)+\sqrt{t}\,i2A\!\!\!/\gamma_{2n+1} \tag{5.4}$$

and I have dropped terms which do not survive the trace/limit combination. Using symmetric integration in $k$, and letting $tk^2=q$, Eq. (5.3) becomes

$$\tfrac{1}{2}\int d\Omega\int_0^{\infty}dq\,q^{n-1}t^{-n}\mathrm{Tr}\left[\beta\gamma_{2n+1}\left[(1-\sqrt{t}\,\overline{\nabla\!\!\!\!/})\frac{1}{1+q-X}+\frac{q}{n}\sqrt{t}\,\gamma^{\mu}\frac{1}{1+q-X}\overline{\nabla}_{\mu}\frac{1}{1+q-X}\right]\right]. \tag{5.5}$$

The integral over angular variables gives

$$\int d\Omega=2\pi^n/(n-1)!\ .$$

Integrating the first term in Eq. (5.5) by parts, one gets

$$\frac{\pi^n}{n!}\int_0^{\infty}dq\,q^nt^{-n}\mathrm{Tr}\left[\beta\gamma_{2n+1}\left[(1-\sqrt{t}\,\overline{\nabla\!\!\!\!/})\frac{1}{(1+q-X)^2}+\frac{1}{1+q-\overline{X}}\sqrt{t}\,\overline{\nabla\!\!\!\!/}\frac{1}{1+q-X}\right]\right], \tag{5.6}$$

where

$$\overline{X}=t(F_+\cdot\sigma P_-+F_-\cdot\sigma P_+)+\sqrt{t}\,i2A\!\!\!/\gamma_{2n+1} \tag{5.7}$$

and I have neglected terms which vanish upon performing the trace/limit. I need the identities

$$\begin{aligned}(\partial\!\!\!/\overline{X})&=X(\nabla\!\!\!\!/-iA\!\!\!/\gamma_{2n+1}+1/\sqrt{t}\,)-(\nabla\!\!\!\!/-iA\!\!\!/\gamma_{2n+1}+1/\sqrt{t}\,)\overline{X}+\cdots\ ,\\ (\partial\!\!\!/X)&=\overline{X}(\nabla\!\!\!\!/+iA\!\!\!/\gamma_{2n+1}-1/\sqrt{t}\,)-(\nabla\!\!\!\!/+iA\!\!\!/\gamma_{2n+1}-1/\sqrt{t}\,)X+\cdots\ .\end{aligned} \tag{5.8}$$

By $\cdots$ I mean terms which contain fewer $\gamma$'s than their dimension, e.g., $\partial_\mu \partial^\mu \not{A}$. Such terms do not contribute to the trace in the $t \to 0$ limit and hence can be neglected. Using the first of these identities, Eq. (5.6) takes the particularly simple form

$$\frac{\pi^n}{n!} \int_0^\infty dq\, q^n t^{-n} \operatorname{Tr}\left[\beta \gamma_{2n+1} \frac{1}{1+q-\bar{X}} \frac{1}{1+q-X}\right] . \tag{5.9}$$

The second term in Eq. (5.1) contributes a similar factor with $X$ and $\bar{X}$ interchanged. Gathering all the intermediate results, the anomaly in flat space is

$$\operatorname{Tr}[\beta(x) G_{2n+1}(x)] = \lim_{t\to 0} \frac{1}{n!(4\pi)^n} \int_0^\infty dq\, q^n t^{-n} \operatorname{Tr}\left[\beta\gamma_{2n+1}\left[\frac{1}{1+q-X}\frac{1}{1+q-\bar{X}} + \frac{1}{1+q-\bar{X}}\frac{1}{1+q-X}\right]\right] . \tag{5.10}$$

One can verify explicitly and laboriously that this satisfies the Wess-Zumino consistency conditions. We shall not do so.

Expanding Eq. (5.10) in powers of $(1+q)^{-j}$ and doing the $q$ integral, Eq. (5.10) can be written as

$$\operatorname{Tr}[\beta(x) G_{2n+1}(x)] = \left[\frac{1}{4\pi}\right]^n \lim_{t\to 0} \frac{1}{t^n} \sum_{m=0}^{n} \frac{m!}{(n+m+1)!} \sum_{j+k=n+m} \operatorname{Tr}\gamma_{2n+1}(X^j \bar{X}^k + \bar{X}^j X^k) . \tag{5.11}$$

The limits on the sums arise since one needs at least $n$ factors of $X$ and $\bar{X}$ to survive the trace over $\gamma$ matrices, while terms with more than $2n$ do not survive the $t \to 0$ limit. The second sum is over all positive values of $j$ and $k$ such that $j+k=m+n$. After rotating to Euclidian space

$$X = t(F_+ \cdot \sigma P_+ + F_- \cdot \sigma P_-) + \sqrt{t}\, 2\not{A}\gamma_{2n+1} ,$$

$$\bar{X} = t(F_+ \cdot \sigma P_- + F_- \cdot \sigma P_+) + \sqrt{t}\, 2\not{A}\gamma_{2n+1} .$$

