Feynman path integral and the photon

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The construction of an overcomplete set of states for both the physical and artificial modes of the photon is examined in a representation with an indefinite metric. The Feynman path integral is then easily derived and the usual Green's functions, kernels, propagators, and Feynman rules follow immediately.

I. INTRODUCTION

In the preceding paper[1] we studied the connection between canonical quantum field theory and the Feynman-path-integral (FPI) formulation for scalar fields. In this paper we address the special problems associated with the photon field in proving the connection between these two formulations of field theory. Quantum electrodynamics (QED) is encumbered with many subtleties relating to the long range of the Coulomb interaction and the infrared divergences which necessitate much care being taken in formulating asymptotic states. These problems have been thoroughly studied[2] and we shall not elaborate on them further. It seems, however, that a simpler problem persists:[3] that of defining suitable coherent states in the Feynman-Gupta-Bleuler form of QED and using them to derive the FPI for the photon aspects of the theory.

Let us begin by reviewing coherent states in a space of negative metric. The negative metric gives rise to the minus signs in the following, otherwise standard, equations:

$$[a,a^\dagger]=-1\ , \tag{1}$$

$$a^\dagger a\,|\,n\,\rangle=n\,|\,n\,\rangle\ , \tag{2}$$

$$a^\dagger\,|\,n\,\rangle=\sqrt{n+1}\,|\,n+1\,\rangle\ , \tag{3}$$

$$a\,|\,n\,\rangle=-\sqrt{n}\,|\,n-1\,\rangle\ , \tag{4}$$

$$|\,n\,\rangle=\frac{(a^\dagger)^n}{\sqrt{n!}}\,|\,0\rangle\ . \tag{5}$$

The unit operator in this space has the decomposition[4]

$$1=\sum_n\,|\,n\,\rangle(-1)^n\langle n\,|\ . \tag{6}$$

One may choose to define a coherent state by[5]

$$a\,|\,z\,\rangle=z\,|\,z\,\rangle \tag{7}$$

and then

$$|\,z\,\rangle=\sum_n\frac{(-za^\dagger)^n}{\sqrt{n!}}\,|\,0\rangle\ . \tag{8}$$

Again the negative metric is responsible for the minus sign in (8).

The scalar product of two such coherent states is

$$\langle z'\,|\,z\,\rangle=e^{-z'^*z} \tag{9}$$

with the minus sign coming from (1). The functional completeness relation for coherent states is[1]

$$\int\mathscr{D}z\,e^{z'^*z}e^{z^*z''}=e^{z'^*z''}\ , \tag{10}$$

where

$$\mathscr{D}z=\frac{d^2z}{\pi}e^{-|z|^2} \tag{11}$$

and

$$d^2z=d\mathscr{R}z\,d\mathscr{I}z\ . \tag{12}$$

The measure in (11) is uniquely determined by (10). It is then obvious that (9) and (10) translate into[6]

$$\int\mathscr{D}z\,|\,z\,\rangle\langle-z\,|=1 \tag{13}$$

as the completeness relation for coherent negative-metric states. Specifically Eq. (10) is

$$\int\mathscr{D}z\langle-z'\,|\,z\,\rangle\langle-z\,|\,z''\rangle=\langle-z'\,|\,z''\rangle\ . \tag{14}$$

II. PHOTONS

We now apply the above results to describe photons in a representation with indefinite metric.[7]

Physical (transverse) photon states are created by operators $a_s^\dagger(\mathbf{k})$, $s=1,2$. The corresponding polarization vectors are $e_\mu^s(\mathbf{k})$, $\mu=1,2,3,0$. For example, with $\mathbf{k}$ in the z direction one might take

$$e_\mu^L(\mathbf{k})=\frac{1}{\sqrt{2}}(1,i,0,0)\ , \tag{15}$$

$$e_\mu^R(\mathbf{k})=\frac{1}{\sqrt{2}}(1,-i,0,0)\ . \tag{16}$$

The initial and final states are of the form

$$|\,a\,\rangle\equiv|\,n_1,n_2,\ldots)\equiv\prod_i\frac{(a_i^\dagger)^{n_i}}{\sqrt{n_i!}}\,|\,0\rangle\ , \tag{17}$$

where $i=1,2,\ldots,\infty$ correspond to the denumerable set $(s,\mathbf{k}_n)$ in box normalization.

The interaction will involve the photon field operators

$\hat{a}_\mu(x)$ that satisfy, in the interaction picture in the Feynman gauge,

$$[\hat{a}_\mu(x),\hat{a}_\nu(y)]=-ig_{\mu\nu}D(x-y) \tag{18}$$

with $g_{\mu\nu}=\mathrm{diag}(1,1,1,-1)$.

When we come to inserting a complete set of intermediate states in a transition-amplitude calculation, this set must contain the unphysical polarizations

$$e_\mu^3(\mathbf{k})=(0,0,1,0)\ , \tag{19}$$

$$e_\mu^0(\mathbf{k})=(0,0,0,1) \tag{20}$$

for the reason that the set of states must be complete in the space spanned by the four components of $\hat{a}_\mu$. Note that

$$e^{\mu s*}e_\mu^{s'}=g^{ss'} \tag{21}$$

and

$$\sum_{s,s'} e_\mu^{s*}e_\nu^{s'}g^{ss'}=g_{\mu\nu}\ . \tag{22}$$

In the Feynman gauge the IP field is given in terms of creation and destruction operators by

$$\hat{a}_\mu(x)=\sum_\alpha\left[a_\alpha\frac{e_\mu^\alpha e^{ikx}}{\sqrt{2\omega}}+a_\alpha^\dagger\frac{e_\mu^{\alpha*}e^{-ikx}}{\sqrt{2\omega}}\right]L^{-3/2} \tag{23}$$

where, with box normalization,

$$k_i\equiv k_{n_i}=n_i\pi/L\quad(i=1,2,3)\ ,$$

$$k_0=(\mathbf{k}^2)^{1/2}=\omega_n$$

$$\alpha=\{s,n_1,n_2,n_3\},\quad s=1,2,3,0\ ,$$

$$n_i=0,\pm1,\pm2,\ldots,$$

and

$$[a_{\mathbf{n},s},a_{\mathbf{m},s'}^\dagger]=g_{ss'}\delta_{n_1m_1}\delta_{n_2m_2}\delta_{n_3m_3}\ . \tag{24}$$

The negative-metric states correspond to the $s=0$ scalar photons.

III. COHERENT STATES

Coherent states are defined to satisfy

$$\hat{a}_\alpha|\xi\rangle=\xi_\alpha|\xi\rangle\ . \tag{25}$$

Noting the effect of the negative metric [Eq. (8)] the solution is

$$|\xi\rangle=\exp(g^{ss'}a_{\mathbf{n}s}^\dagger\xi_{\mathbf{n}s'})|0\rangle\ , \tag{26}$$

where a summation over repeated indices s,s' and $\mathbf{n}$ is called for. The results of Sec. I can now be generalized to

$$\int\mathscr{D}\xi|\xi\rangle\langle\bar{\xi}|=1\ , \tag{27}$$

where

$$\langle\bar{\xi}|\equiv\langle0|\exp(a_{\mathbf{n},s}\xi_{\mathbf{n}s}^*) \tag{28}$$

and

$$\mathscr{D}\xi\equiv\prod_{\mathbf{n},s}\left[\frac{d^2\xi_{\mathbf{n}s}}{\pi}e^{-|\xi_{\mathbf{n},s}|^2}\right]\ . \tag{29}$$

Note that the bra, $\langle\bar{\xi}|$, is just $\langle\xi|$ with ξ_0^* replaced by $-\xi_0^*$ so that

$$\langle\bar{\xi}|a_{\mathbf{n},s}^\dagger=\langle\bar{\xi}|\xi_{\mathbf{n},s'}^*g^{ss'} \tag{30}$$

and

$$\langle\bar{\xi}'|\xi\rangle=e^{\xi_{\mathbf{n},s}'^*\xi_{\mathbf{n},s}}\ . \tag{31}$$

IV. TRANSITION AMPLITUDES

To calculate the transition amplitude $\langle bt_b|at_a\rangle$ one proceeds in the usual way: break up the interval $T=t_b-t_a$ into $N+1$ equal pieces of length ϵ and label the times $t_j=t_a+j\epsilon$. Insert a complete set of states

$$1=\int\mathscr{D}\xi|\xi\rangle\langle\bar{\xi}|=\int\mathscr{D}\xi|\xi t\rangle\langle\bar{\xi}t| \tag{32}$$

for times t_b and t_a. Here $|\xi t\rangle$ is the Heisenberg state related to $|\xi\rangle$ by[1]

$$|\xi t\rangle=e^{i\hat{H}t}|\xi\rangle\ . \tag{33}$$

One easily gets

$$\begin{aligned}\langle bt_b|at_a\rangle&=\int\mathscr{D}\xi_b\mathscr{D}\xi_a\langle b|\xi_b\rangle\langle\bar{\xi}_bt_b|\xi_at_a\rangle\langle\bar{\xi}_a|a\rangle\\&\equiv\int\mathscr{D}\xi_b\mathscr{D}\xi_a\Psi_b^*(\xi_b)\langle\bar{\xi}_bt_b|\xi_at_a\rangle\Psi_a(\xi_a)\ ,\end{aligned} \tag{34}$$

where the wave functions entering are

$$\Psi_a(\xi_a)\equiv\langle\xi_a|a\rangle=\langle\bar{\xi}_a|a\rangle=\langle\bar{\xi}_at_a|at_a\rangle\ . \tag{35}$$

Note that $\bar{\xi}$ can be replaced by ξ in the initial- and final-state wave functions since these states contain only transverse photons [Eq. (17)].