The two- and four-dimensional anomalies are of particular interest. The anomaly in two dimensions is

$$\operatorname{Tr}(\beta G_3) = \frac{1}{\pi} \operatorname{Tr}[\beta(\partial_\mu V_\nu + V_\mu V_\nu - A_\mu A_\nu)\epsilon^{\mu\nu}] . \tag{5.12}$$

In four dimensions the anomaly is

$$\operatorname{Tr}(\beta G_5) = \frac{1}{(4\pi)^2} \operatorname{Tr}\{\beta \epsilon^{\alpha\beta\mu\nu}[V_{\alpha\beta}V_{\mu\nu} + \tfrac{1}{3} A_{\alpha\beta}A_{\mu\nu} - \tfrac{8}{3}(A_\alpha A_\beta V_{\mu\nu} A_\alpha + A_\alpha V_{\beta\mu} A_\nu + V_{\alpha\beta} A_\mu A_\nu) + \tfrac{32}{3} A_\alpha A_\beta A_\mu A_\nu]\} , \tag{5.13}$$

where

$$V_{\mu\nu} = \tfrac{1}{2}[(F_+)_{\mu\nu} + (F_-)_{\mu\nu}], \quad A_{\mu\nu} = \tfrac{1}{2}[(F_+)_{\mu\nu} - (F_-)_{\mu\nu}] .$$

This agrees with the perturbative calculations of Ref. 2.

In arbitrary dimensions there are two cases which are of special interest. If $A=0$, corresponding to pure vector coupling, then

$$X = \bar{X} = tF \cdot \sigma .$$

Only the $m=0$ term in (5.11) survives. The sum can be done to yield

$$\operatorname{Tr}[\beta(x) G_{2n+1}(x)] = \frac{2}{n!(4\pi)^n} \operatorname{Tr}[\beta\gamma_{2n+1}(F\cdot\sigma)^n] \tag{5.14}$$

which agrees with the results of Refs. 9 and 14.

The second case of special interest corresponds to pure chiral coupling. Suppose $A=-V$, which corresponds to pure left-handed coupling. Then

$$X = tF\cdot\sigma P_- - \sqrt{t}\,\not{L}\gamma_{2n+1}, \quad \bar{X} = tF\cdot\sigma P_+ - \sqrt{t}\,\not{L}\gamma_{2n+1} , \tag{5.15}$$

where

$$2\not{A} = \not{L}, \quad F_{\mu\nu} = \partial_\mu L_\nu - \partial_\nu L_\mu + [L_\mu, L_\nu] .$$

Using Eqs. (B8) and (B9), the trace in Eq. (5.10) can be written as

$$2\operatorname{Tr}\left[\beta\gamma_{2n+1}\left[\frac{1}{1+q}\frac{1}{1+q-tF(q)\cdot\sigma} - \frac{1}{1+q} t\not{L} \frac{1}{1+q-tF(q)\cdot\sigma}\frac{1}{1+q}\not{L}\frac{1}{1+q-tF(q)\cdot\sigma}\right.\right.$$
$$\left.\left. - \frac{1}{1+q-tF(q)\cdot\sigma}\frac{1}{1+q} t\not{L}\frac{1}{1+q-tF(q)\cdot\sigma}\not{L}\frac{1}{1+q}\right]\right] , \tag{5.16}$$

where

$$F(q)_{\mu\nu}=\partial_\mu L_\nu-\partial_\nu L_\mu+q/(1+q)[L_\mu,L_\nu] \ . \tag{5.17}$$

Expanding Eq. (5.16) in inverse powers of $(1+q)$, we obtain

$$2t^n(1+q)^{-n-2}\mathrm{Tr}\left[\beta\gamma_{2n+1}[F(q)\cdot\sigma]^n-\frac{1}{(1+q)}\sum_{k=0}^{n-1}\{\not{L}[F(q)\cdot\sigma]^k\not{L}[F(q)\cdot\sigma]^{n-k-1}+[F(q)\cdot\sigma]^k\not{L}[F(q)\cdot\sigma]^{n-k-1}\not{L}\}\right] + O(t^{n+1}) \ . \tag{5.18}$$

Substituting this trace in Eq. (5.10), and changing the integration variable from $q$ to $s=q/(1+q)$ we obtain

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=\frac{2}{n!(4\pi)^n}\int_0^1 ds\, s^n\,\mathrm{Tr}\left[\beta\gamma_{2n+1}[F(s)\cdot q]^n-(1-s)\sum_{k=0}^{n-1}\{\not{L}[F(s)\cdot\sigma]^k\not{L}[F(s)\cdot\sigma]^{n-k-1} + [F(s)\cdot\sigma]^k\not{L}[F(s)\cdot\sigma]^{n-k-1}\not{L}\}\right] \ . \tag{5.19}$$

Up to an overall normalization factor, this is identical to the anomaly in the left-handed current due to purely left-handed coupling as calculated in Ref. (16). The difference arises since we regulate differently. I regulate so that the vector current has no anomaly, while the regulate so that the right-handed current has no anomaly.