Now insert a complete set at each intermediate time to get, with $t_0=t_a$, $t_{N+1}=t_b$, $\xi_0=\xi_a$, $\xi_{N+1}=\xi_b$,

$$\langle\bar{\xi}_bt_b|\xi_at_a\rangle=\int\prod_{j=0}^N\langle\bar{\xi}_{j+1}t_{j+1}|\xi_jt_j\rangle\prod_{j=1}^N\mathscr{D}\xi_j\ . \tag{36}$$

We use (32) to write this in the form

$$\langle\bar{\xi}_bt_b|\xi_at_a\rangle=\int\prod_1^N\mathscr{D}\xi_j\prod_0^N\langle\bar{\xi}_{j+1}|e^{-i\epsilon\hat{H}}|\xi_j\rangle \tag{37}$$

with $\hat{H}=\hat{H}_{\mathrm{IP}}(0)$.

With the transition amplitude in the form (37) it is easy to see why the representation of the unit operator (32) must involve a sum over both physical and artificial modes of the photon: while $|a\rangle$ is a vector in the physical subspace, the ket $e^{-i\epsilon\hat{H}}|a\rangle$ has components in all of the photon modes, physical *and* artificial.

Finally, one can see that in the limit $\epsilon\to0$, $N\epsilon\to T$, the transition amplitude is exactly

$$\langle\bar{\xi}_bt_b|\xi_at_a\rangle=\int\prod_1^N\mathscr{D}\xi_j\prod_0^N\langle\bar{\xi}_{j+1}|\xi_j\rangle\exp[-i\epsilon\langle\bar{\xi}_{j+1}|\hat{H}_{\mathrm{IP}}(0)|\xi_j\rangle/\langle\bar{\xi}_{j+1}|\xi_j\rangle]\ . \tag{38}$$

It is then straightforward to derive the Feynman-path-integral expression for the transition amplitude; we are interested primarily in the dependence of (38) on the expectation value of the photon field operator $\hat{a}_\mu(x)$. This dependence will be investigated in the following sections.

V. FREE-PHOTON KERNEL

The free-photon Hamiltonian in normal-mode form is

$$\hat{H}^0_{\rm IP}(0)=\sum_{n,s}\omega_n a^\dagger_{ns}a_{ns'}g^{ss'} \tag{39}$$

so, remembering (30)

$$\langle\bar{\xi}_{j+1}|\hat{H}^0_{\rm IP}(0)|\xi_j\rangle=\sum_{n,s}\omega_n\xi^*_{j+1ns}\xi_{jns}\langle\bar{\xi}_{j+1}|\xi_j\rangle . \tag{40}$$

Therefore, using (31), the free particle kernel is

$$\langle\bar{\xi}_b t_b|\xi_a t_a\rangle_0=\int\prod_{j=1}^{N}\mathscr{D}\xi_j\prod_{j=0}^{N}\exp\left[\sum_{ns}\gamma_n\xi^*_{j+1ns}\xi_{jns}\right], \tag{41}$$

where

$$\gamma_n=1-i\epsilon\omega_n . \tag{42}$$

Using (10) repeatedly, one easily obtains the result

$$\langle\bar{\xi}_b t_b|\xi_a t_a\rangle_0=\prod_{ns}\exp(\xi^*_{bns}\xi_{ans}e^{-i\omega_n T}) . \tag{43}$$

VI. EXTERNAL SOURCES

We write the free kernel in the form

$$\langle\xi_b t_b|\xi_a t_a\rangle_0=e^{|\xi_a|^2+|\xi_b|^2}\int\left[\prod_1^N\frac{d^2\xi_j}{\pi}\right]e^{-\xi^\dagger M\xi} . \tag{44}$$

The dependence on normal-mode indices n,s is suppressed as we have seen in (43) so that all modes contribute equally. Here

$$\xi=\begin{bmatrix}\xi_0\\ \xi_1\\ \vdots\\ \xi_{N+1}\end{bmatrix}, \tag{45}$$

$$M=\begin{bmatrix}1 & 0 & 0 & \cdots\\ -\gamma & 1 & 0 & \cdots\\ 0 & -\gamma & 1 & \cdots\\ \vdots & & & \end{bmatrix}. \tag{46}$$