Kawai and Tye,[21] and Lott[22] give a form for the axial-vector anomaly in arbitrary even dimensions which differs from mine given in Eq. (5.10). In my notation, their formula is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=\frac{2}{n!(4\pi)^n}\int_0^1 ds\,\mathrm{Tr}\left[\beta\gamma_{2n+1}[F(s)\cdot\sigma]^n-s(1-s)\sum_{k=0}^{n-1}\{\not{L}[F(s)\cdot\sigma]^k\not{L}[F(s)\cdot\sigma]^{n-k-1} + [F(s)\cdot\sigma]^k\not{L}[F(s)\cdot\sigma]^{n-k-1}\not{L}\}\right] , \tag{5.20}$$

where

$$F(s)=[sF_+ +(1-s)F_- -s(1-s)(2A)(2A)] \ .$$

This reproduces Eq. (5.19) in the case of chiral coupling. One can verify that this satisfies the Wess-Zumino consistency conditions. Kawai and Tye, following Zumino, Wu, and Zee, derive the anomaly by starting with the "Abelian" anomaly in $2n+2$ dimensions. Lott derives the anomaly using cohomology.

As mentioned in the Introduction, the solution to the Wess-Zumino consistency conditions with given boundary conditions is unique. Comparing Eqs. (5.10) and (5.20) it is clear that the terms of highest order in $F_+$ and $F_-$ differ. My terms are identical to what one would obtain in perturbation theory if one treated all vertices symmetrically. The formula given in Eq. (5.10) can be transformed into Eq. (5.20) if one totally symmetrizes the trace. Lott claims that this is essentially Bose symmetrization. However, this is not the case, since in both the ordinary perturbative calculations and in my path-integral derivation, Bose symmetry is guaranteed by the formalism.

## VI. EVALUATION OF THE ANOMALY IN CURVED SPACE

In this section I will derive the general formula for the anomaly in curved space. The regularized anomaly is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(x)]=\lim_{t\to 0}(2\pi)^{-2n}g^{-1/2}(x)\int d^{2n}k\,\mathrm{Tr}\{\beta\gamma_{2n+1}[1+\sqrt{t}\,(\not{\nabla}+i\not{k})]^{-1}+(\not{\nabla}\to\bar{\not{\nabla}})\} \ . \tag{6.1}$$

I will concentrate on evaluating the first term in the integral, an alternative form of which is given in Eq. (4.3):

$$\int d^{2n}k\,\mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}\,(\bar{\not{\nabla}}+i\not{k})]\frac{1}{1-t(\nabla_+ +ik)^2P_+ -t(\nabla_- +ik)^2P_- -2\sqrt{t}\,i\not{A}\gamma_{2n+1}}\right] \ . \tag{6.2}$$

The denominator of this can be written as

$$1-t(D+ik+A_+P_+ +A_-P_-)^2-[it(F_+\cdot\sigma P_+ +F_-\cdot\sigma P_-)-2\sqrt{t}\,i\not{A}\gamma_{2n+1}]+tr/4 \ .$$

The last term contributes nothing to the trace in the $t\to 0$ limit and will be dropped. The third term is defined as $X$ in Eq. (5.4). With these observations, the denominator of Eq. (6.2) effectively becomes

$$1-t(D+ik+A_+P_+ +A_-P_-)^2-X \ . \tag{6.3}$$

Introducing an auxiliary variable $u$, Eq. (6.2) can be rewritten as

$$\int_0^\infty du \exp -u \int d^{2n}k \, \mathrm{Tr}\{\beta\gamma_{2n+1}[1-\sqrt{t}(\overline{\nabla}+ik)]\exp[ut(D+ik+A_+P_++A_-P_-)^2+uX]\} \; . \tag{6.4}$$

The $k$ integration can be rewritten in the form

$$\begin{aligned}\int d^{2n}k \, \mathrm{Tr}(\cdots) &= \int d^{2n}k \, \mathrm{Tr}\{\beta\gamma_{2n+1}[1-\sqrt{t}(\overline{\nabla}+ik)E_1(k)E_2(k)]\} \\ &= \mathrm{Tr}\left[\beta\gamma_{2n+1}[1-\sqrt{t}(\overline{\nabla}+i\partial^y)]E_1(\partial^y)\int d^{2n}k \exp(k\cdot y)E_2(k)\right]_{y=0} .\end{aligned} \tag{6.5}$$

Here

$$\begin{aligned}&E_1(k)=\exp[ut(D+ik+A_+P_++A_-P_-)^2+uX]\exp[-ut(D+ik)^2] \; , \\ &E_2(k)=\exp[ut(D+ik)^2] \; .\end{aligned} \tag{6.6}$$

The first of these incorporates the gauge effects, while the second incorporates the effects of gravity.