Now define the free kernel with external sources

$$\langle\xi_b t_b|\xi_a t_a\rangle_\zeta=e^{|\xi_a|^2+|\xi_b|^2}\int\left[\prod_1^N\frac{d^2\xi_j}{\pi}\right]e^{-\xi^\dagger M\xi+\xi^\dagger\zeta+\bar{\zeta}\xi} . \tag{47}$$

Set

$$\xi=\tilde{\xi}+M^{-1}\zeta \tag{48}$$

and identify

$$\bar{\zeta}=\zeta^\dagger(M^{-1})^\dagger M \tag{49}$$

and one has

$$\langle\xi_b t_b|\xi_a t_a\rangle_\zeta=\langle\xi_b t_b|\xi_a t_a\rangle_0 e^{\bar{\zeta}M^{-1}\zeta}$$
$$\equiv\langle\xi_b t_b|\xi_a t_a\rangle_0\prod_{ns}e^{\bar{\zeta}_{ns}M^{-1}{}_{ns}\zeta_{ns}} . \tag{50}$$

VII. GREEN'S FUNCTION

Equation (21) gives us the interaction-picture field in terms of destruction and creation operators:

$$\hat{a}(x)=\hat{a}^{(+)}(x)+\hat{a}^{(-)}(x) . \tag{51}$$

According to (25) and (30) their eigenvalues with coherent states are

$$\hat{a}^{(+)}_\mu(x)|\xi\rangle=L^{-3/2}\sum_{n,s}\xi_{ns}e^s_\mu(\mathbf{k}_n)(e^{ik_n\cdot x}/\sqrt{2\omega_n})|\xi\rangle \tag{52}$$

$$\equiv\xi_\mu(x)|\xi\rangle \tag{53}$$

and

$$\begin{aligned}\langle\bar{\xi}|\hat{a}^{(-)}_\mu(x)&\\ &=\langle\bar{\xi}|L^{-3/2}\sum_{n,s,s'}g^{ss'}\xi^*_{n,s'}e^{s*}_\mu(\mathbf{k}_n)e^{-ik_n\cdot x}/\sqrt{2\omega_n}\\ &\equiv\langle\bar{\xi}|\bar{\xi}_\mu(x) .\end{aligned} \tag{54}$$

Note that, of course, $\bar{\xi}_\mu$ is not the complex conjugate of ξ_μ because $\langle\bar{\xi}|$ is not the Hermitian conjugate of $|\xi\rangle$. The eigenvalues of the Heisenberg fields

$$\hat{A}^{H(\pm)}_\mu(\mathbf{x},t)\equiv e^{i\hat{H}t}\hat{a}^{(\pm)}_\mu(\mathbf{x},0)e^{-i\hat{H}t} \tag{55}$$

with the Heisenberg states (33) are accordingly given by

$$\hat{A}^{H(+)}_\mu(\mathbf{x},t)|\xi,t\rangle=\xi_\mu(\mathbf{x},0)|\xi,t\rangle , \tag{56}$$

$$\langle\bar{\xi},t|\hat{A}^{H(-)}_\mu(\mathbf{x},t)=\langle\bar{\xi},t|\bar{\xi}^\mu(\mathbf{x},0) . \tag{57}$$

Now consider the ba matrix element of $\hat{A}^H_\mu$:

$$\langle\xi_b t_b|\hat{A}^H_\mu(\mathbf{x},t)|\xi_a t_a\rangle$$

for $t_a<t<t_b$. As above we can break up the interval (t_b,t_a) into N intermediate times and insert a complete set of Heisenberg states at each time. Make sure that $\hat{A}^{H(+)}_\mu(\mathbf{x},t_k)$ is between $\langle\bar{\xi}_{k+1}|$ and $|\xi_k t_k\rangle$ and that $\hat{A}^{H(-)}_\mu(\mathbf{x},t_k)$ is between $\langle\bar{\xi}_k t_k|$ and $|\xi_{k-1}t_{k-1}\rangle$. Then we have

$$\langle\xi_b t_b|\hat{A}^H_\mu(\mathbf{x},t_k)|\xi_a t_a\rangle=\int\prod_1^N\mathscr{D}\xi_j\mathscr{A}^k_\mu(\mathbf{x},0)\prod_0^N\langle\bar{\xi}_{j+1}t_{j+1}|\xi_j t_j\rangle , \tag{58}$$

where

$$\mathscr{A}^k_\mu(\mathbf{x},0)\equiv\bar{\xi}^k_\mu(\mathbf{x},0)+\xi^k_\mu(\mathbf{x},0) . \tag{59}$$