By shifting the origin in $k$ space, one can move the $y$ dependence of the $k$ integral in (6.5) outside the integral. Letting $k \to k+z$, and using Eq. (C10) from Appendix C, we find

$$\begin{aligned}&\int d^{2n}k \exp[ut(D+ik)\cdot(D+ik)]\exp(y\cdot k) \\ &= \int d^{2n}k \left\{\exp\left[-utz^T\left[\frac{\sinh(Rut)}{Rut}\right]z+2iutz^T\left[\frac{1-\exp(-Rut)}{Rut}\right](D+ik)+y\cdot(k+z)\right]\exp[ut(D+ik)\cdot(D+ik)]\right\} .\end{aligned} \tag{6.7}$$

The demand that the first term be independent of $k$ is equivalent to

$$z=-\frac{1}{2}\frac{R}{1-\exp(Rut)}y \; , \tag{6.8}$$

where we have used the antisymmetry of $R_{\mu\nu}$. Making this substitution, Eq. (6.7) becomes

$$\begin{aligned}&\int d^{2n}k \exp[ut(D+ik)^2]\exp(y\cdot k) \\ &=\exp\left[\tfrac{1}{8}y^T\left[\frac{R}{1-\exp(Rut)}+\frac{3R}{1-\exp(-Rut)}\right]y+iy\cdot D\right]\int d^{2n}k \exp[ut(D+ik)\cdot(D+ik)] \\ &=\exp\left[\tfrac{1}{8}y^TR\coth\left[\frac{Rut}{2}\right]y+iy\cdot D\right]\int d^{2n}k \exp[ut(D+ik)\cdot(D+ik)] \; .\end{aligned} \tag{6.9}$$

[In going from the second to the third line I have used $y^TAy=\frac{1}{2}y^T(A+A^T)y$ and the antisymmetry of $R$.]

The term multiplying the integral will occur so often that it is useful to give it its own symbol

$$Z(y)=\exp\left[\tfrac{1}{8}y^TR\coth\left[\frac{Rut}{2}\right]y+iy\cdot D\right] . \tag{6.10}$$

We have the following identities:

$$Z^{-1}D_\mu Z=D_\mu-i\frac{R_{\mu\nu}}{2}y^\nu \tag{6.11}$$

and

$$Z^{-1}\partial^y_\mu Z=\partial^y_\mu+\frac{1}{4}\left[R\coth\left[\frac{Rut}{2}\right]+R\right]_{\mu\nu}y^\nu+iD_\mu \; . \tag{6.12}$$

The first of these follows from

$$e^ABe^{-A}=B+\frac{1}{1!}[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots \tag{6.13}$$

valid for any matrices $A$, $B$. The second follows from

$$(\partial^y_\mu U)=\left[\int_0^1 dq \, e^{q\ln U}(\partial^y_\mu \ln U)e^{-q\ln U}\right]U \; . \tag{6.14}$$

We are now in a position to evaluate

$$G(u)=\int d^{2n}k \exp[ut(D+ik)\cdot(D+ik)] \; .$$

Now, note that

$$\frac{dG}{du}=\int d^{2n}k[t(D+ik)\cdot(D+ik)]\exp[ut(D+ik)\cdot(D+ik)]$$
$$=t(D+i\partial^y)\cdot(D+i\partial^y)\int d^{2n}k\, e^{y\cdot k}\exp[ut(D+ik)\cdot(D+ik)]\,|_{y=0}$$
$$=t(D+i\partial^y)\cdot(D+i\partial^y)Z(y)\,|_{y=0}G(u)\,, \tag{6.15}$$

where I have used (6.9) and (6.10) to go from the second to the third line. Using (6.10) and (6.11), we find

$$\frac{dG}{du}=\frac{t}{4}\left[R\coth\left[\frac{Rut}{2}\right]\right]_{\mu\mu}G(u)\,. \tag{6.16}$$

(Note that $R_{\mu\mu}=0$.) Using

$$\frac{d}{du}\mathrm{Det}A=\mathrm{Det}A\ \mathrm{Tr}\left[\frac{dA}{du}A^{-1}\right] \tag{6.17}$$

valid for any matrix $A$, it follows that

$$G(u)=\mathrm{Det}\left[\sinh\left[\frac{Rut}{2}\right]\right]^{-1/2}\times\mathrm{const}\,. \tag{6.18}$$

On dimensional grounds the constant is proportional to $\mathrm{Det}R^{1/2}$. The constant of proportionality can be determined by looking at the $R\to 0$ limit of $G(u)$

$$G(y;R=0)=\int d^{2n}k\exp(-utk^2)=\left[\frac{\pi}{ut}\right]^n\,. \tag{6.19}$$

It follows that

$$G(u)=\left[\frac{\pi}{2}\right]^n\mathrm{Det}\left[\frac{R}{\sinh(Rut/2)}\right]^{1/2}\,. \tag{6.20}$$

We are now in a position to evaluate the gauge contribution to the anomaly. From (6.5)–(6.10), this piece is

$$[1-\sqrt{t}(\bar{\bar{\nabla}}+i\partial^y)]E_1(\partial^y)Z(y)\,|_{y=0}\,.$$

Making repeated use of (6.11) and (6.12), this can be rewritten as

$$\{1-\sqrt{t}[i(\partial^y_\mu+C_{\mu\nu}y^\nu)\gamma^\mu+A_+P_++A_-P_-]\}$$
$$\times\exp(-ut\{\partial^y+Cy-i(A_+P_++A_-P_-)+i[y\cdot D,A_+P_++A_-P_-]\}^2-uX+i[y\cdot D,uX])$$
$$\times\exp[ut(\partial^y+Cy)^2]\,|_{y=0}\,, \tag{6.21}$$

where

$$C_{\mu\nu}=\frac{1}{4}\left[R\coth\left[\frac{Rut}{2}\right]-R\right]_{\mu\nu}\,.$$

The terms involving $[y\cdot D,A^\mu_+P_+ + A^\mu_-P_-]$ in Eq. (6.21) do not contribute to the trace and will be set equal to 0. Equation (6.21) becomes

$$[1-\sqrt{t}(i\partial^y+A_+P_++A_-P_-)]\exp\{-ut[\partial^y+Cy-i(A_+P_++A_-P_-)]^2+uX-i[y\cdot D,uX]\}\exp[ut(\partial^y+Cy)^2]\,. \tag{6.22}$$

We shall now evaluate the product of the exponentials appearing in Eq. (6.22). Note that this product can be written as

$$\exp[+(H_0+H_1)]\exp(-H_0)\,, \tag{6.23}$$

where

$$H_0=-ut(\partial^y+Cy)^2\,,$$
$$H_1=+2ut(\partial^y+Cy)\cdot i(A_+P_++A_-P_-)+uX-i[y\cdot D,uX]\,. \tag{6.24}$$

If we define

$$W(s)=e^{sH_0}H_1e^{-sH_1} \tag{6.25}$$

then

$$e^{(H_0+H_1)}e^{-H_0}=I+\int_0^1 ds\, W(s)+\int_0^1 ds\int_0^s ds' W(s')W(s)+\int_0^1 ds\int_0^s ds'\int_0^{s'} ds'' W(s'')W(s')W(s)+\cdots . \tag{6.26}$$

Now, using the identity

$$[\partial_\mu^y+C_{\mu\nu}y^\nu,\partial_\alpha^y+C_{\alpha\beta}y^2]=-\tfrac{1}{2}R_{\mu\alpha} \tag{6.27}$$

and Eq. (6.13) one can prove that

$$\exp[-sut(\partial^y+Cy)^2](\partial_\mu^y+C_{\mu\nu}y^\nu)\exp[sut(\partial^y+Cy)^2]=[\exp(Rsut)]_{\mu\nu}(\partial_\mu^y+C_{\mu\alpha}y^\alpha) \tag{6.28}$$

and

$$\exp[-sut(\partial^y+Cy)^2]y_\mu\exp[sut(\partial^y+Cy)^2]=y_\mu-2sut\left[\frac{\exp(Rsut)-1}{Rsut}\right]_{\mu\nu}(\partial_\nu^y+C_{\nu\alpha}y^\alpha) . \tag{6.29}$$

With these two identities the evaluation of $W(s)$ is immediate. Note that by Eq. (4.20) it is sufficient to use the $R\to 0$ limit of $W(s)$. In this limit, $C_{\mu\nu}(ut)=(1/2ut)\delta_{\mu\nu}$, and

$$\lim_{R\to 0} W(s)=i(2ut\partial^y+y)\cdot(A_+P_++A_-P_-)+uX-i[(y(1-s)-2sut\partial^y)\cdot D,uX] . \tag{6.30}$$

From Eq. (6.21), we only need to keep the terms linear in $y$ in the evaluation of Eq. (6.26). Keeping these terms one finds

$$e^{(H_0+H_1)}e^{-H_0}=e^{uX}+i\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(uX)^j y\cdot(A_+P_++A_-P_-)(uX)^m}{(j+m+1)!}-i\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(uX)^j[y\cdot D,uX](uX)^m(m+1)}{(j+m+2)!} . \tag{6.31}$$

Substitute this expression into (6.22), one finds for the gauge contribution

$$[1-\sqrt{t}(\mathbf{A}_+P_++\mathbf{A}_-P_-)]\exp(uX)+\sqrt{t}\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(u\bar{X})^j(\mathbf{A}_+P_++\mathbf{A}_-P_-)(uX)^m}{(j+m+1)!}-\sqrt{t}\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(u\bar{X})^j[\not{D},uX](uX)^m(m+1)}{(j+m+2)!} . \tag{6.32}$$

Using the identity

$$[\not{D},X]=\bar{X}\left[\mathbf{A}_+P_++\mathbf{A}_-P_--\frac{1}{\sqrt{t}}\right]-\left[\mathbf{A}_+P_++\mathbf{A}_-P_--\frac{1}{\sqrt{t}}\right]X , \tag{6.33}$$

where we have neglected terms which vanish on taking the trace. The last sum can be evaluated:

$$\sqrt{t}\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(u\bar{X})^j[\not{D},uX]}{(j+m+2)!}(uX)^m(m+1)=-\left[\mathbf{A}_+P_++\mathbf{A}_-P_--\frac{1}{\sqrt{t}}\right]\exp(uX)+\sqrt{t}\sum_{\substack{j=0\\m=0}}^{\infty}\frac{(u\bar{X})^j(\mathbf{A}_+P_++\mathbf{A}_-P_--1/\sqrt{t})(uX)^m}{(j+m+1)!} . \tag{6.34}$$