Repeating the above steps gives us the following expression for the Green's function:

$$\langle \bar{\xi}_b t_b \mid [\hat{A}^H_{\mu_1}(x_1)\cdots \hat{A}^H_{\mu_n}(x_n)]_+ \mid \xi_a t_a \rangle = \int \prod_1^N \mathscr{D}\xi_j \mathscr{A}^{k_1}_{\mu_1}(\mathbf{x}_1)\cdots \mathscr{A}^{k_n}_{\mu_n}(\mathbf{x}_n) \times \prod_0^N \langle \bar{\xi}_{j+1} t_{j+1} \mid \xi_j t_j \rangle , \tag{60}$$

where $x_i = (\mathbf{x}_i, t_{ki})$.

VIII. FREE-PHOTON PROPAGATORS

We can use (60) and (51) to evaluate the propagator in the well-known fashion. Briefly the operation of $\hat{A}^H_\mu(\mathbf{x}, t_k)$ on the left of (60) is equivalent to the operation of

$$L^{-3/2}\sum_{n,s}\left[\frac{e^s_\mu(\mathbf{k}_n)e^{i\mathbf{k}_n\cdot\mathbf{x}}}{\sqrt{2\omega_n}}\frac{\partial}{\partial\bar{\xi}_{kns}} + \frac{g^{ss'}e^{s'*}_\mu(\mathbf{k}_n)e^{-i\mathbf{k}_n\cdot\mathbf{x}}}{\sqrt{2\omega_n}}\frac{\partial}{\partial\xi_{kns}}\right] \tag{61}$$

on (51). Then since

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \gamma & 1 & 0 & 0 & \cdots \\ \gamma^2 & \gamma & 1 & 0 & \cdots \\ \gamma^3 & \gamma^2 & \gamma & 1 & \cdots \\ \vdots \end{bmatrix} \tag{62}$$

one has

$$M^{-1}{}_{kl} = \gamma^{k-l} \text{ if } k > l , \quad M^{-1}{}_{kl} = 0 \text{ if } k < l , \tag{63}$$

and

$$\gamma^{k-l} \to e^{-i\omega(t_k - t_l)} \text{ as } \epsilon \to 0 . \tag{64}$$

Then one easily finds with (22)

$$\langle \bar{\xi}_b t_b \mid [\hat{A}^H_\mu(\mathbf{x}, t_k)\hat{A}^H_\nu(\mathbf{y}, t_l)]_+ \mid \xi_a t_a \rangle_0 = g_{\mu\nu}L^{-3}\sum_n \frac{e^{i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y}) - i\omega_n|t_k - t_l|}}{2\omega_n}\langle \bar{\xi}_b t_b \mid \xi_a t_a\rangle_0 = -ig_{\mu\nu}D_c(\mathbf{x}-\mathbf{y}, t_k - t_l)\langle \bar{\xi}_b t_b \mid \xi_a t_a \rangle_0 \tag{65}$$

as one should hope.

These calculations are, of course, elementary; the reason we show them is to demonstrate that the usual Feynman rules follow from the formalism of Sec. III without further ado. There is no need to impose the subsidiary condition on the intermediate states as suggested in Ref. 3. In conventional QED current conservation in the Heisenberg picture prevents the generation of the artificial modes (briefly, the subsidiary condition prohibits the $3-0$ combination of modes from the initial and final states and current conservation prevents the combination $3+0$ from coupling); it also does the job in the FPI formalism. A bit of care is required, however, because the current appearing in (38) via

$$\langle \bar{\xi}_{j+1} \mid \hat{H}^{(1)}_{\mathrm{IP}}(0) \mid \xi_j \rangle = ie\int d^3x \langle \bar{\xi}_{j+1} \mid \hat{a}_\mu(\mathbf{x})\hat{j}^u(\mathbf{x}) \mid \xi_j \rangle$$

i.e., the current

$$J^u(\mathbf{x}, t_j) \equiv \langle \bar{\xi}_{j+1} \mid \hat{j}^u(\mathbf{x}) \mid \xi_j \rangle$$

(with the kets now including the charged-particle states) is *not* conserved simply because the parameters ξ_j may be arbitrarily chosen. On the other hand, once the integrations over all the ξ_j's have been performed the free-particle time dependence is restored via the charged-particle equivalent of Eq. (64) and the current to which $\hat{a}_\mu(\mathbf{x})$ couples is again conserved.

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