Therefore, the gauge contribution of Eq. (6.21) is

$$\sum_{\substack{j=0\\m=0}}\frac{(u\bar{X})^j(uX)^m}{(j+m+1)!} . \tag{6.35}$$

Gathering all the intermediate results, the total anomaly, taking into account both gauge and gravitational effects and both terms in Eq. (6.1), is

$$\mathrm{Tr}[\beta(x)G_{2n+1}(X)]=\lim_{t\to 0}g^{-1/2}(x)\frac{1}{(4\pi)^n}\int_0^\infty du\exp(-u)\left[\frac{1}{ut}\right]^n\times\mathrm{Tr}\left[\gamma_{2n+1}\det\left[\frac{Rut/2}{\sinh(Rut/2)}\right]^{1/2}\sum_{j=m=0}^{\infty}\frac{\bar{X}^jX^m+X^j\bar{X}^m}{(j+m+1)!}u^{j+m}\right] .$$

In the limit of vanishing curvature, $R_{\alpha\beta}=0$, this reduces to Eq. (5.11). This satisfies the Wess-Zumino consistency condition since each term in the sum does.

## VII. CONCLUSIONS

In this paper I have derived the general formula for the non-Abelian anomaly of fermions coupled to vector and axial-vector fields in $2n$-dimensional curved spaces. Special cases of the general formula agree with all previously published results. The anomalies are due to the noninvariance of the functional measure under the linear transformations which generate the "naively" conserved currents. By general arguments, the anomalies must satisfy the Wess-Zumino consistency conditions. Coupling these two statements together puts constraints on the allowed definition of the function determinant. In particular, if one expands the measure in terms of eigenfunctions of the complete Dirac operator, one obtains anomalies which satisfy the consistency conditions. Other definitions of the functional measure give rise to incorrect results.

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## APPENDIX A: SPINORS IN CURVED SPACE

I work in a $2n$-dimensional curved Euclidean space. Both greek and latin indices run from 1 to $2n$. In terms of the vielbeins $e_\mu{}^a$ the metric $g_{\mu\nu}$ is

$$g_{\mu\nu}=e_\mu{}^a e_\mu{}^b \delta_{ab} \ . \tag{A1}$$

Let $\gamma_a$ be the (space-time-independent) generators of the Clifford algebra:

$$\{\gamma_a,\gamma_b\}=2\delta_{ab} \ . \tag{A2}$$

The $\gamma_a$ can always be chosen to be Hermitian. In terms of these we can define the generalization of the Dirac matrices appropriate for curved space:

$$\gamma_\mu=e_\mu{}^a\gamma_a \ . \tag{A3}$$

These satisfy

$$\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu} \ . \tag{A4}$$

Related to $\gamma_\mu$ are

$$\sigma^{\alpha\beta}=\tfrac{1}{4}[\gamma^\alpha,\gamma^\beta] \tag{A5}$$

which form a representation of the rotation group:

$$[\sigma^{\alpha\beta},\sigma_{\mu\nu}]=\sigma^{\alpha\nu}g^{\beta\mu}+\sigma^{\beta\mu}g^{\alpha\nu}-\sigma^{\alpha\mu}g^{\beta\nu}-\sigma^{\beta\nu}g^{\alpha\mu} \ . \tag{A6}$$

The commutator of $\sigma$ with $\gamma$ is also useful:

$$[\sigma^{\alpha\beta},\gamma^\mu]=\gamma^\alpha g^{\beta\mu}-\gamma^\beta g^{\alpha\mu} \ . \tag{A7}$$

In analogy with $\gamma_5$ in four dimensions, define

$$\gamma_{2n+1}=i^n\gamma_1\gamma_2,\ldots,\gamma_{2n} \ , \tag{A8}$$

where the $\gamma$'s on the RHS are the flat space $\gamma$'s. By construction

$$\{\gamma_{2n+1},\gamma_\mu\}=0,\quad (\gamma_{2n+1})^2=1 \ ,\quad \gamma_{2n+1}^{+}=\gamma_{2n+1} \ . \tag{A9}$$

$\gamma_{2n+1}$ obeys the trace identities

$$\begin{aligned}\mathrm{Tr}\{\gamma_{2n+1}\gamma_{\mu_1}\gamma_{\mu_2},\ldots,\gamma_{\mu_j}\}&=0 \quad \text{for } j<2n\\ &=2^n g^{1/2}(x)\epsilon_{\mu_1\mu_2,\ldots,\mu_{2n}} \\ &\qquad\qquad \text{for } j=2n \ ,\end{aligned} \tag{A10}$$

where $g(x)$ is the determinant of the metric.

In the absence of gauge fields the Dirac equation for a massless particle is

$$\gamma_\mu D^\mu\Psi=0 \ . \tag{A11}$$

$D_\mu$ is the generalization of the covariant (spatial) derivative appropriate for spinors:

$$D_\mu\equiv\partial_\mu+\omega_\mu=\partial_\mu+(\Gamma_{\alpha\nu\mu}+g_{\alpha\beta}e^a{}_\nu\partial_\mu e^\beta{}_a)\sigma^{\alpha\nu} \ . \tag{A12}$$

$D_\mu$ obeys the important identity

$$[D_\mu,\gamma_\nu]=0 \tag{A13}$$

which can be used as the definition of $D_\mu$. The commutator of two $D$'s generates the Riemann curvature tensor:

$$[D_\mu,D_\nu]=\tfrac{1}{2}R_{\mu\nu\alpha\beta}\sigma^{\alpha\beta}\equiv\tfrac{1}{2}R_{\mu\nu} \ . \tag{A14}$$

From the above identities it follows that

$$D^2=D_\mu D^\mu+\tfrac{1}{2}R_{\mu\nu}\sigma^{\mu\nu}=D_\mu D^\mu+\tfrac{1}{4}r \ , \tag{A15}$$

where $r$ is the scalar curvature.

The Dirac equation in the presence of gauge fields is

$$\gamma_\mu\nabla^\mu\Psi\equiv\gamma_\mu(D^\mu+iA_\mu{}^jT_j)\Psi=0 \ , \tag{A16}$$

where $T^j$ are the Hermitian generators of the gauge group in the representation of the fermions and $A_\mu{}^j$ is Hermitian. We then have

$$\nabla\!\!\!/^2=g_{\mu\nu}\nabla^\mu\nabla^\nu+\tfrac{1}{4}r+F_{\mu\nu}\sigma^{\mu\nu} \ , \tag{A17}$$

where $F_{\mu\nu}$ is the gauge field strength tensor in the representation of the fermions:

$$F_{\mu\nu}\equiv[\partial_\mu+A_\mu,\partial_\nu+A_j]=iF_{\mu\nu}{}^jT_j$$

and (A18)

$$A_\mu=A_\mu{}^jT^j \ .$$

## APPENDIX B: USEFUL FORMULAS

Let $C$ and $D$ be two operators involving an even number of $\gamma$ matrices, and $E$ an operator involving an odd number of $\gamma$ matrices. Consider

$$\left[1-E\gamma_{2n+1}\frac{1}{1-CP_+-DP_-}\right]\times(1-CP_+-DP_-+E\gamma_{2n+1})=1-CP_+-DP_-+E\frac{1}{1-CP_+-DP_-}E\ . \qquad \text{(B1)}$$

Hence, one can conclude that

$$(1-CP_+-DP_-+E\gamma_{2n+1})^{-1}=\left[1-CP_+-DP_-+E\frac{1}{1-CP_+-DP_-}E\right]^{-1}\times\left[1-E\gamma_{2n+1}\frac{1}{1-CP_+-DP_-}\right]. \qquad \text{(B2)}$$

Making repeated use of

$$(1-CP_+-DP_-)^{-1}=(1-C)^{-1}P_+ +(1-D)^{-1}P_- \qquad \text{(B3)}$$

and similar identities, one finds

$$(1-CP_+-DP_-+E\gamma_{2n+1})^{-1}=\left[1-C+E\frac{1}{1-D}E\right]^{-1}P_+\left[1+E\frac{1}{1-D}\right]+\left[1-D+E\frac{1}{1-C}E\right]^{-1}P_-\left[1-E\frac{1}{1-C}\right]. \qquad \text{(B4)}$$

A similar analysis leads to the identity

$$(1-CP_+-DP_-+E\gamma_{2n+1})^{-1}=\left[1-\frac{1}{1-D}E\right]P_+\left[1-C+E\frac{1}{1-D}E\right]^{-1}+\left[1+\frac{1}{1-C}E\right]P_-\left[1-D+E\frac{1}{1-C}E\right]^{-1}. \qquad \text{(B5)}$$

There are alternative forms of Eqs. (B4) and (B5) which are useful. I need the preliminary identities

$$\left[1-C+E\frac{1}{1-D}E\right]^{-1}=\frac{1}{1-C}-\frac{1}{1-C}E\left[1-D+E\frac{1}{1-C}E\right]^{-1}E\frac{1}{1-C} \qquad \text{(B6)}$$

and

$$\left[1-C+E\frac{1}{1-D}E\right]^{-1}E\frac{1}{1-D}=\frac{1}{1-C}E\left[1-D+E\frac{1}{1-C}E\right]^{-1} \qquad \text{(B7)}$$

which can be proved directly or by expanding the LHS in powers of $(1-C)^{-1}$, and the RHS in powers of $(1-D)^{-1}$. Using these identities, Eq. (B4) can be rewritten as

$$(1-CP_+-DP_-+E\gamma_{2n+1})^{-1}=\left[1-C+E\frac{1}{1-D}E\right]^{-1}P_+-\frac{1}{1-D}E\left[1-C+E\frac{1}{1-D}E\right]^{-1}P_+ +\frac{1}{1-D}P_- -\frac{1}{1-D}E\left[1-C+E\frac{1}{1-D}E\right]^{-1}E\frac{1}{1-D}P_- +\left[1-C+E\frac{1}{1-D}E\right]^{-1}E\frac{1}{1-D}P_-\ . \qquad \text{(B8)}$$

Similarly, Eq. (B5) becomes

$$(1-CP_+-DP_-+E\gamma_{2n+1})^{-1}=P_+\frac{1}{1-C}-P_+\frac{1}{1-C}E\left[1-D+E\frac{1}{1-C}E\right]^{-1}E\frac{1}{1-C}+P_+\frac{1}{1-C}E\left[1-D+E\frac{1}{1-C}E\right]^{-1} +P_-\left[1-D+E\frac{1}{1-C}E\right]^{-1}-P_-\left[1-D+E\frac{1}{1-C}E\right]^{-1}E\frac{1}{1-C}\ . \qquad \text{(B9)}$$

## APPENDIX C: SOME MORE USEFUL FORMULAS

In this appendix we prove that if $\{D_\alpha\}$ are any family of operators such that

$$[D_\alpha,D_\beta]=\tfrac{1}{2}R_{\alpha\beta}\ ,\quad [D_\alpha,R_{\mu\nu}]=0\ , \qquad \text{(C1)}$$

then

$$\exp[s(D+iZ)\cdot(D+iZ)]=\exp\left\{s\left[-Z^T\left[\frac{\sinh(Rs)}{Rs}\right]Z+2iZ\left[\frac{1-\exp(-Rs)}{Rs}\right]D\right]\right\}\exp(sD^2)\ . \qquad \text{(C2)}$$

Note that (C1) implies

$$[Z^T A(R)D, Z^T B(R)D] = \tfrac{1}{2} Z^T A(R) R B(-R) Z = -\tfrac{1}{2} Z^T A(-R) R B(R) Z \quad \text{(C3)}$$

for any matrices $A(R)$ and $B(R)$.

To prove (C2), let

$$U(s) = \exp[s(D+iZ)\cdot(D+iZ)]\exp(-sD^2) \ . \quad \text{(C4)}$$

Using Eq. (C1) and the Baker-Campbell-Hausdorf formula, this can be written as

$$U(s) = \exp[sB_{\mu\nu}(Rs)Z^\mu Z^\nu + sC_{\mu\nu}(Rs)Z^\mu D^\nu] \ , \quad \text{(C5)}$$

where $B$ and $C$ are functions of $sR_{\mu\nu}$. These can be determined by brute force, or by solving the differential equation

$$\frac{dU}{ds}U^{-1} = -Z^2 + \exp\{+[s(D+iZ)^2]\}2iZ\cdot D\exp\{-[s(D+iZ)^2]\}$$
$$= Z[1-2\exp(-Rs)]Z + 2iZ\exp(-Rs)D \quad \text{(C6)}$$

with the boundary condition $U(0)=1$. In this expression

$$(\exp(-Rs))^\mu{}_\nu = \delta^\mu{}_\nu - sR^\mu{}_\nu/1! + s^2 R^\mu{}_\alpha R^\alpha{}_\nu/2 \ldots .$$

Using the identity

$$\frac{dU}{ds}U^{-1} = \int_0^1 dq\, e^{q\ln U} d(\ln U)/ds\, e^{-q\ln U} \quad \text{(C7)}$$

and the form for $U(s)$ given in Eq. (6.10), we find

$$\frac{dU}{ds}U^{-1} = Z[d(Bs)/ds]Z + Z[d(Cs)/ds]D + \tfrac{1}{2}\{Z(Cs)D, Z[d(Cs)/ds]D\}$$
$$= Z([dB(Rs)]/ds + \tfrac{1}{4}\{C(Rs)sd[C(-Rs)s]/ds\})Z + Z\{d[C(Rs)s]/ds\}D \ . \quad \text{(C8)}$$

In going from the first to the second lines I have used Eq. (C3). Comparing Eqs. (C6) and (C8), we conclude

$$B(Rs) = -\sinh(Rs)/(Rs) \ ,$$
$$C(Rs) = 2i[1-\exp(-Rs)]/(Rs) \ . \quad \text{(C9)}$$

This proves Eq. (C2). Note that if $D$ is replaced by $(D+ik)$ on both sides of Eq. (C2) the equation remains valid since the new operators $[(D+ik)_\alpha]$ satisfy Eq. (C1). It is this translated version which is used in the text:

$$\exp\{s[D+i(Z+k)]^2\} = \exp\left\{s\left[-Z^T\left[\frac{\sinh(Rs)}{Rs}\right]Z + 2iZ\left[\frac{1-\exp(-Rs)}{Rs}\right](D+ik)\right]\right\} \times \exp[s(D+ik)^2] \ . \quad \text{(C10)}$$


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